

Homework 9 (Ling 726) – Model Theory

October 22, 2006

I. The elementary formal system L in Section 8.5.4. of PtMW

1. Find three models for L other than the one given in the text.

M_0 : the model given in the text

$$D = \{1^*, 2^*, 3^*\}, R^* = \{\langle 1^*, 2^* \rangle, \langle 2^*, 3^* \rangle, \langle 3^*, 1^* \rangle\},$$

$$\text{Interpretation function } I_0(1) = 1^*, I_0(2) = 2^*, I_0(3) = 3^*, I_0(R) = R^*.$$

M_1 :

$$I_1(1) = 2^*, I_1(2) = 3^*, I_1(3) = 1^*.¹$$

M_2 :

$$I_2(1) = 3^*, I_2(2) = 1^*, I_2(3) = 2^*.$$

M_3 :

$$R_3^* = \{\langle 2^*, 1^* \rangle, \langle 1^*, 3^* \rangle, \langle 3^*, 2^* \rangle\}, I_3(1) = 3^*, I_3(2) = 2^*, I_3(3) = 1^*.$$

2. If the deletion of a certain axiom δ from a formally complete system Δ changes the system into one which is not formally complete, then that axiom is independent. Why?

A system Δ is formally complete iff its deductive closure Δ^c is maximally consistent. Suppose that the axiom to be deleted, δ , is not independent. Then the deductive closure of the system $\Delta - \{\delta\}$ does not change from Δ^c , so it must be also formally complete. Hence the supposition that δ is not independent cannot be maintained.

3. If the deletion of a certain axiom δ changes a formal system Δ from categorical to non-categorical, must that axiom be independent? Why?

Yes, it must be independent. The reason is as follows. A formal system Δ is categorical if all of its models are isomorphic. The class of all models for $\Delta - \{\delta\}$, $\text{MOD}(\Delta - \{\delta\})$, is given by

$$\begin{aligned} \text{MOD}(\Delta - \{\delta\}) &:= \{M \mid M \models \Delta - \{\delta\}\} \\ &= \{M \mid M \models \Delta - \{\delta\} \text{ and } M \models \delta\} \cup \{M \mid M \models \Delta - \{\delta\} \text{ and } M \not\models \delta\}. \end{aligned}$$

The first term of the last line is just $\text{MOD}(\Delta)$, so all the members of it are isomorphic. Therefore, if $\Delta - \{\delta\}$ is non-categorical, the set $\{M \mid M \models \Delta - \{\delta\} \text{ and } M \not\models \delta\}$ must be non-empty. It means that δ must be independent.

¹ Throughout Part I of this homework, I show only changes from M_0 to specify a model.

4. Find two models for axioms A2-A6 which are not isomorphic to the models of L nor two each other. What does this tell you about axiom A1?

M_4 :

$$D = \{1^*, 2^*, 3^*, 4^*\}, R^* = \{\langle 1^*, 2^* \rangle, \langle 2^*, 3^* \rangle, \langle 3^*, 4^* \rangle, \langle 4^*, 1^* \rangle\},$$

$$I_4(1) = 1^*, I_4(2) = 2^*, I_4(3) = 3^*, I_4(4) = 4^*.$$

M_5 :

$$D = \{1^*, 2^*, 3^*, 4^*, 5^*\}, R^* = \{\langle 1^*, 2^* \rangle, \langle 2^*, 3^* \rangle, \langle 3^*, 4^* \rangle, \langle 4^*, 5^* \rangle, \langle 5^*, 1^* \rangle\},$$

$$I_5(1) = 1^*, I_5(2) = 2^*, I_5(3) = 3^*, I_5(4) = 4^*, I_5(5) = 5^*.$$

It tells that axiom A1 is independent (from the result of **Question 3** above).

5. If A2 is replaced by A2': $\forall x Rxx$, is the resulting system consistent? If so, find a model for it. If not, deduce a contradiction from the new set of axioms.

No, the resulting system is inconsistent. Deduction of a contradiction is following (here, I follow the notational conventions of PtMW).

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|----------------------------------------------------------------------------------------|----------------------------|
| 1. $\forall x (x=1 \vee x=2 \vee x=3) \wedge 1 \neq 2 \wedge 1 \neq 3 \wedge 2 \neq 3$ | axiom A1 |
| 2. $\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow y = z)$ | axiom A3 |
| 3. $\forall x Rxx$ | axiom A2' |
| 4. $R12$ | axiom A6 |
| 5. $R11$ | 3, Universal Instantiation |
| 6. $R11 \wedge R12 \rightarrow 1=2$ | 2, Universal Instantiation |
| 7. $1=2$ | 4, 5, 6, Modus Ponens |
| 8. $1 \neq 2$ | 1, Simplification |
| 9. $1=2 \wedge 1 \neq 2$ | 7, 8, Conjunction |

6. If A2 is replaced by A2' as above and A3 and A4 are deleted, is the resulting system consistent? Justify as above. Is the resulting system categorical? If not, find two non-isomorphic models for it.

