9.3 VENN16

Take the usual $\Omega_{BA} = \{1, 0, \&, V, -\}$.

Let $B$ be the set of distinct Venn diagrams with just two circles (where a Venn diagram $A$ is distinct from $B$ iff their shaded areas are distinct). These are the members of $B$:

Ssh

1

2

3

4
We consider the $\Omega_{BA}$-algebra Venn16 on the carrier B with the following interpretation for the operators in the signature:

(i) $[[1]] = \text{the Venn diagram in which the whole rectangle is shaded.}$

(ii) $[[0]] = \text{the Venn diagram with no shaded area}$

(iii) $[[\&]] = \text{the operation consisting on replacing two Venn diagrams with the one which has as the only shaded areas those that are shaded in both of the original diagrams.}$

(iv) $[[V]] = \text{the operation consisting on replacing two Venn diagrams with the one which has shaded everything shaded in either of the original diagrams.}$

(v) $[[\sim]] = \text{the operation of replacing a Venn diagram by one in which the only shaded areas are those that are not shaded in the original.}$

Venn2 is a subalgebra of Venn16

Consider $B_1 = \{\text{Shaded}, \text{Ssh}\}$

(i) $B_1 \subseteq B$

(ii) The operations denoted by the members of the signature are defined as follows:

one = Shaded

zero = Ssh

$\sim (\text{Shaded}) = \text{Ssh}$

$\sim (\text{Ssh}) = \text{Shaded}$

Shaded $\&$ Shaded = Shaded

Shaded $\&$ Ssh = Ssh $\&$ Shaded = Ssh

Ssh $\&$ Ssh = Ssh

Shaded V Shaded = Shaded

Shaded V Ssh = Ssh V Shaded = Shaded

Ssh V Ssh = Ssh

(iii) it is easily seen that $B_1$ is closed with respect all the operations. 1 and 0 are in $B_1$. If $\text{Shaded} \& \text{Ssh} = \text{Ssh}$, and $\text{Ssh} \in B_1$. Shaded V Ssh = Shaded $\in B_1$, $\sim \text{Shaded} = \text{Ssh} \in B_1$ and $\sim \text{Ssh} = \text{Shaded} \in B_1$.

Hence the $\Omega_{BA}$-algebra Venn2 on $B_1$ for the given interpretation for the operations is a subalgebra of Venn16.

h: Venn16 $\rightarrow$ Venn2
Consider the set of variables \(\{X,Y\}\) ranging over members of \(B\). We define a function \(h: \{X,Y\} \rightarrow B1\), as follows: \(h = \{\langle X, Ssh \rangle, \langle Y, \text{Shaded} \rangle\}\)

Consider the following mapping \(h^\#: \text{Venn16} \rightarrow \text{Venn2}\)

\[
h^\#(\text{Shaded}) = \text{Shaded}
\]

\[
h^\#(X) = h(X)
\]

\[
h^\#(Y) = h(Y)
\]

\[
h^\#(\neg X) = \neg (h(X))
\]

\[
h^\#(\neg Y) = \neg (h(Y))
\]

\[
h^\#(X \cap Y) = h(X) \cap h(Y)
\]

\[
h^\#(X \cup Y) = h(X) \cup h(Y)
\]

\(h^\#\) is a homomorphism, since it can be verified that:

(i) \(h(\text{zero}_{\text{Venn16}}) = \text{zero}_{\text{Venn2}}\)

(ii) \(h(\text{one}_{\text{Venn16}}) = \text{one}_{\text{Venn2}}\)

(iii) \(\forall X \in \text{Venn16} \ h^\#(\neg_{\text{Venn16}} X) = \neg_{\text{Venn2}} (h^\#(X)).\)

Since \(h^\#(\neg_{\text{Venn16}} X) = \neg (Ssh) = \text{Shaded}\)

(iv) \(\forall X \in \text{Venn16} \ h^\#(X \cap_{\text{Venn16}} Y) = h^\#(X) \cap_{\text{Venn2}} h^\#(Y)\)

Since \(h^\#(X \cap_{\text{Venn16}} Y) = h(X) \cap_{\text{Venn16}} h(Y) = Ssh\)

(v) \(\forall X \in \text{Venn16} \ h^\#(X \cup_{\text{Venn16}} Y) = h^\#(X) \cup_{\text{Venn2}} h^\#(Y)\)

Since \(h^\#(X \cup_{\text{Venn16}} Y) = h(X) \cup_{\text{Venn16}} h(Y) = \text{Shaded}\)