Homework 10. Model Theory.

0. PROVING FACT 2

<u>SHOW:</u> If Δ is maximally consistent, then Δ is deductively closed.

Assume to the contrary that Δ is not deductively closed, then there is at least one ϕ such that $\phi \notin \Delta$ ans $\Delta \vdash \phi$. Δ is consistent, so Δ U {ϕ} is consistent. $\Delta \subset \Delta$ U {ϕ}, so Δ is NOT maximally consistent, hence Δ must be deductively closed. ☺

1. THREE INTERPRETED MODELS FOR L

<u>M1</u>

M1 = <{0,1,2}, I>

I(**R**) = {<0,1>,<1,2>, <2,0>}

I(**1**) = 0, I(**2**) = 1, I(**3**) = 2

<u>M2</u>

M2 = <{Kurt, Alfred, Gottlob}, I>

I (**R**) = {<Kurt, Alfred>, <Alfred, Gottlob>, <Gottlob, Kurt>}

I (**1**) = Kurt, I(**2**) = Alfred, I(**3**) = Gottlob

<u>M3</u>

M2 = <{  **,**  **,**   }, I >

I (**R**) = {<  ,  >, <  ,  >, <  ,  > }

I (**1**) =  , I(**2**) =  , I(**3**) =

2. INDEPENDENCE OF AXIOMS AND FORMAL COMPLETENESS

Assume for any Δ that Δ^c is maximally consistent. Let Δ'^c U {ϕ} = Δ^c and assume that Δ'^c is not maximally consistent. Assume to the contrary that $\Delta'^c \vdash \phi$, then $\phi \in \Delta'^c$ and Δ'^c U {ϕ} = Δ'^c = Δ^c, and then Δ^c would be and would not be maximally consistent.☺

3. Independence of Axioms and Categoricity

Consider $\mathit{m} =_{df} \{M: \forall\partial \in \Delta\, [M \vDash \partial]\ \}$. Assume that all $M \in \mathit{m}$ are isomorphic.

Consider $\mathit{m1} =_{df} \{M: \forall\partial: \partial \neq \partial^{*}\ \&\ \partial \in \Delta\ [M \vDash \partial]\ \}$. Assume that not all $M \in \mathit{m1}$ are isomorphic.

Assume that ∂^{*} is not independent. Then, $\mathit{m} = \mathit{m1}$ contra the assumption that all $M \in \mathit{m}$ are isomorphic and not all $M \in \mathit{m1}$ are isomorphic. ☺

4. Models for A2-A6

Let L1 be the formal system that results from A2-A6. Consider the following models:

<u>M4</u>

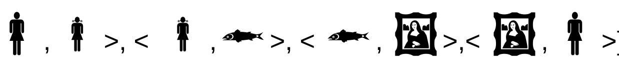
M4 = <{Kurt, Alfred, Gottlob, Bertrand}, I>

I (**R**) = {<Kurt, Alfred>, <Alfred, Gottlob>, <Gottlob, Bertrand>, <Bertrand, Kurt>}

I (**1**) = Kurt, I(**2**) = Alfred, I(**3**) = Gottlob

<u>M5</u>

M5 = <{  }, I >

I (**R**) = {<  >}

I (**1**) =  , I(**2**) =  , I(**3**) =

M1-M5 are models for L1. It is obvious that neither M4 nor M5 are isomorphic to any of the M1-M3, for the cardinality of the domains of M4 and M5 is 4. L1 is not categorical.

L is a categorical system (PMW 205), so deleting A1 from L, changes the system from categorical to non-categorical. Since M4-M5 satisfy all A2-A6 but do not satisfy A1, A1 is independent.

5. A2'

Replacing A2 with A2' results in an inconsistent system, to wit:

0.	$\forall x\ (x = 1 \lor x = 2 \lor x = 3)\ \&\ 1 \neq 2\ \&\ 1 \neq 3\ \&\ 2 \neq 3$	A1
1.	$\forall x[Rxx]$	A2'
2.	$\forall x \forall y \forall z\ [\ (Rxy\ \&\ Rxz) \rightarrow y = z]$	A3
3.	R12	A6

4.	R11	1, ∀-out
5.	R11 & R12 → 1 = 2	2, ∀-out
6.	1 = 2	3,4, → out
7.	1 ≠ 2	A1, SL
8.	⊥	6,7,SL

6. L + A2' – (A3-A4) = L2

Let L2 be the result of replacing A2 with A2' and deleting A3 and A4. L2 is consistent, since it is obvious that M6 is a model for L2.

M6

M6 = <{0,1,2}, I>

I (**R**) = R1 = {<0,0>, <0,1>, <1,1>,<2,2>}

I(**1**) = 0, I(**2**) = 1, I(**3**) = 2

L2 is not categorical, for consider M7.

M7

M7 = <{1,2,3}, I>

I (**R**) = R2 = {<1,1>, <1,2>, <2,2>, <2,3>, <3,3>}

I(**1**) = 1, I(**2**) = 2, I(**3**) = 3

M7 is a model for L2. It has only three distinct individuals. Every individual bears R to itself, so every individual bears R to some individual and R12. But, obviously, M6 is not isomorphic to M7, since, although the domains are of the same cardinality, R1 is not 'preserved' in M7.