The resulting system is consistent. We can find a model for it.

M_6 :

$$R_3^* = \{\langle 1^*, 2^* \rangle, \langle 1^*, 1^* \rangle, \langle 2^*, 2^* \rangle, \langle 3^*, 3^* \rangle\}.$$

The system is not categorical because we can find a model which is not isomorphic to the above model:

M_7 :

$$R_3^* = \{\langle 1^*, 2^* \rangle, \langle 1^*, 3^* \rangle, \langle 1^*, 1^* \rangle, \langle 2^*, 2^* \rangle, \langle 3^*, 3^* \rangle\}.$$

7. What happens if we replace A5 by A5': $\exists y \forall x Rxy$? Is the resulting axiom system consistent or inconsistent? Justify as above.

The resulting system is inconsistent. Deduction of a contradiction is follows.

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|----------------------------------------------------------------------------------------|---------------|
| 1. $\forall x (x=1 \vee x=2 \vee x=3) \wedge 1 \neq 2 \wedge 1 \neq 3 \wedge 2 \neq 3$ | axiom A1 |
| 2. $\forall x \forall y \forall z ((Ryx \wedge Rzx) \rightarrow y = z)$ | axiom A4 |
| 3. $\exists y \forall x Rxy$ | axiom A5' |
| 4. $\forall x Rxxw$ | 3, E.I. |
| 5. $R1w$ | 4, U.I. |
| 6. $R2w$ | 4, U.I. |
| 7. $\forall y \forall z ((Ryw \wedge Rzw) \rightarrow y = z)$ | 2, U.I. |
| 8. $(R1w \wedge R2w) \rightarrow 1=2$ | 7, U.I. |
| 9. $1=2$ | 5, 6, 8, M.P. |
| 10. $1 \neq 2$ | 1, Simp. |
| 11. $1=2 \wedge 1 \neq 2$ | 9, 10, Conj. |

8. Let axioms A2-A6 be replaced by the single axiom A2'': $\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow y \neq z)$. Is this system consistent? Is it categorical? Justify your answers.

This system is consistent. Consider a model M_8 , whose binary relation R_8^* is an empty set.

M_8 :

$$R_8^* = \emptyset.$$

This model satisfies all the axioms of the system, A1 and A2''.

Moreover, to satisfy axiom A2'', R^* must be an empty set. So the difference of all the models for this system can be seen as the difference of interpretation functions only. If we define a mapping $f: D_M \rightarrow D_N$ from a model for this system \mathbf{M} to any other model for this system \mathbf{N} as $f(I_M(c)) = I_N(c)$ for any $c \in \{1, 2, 3\}$, then f is an isomorphism between \mathbf{M} and \mathbf{N} . Hence this system is categorical.

9. Show that axiom A3 is not independent in the system L .

We can obtain A3 from the other axioms.

1. $\sim R11 \wedge \sim R22 \wedge \sim R33$ (A2)

- 2. $R12$ (A6)
- 3. $\sim R32$ (2. and A4)
- 4. $R31$ (1., 3. and A5)
- 5. $\sim R21$ (4. and A4)
- 6. $R23$ (1., 5. and A5)
- 7. $\sim R13$ (6. and A4)
- 8. $\forall x \forall y \forall z ((Rxy \wedge Rxz) \rightarrow y = z)$ (1.-7. and A1) This is A3!

II. The axiom system W in PtMW Ch.8, p.235, question 13.

10. Is the axiom system W consistent? Is it categorical?

We know that W is consistent because we found a model for it, namely $P = \{1, 2, 3, 4\}$, $L = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}, \{1,3\}, \{2,4\}\}$, in the last homework.

Is it categorical? In my opinion, it is not categorical because there is a model quite different (so not isomorphic) from the above one: we can consider a flat plane as a model for the formal system W . Precisely, if we take points and straight lines² on a flat plane as the primitives of W , namely “points” and “lines”, then all the axioms hold.

11. What do your answers to the questions 13a-e tell you about the independence of various of the axioms of W ?

Nothing.

(After finishing this homework, I checked Student Solution on the web and found that it claims as follows:

It can be shown that there can be no model for W in which P has exactly two members, so if A1, A3, A4, A5 hold, A2 must hold. A2 can be proved from the remainder axioms, so it is not independent.

But this does not make sense because we had to resort to A2 when we derived the result (13d), that there can be no model for W in which P has exactly two members. In fact, consider a model which has only one point and no line. This model satisfies A1, A3, A4 and A5, but does not satisfy A2. Hence A2 is independent.)

² Note that not straight *line segments*, but straight *lines* correspond to “lines” in W because of axiom A3 (axiom A5 also requires that).