7. L3

L3 results from L by subtituting A5 for A5'. L3 is inconsistent:

0.	∀x (x = 1 V x = 2 V x = 3) & 1 ≠ 2 & 1 ≠ 3 & 2 ≠ 3	A1
1.	∃y∀x[Rxy]	A5'
2.	∀x∀y∀z [(Ryx & Rzx) → y = z]	A4
3.	R12	A6
4.	∀xRx2	1, ∃-out
5.	R32	1, ∀-out

6.	$R12 \,\&\, R32 \rightarrow 1 = 3$	2, $\forall$-out
7.	$1 = 3$	3,4,5 SL
8.	$1 \neq 3$	A1, SL
9.	$\perp$	6,7,SL

8. L4

L4 is the result of substituting A2-A6 with A2'':

A2'': $\forall x \forall y \forall z\, ((Rxy \,\&\, Rxz) \rightarrow y \neq z)$

L4 is consistent, since M8 is a model for it:

<u>M8</u>

M8 = <{1,2,3}, I>

I (**R**) = {<1,1>, <1,2>, <2,2>,<3,3>, <3,1>, <3,2>}

I(**1**) = 1, I(**2**) = 2, I(**3**) = 3

L4 is not categorical, since M9 is also a model for L4, but it is not isomorphic to M8.

> *****Instructors' comments**: No, A2'' is incompatible with any model with a non-empty R. For instance, your model M8 includes R12. But by A2'', R12 & R12 → 2 not-equal 2. And 2 not-equal 2 is a contradiction. So the relation must be empty. And the system is therefore categorical.

<u>M9</u>

M9 = <{1,2,3}, I>

I (**R**) = {<1,1>, <1,2>, <2,2>, <3,3>}

I(**1**) = 1, I(**2**) = 2, I(**3**) = 3

9. A3 IS NOT INDEPENDENT

A1.	$\forall x\, (x = 1 \vee x = 2 \vee x = 3) \,\&\, 1 \neq 2 \,\&\, 1 \neq 3 \,\&\, 2 \neq 3$
A2.	$\forall x \sim[Rxx]$
A3.	$\forall x \forall y \forall z\, [(Ryx \,\&\, Rzx) \rightarrow y = z]$
A4.	$\forall x \exists y\, [Rxy]$
A5.	R12

We derive A3 from the previous set of premises as follows:

1. From A2, A3, A4 and A6 we have ~R32.

2. From A2, A3 and A4, we have R31 v R32. From 1, we have R31.

3. Since R31, and A2 and A3 hold, we have ~R21.

4. From A2, A3 and A4, we have R21 v R23. From 3 we have R23.

5. From A2, we have ~R11, since R23 and A3, then ~R13

6. We have ~R21, from A2 we have ~R22, and from 4 we have R23.

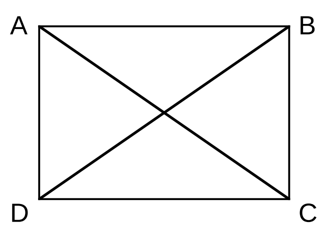
7. From 2 we have R31, from 1 we have ~R32 and from A2 we have ~R33

8. From 5-7 and A1 we conclude that $\forall x \forall y \forall z$ [(Rxy & Rxz) $\rightarrow$ y = z] Q.E.D. ☺

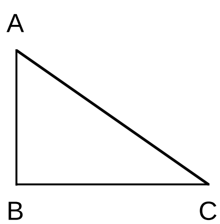
10. Is W consistent? Is it categorical?

W is consistent, since we have at least one model that satisfies all axioms of W, namely:

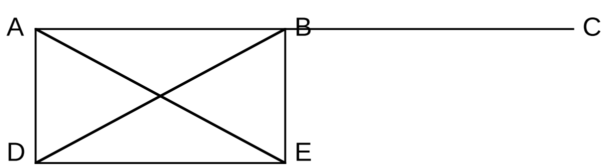
<u>M*</u>



Is W categorical? I models for W are isomorphic to M* . How can a model for W fail to be isomorphic to the one above? Well, maybe there are less or more primitives than in M*. First, models with none or just one point fail to satisfy A2. There is no model for W with three points, because for any line l, the only distinct point not in l is in a l' that is not disjoint from l.

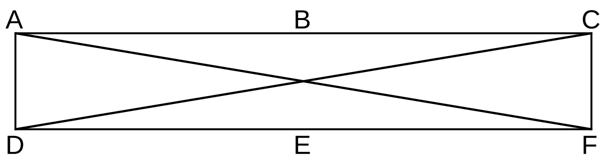


Are there any models with more than 4 points? Not with 5. In M*1 below, there is no unique line between C and any two distinct points:

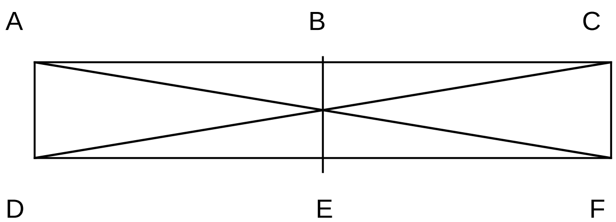


[**Instructors' comment**: But showing that one putative model fails doesn't constitute a proof that there can't be a model with 5 points. Similarly with your suggestive but not conclusive remarks about the 6-point case -- that's not a proof. (And we don't have one.)

Not with 6 points. In the model below A3 fails to hold.



In the following variant, A5 fails to hold: {A,B,C} is a line, but there is no unique line containing E disjoint from {A,B,C}, since {B,E} is not disjoint.



I am not in a position to *prove* it, but it seems to me obvious that the only way to satisfy all axioms is by considering models with 4 points.

Then since all axioms are independent, any model with 4 points cannot fail to satisfy all axioms of W.

11. On the Independence of the axioms of W.

It can be shown that there can be no model for W in which P has exactly two members, so if A1, A3, A4, A5 hold, A2 must hold. A2 can be proved from the remainder axioms, so it is not independent.