9.2 Defining the congruence \( \equiv \) in algebraic terms.

Consider language SL1.

Syntax for SL1

Let \( AT = \{ p, q \} \)

We define the set \( W \) of well formed formulas of SL1 as the smallest set such that:

(i) \( AT \cup \{ \text{one, zero} \} \subset W \)

(ii) If \( x, y \in W \), then \( \neg x, (x \land y), (x \lor y) \in W \)

Given \( \Omega_{\text{Numb}} = \{ \text{zero, one, \( \neg \)}, \land, \lor \} \) where \( \text{zero} \) and \( \text{one} \) are 0-ary operators, \( \neg \) is a unary operator and \( \land, \lor \) are binary operators, we consider the word algebra \( W_{\OmegaBA}(\text{Atom}) \) on the carrier \( AT \cup \{ \text{one, zero} \} \) with operations defined in a natural way ( \( \text{zero} \land \text{zero} = \text{zero} \land \text{zero} \) …)

Semantics for SL1. Defining logical equivalence.

We now consider the \( \Omega_{\text{Numb}} \)-algebra \( T \) on the carrier \( \{0,1\} \) with the usual interpretation for the operations ( \( 0 \land 0 = 0, \neg 1 = 0 \)…)

- \( F = \{ f_1, f_2, f_3, f_4 \} \) is the set of possible mappings \( f: AT \to \{0,1\} \), where 
  \[ f_1 = \{ < p,0 >, < q,0 > \}, f_2 = \{ < p,1 >, < q,1 > \}, f_3 = \{ < p,1 >, < q,0 > \}, f_4 = \{ < p,0 >, < q,1 > \} \]

- Every \( f_n \in F \) defines the corresponding homomorphism \( f_n^\#: W_{\OmegaBA}(\text{Atom}) \to T \)

- Each homomorphism \( f_n^\#: W_{\OmegaBA}(\text{Atom}) \to T \) defines a congruence \( Q_n \) on \( W_{\OmegaBA}(\text{Atom}) \) : for any two formulas \( \phi, \phi' \in W_{\OmegaBA}(\text{Atom}) \) (= W), \( < \phi, \phi' > \in Q_n \) iff \( f_n^\#(\phi) = f_n^\#(\phi') \).

Examples: \( < p, p \land p > \in Q_1, < \neg p, \neg p \land p > \in Q_1, < p \& q, p > \in Q_2, < p \lor q, p > \in Q_3, < p, \neg q > \in Q_4 \… \)

- For any \( f_n \), \( \ker f_n = f_n \circ f_n^{-1} \) induces a partition on \( W_{\OmegaBA}(\text{Atom}) (=W) \) into two classes: the class of formulas that map to 1 and the set of formulas that map to 0.

Examples: \( < p, p \& p > \in \ker f_1, < p, p \& q > \notin \ker f_2, < \text{one}, p > \in \ker f_3, < \text{one}, q > \in \ker f_4 \… \)

\( \ker f_1 \) induces a partition on \( W \) into two equivalence classes \( \{ [\text{one}] \} \) and \( \{ [\text{zero}] \} \).
Some members of $[[\text{zero}]]$ are: zero, $(p \& p)$, $(q \& q)$, $(p \& q)$, $(\text{one} \& \text{zero})$, $(\text{zero} \& \text{zero})$, $(p \& \text{one})$, $(\text{zero} \lor \text{zero})$.

Some members of $[[\text{one}]]$ are: one, $\text{one} \& \text{one}$, $\neg p$, $\neg q$, $(\neg p \& \neg q)$, $(\neg p \lor q)$ …

For $\ker f_2$ we have $\text{zero} \in [[\text{zero}]]$, $\neg p \in [[\text{zero}]]$, $\neg q \in [[\text{zero}]]$, $p \in [[\text{one}]]$, $p \lor p \in [[\text{one}]]$ …

For $\ker f_3$ we have $\text{zero} \in [[\text{zero}]]$, $\neg p \in [[\text{zero}]]$, $\neg q \in [[\text{one}]]$, $p \in [[\text{one}]]$, $p \lor p \in [[\text{one}]]$ …

For $\ker f_4$ we have $\text{zero} \in [[\text{zero}]]$, $\neg p \in [[\text{one}]]$, $\neg q \in [[\text{zero}]]$, $p \in [[\text{zero}]]$, $p \lor p \in [[\text{zero}]]$ …

• $\bigcap \{\ker f_1 \ldots \ker f_4\}$ contains pairs of formulas that have the same truth value for any valuation.

Illustration. $\bigcap \{\ker f_1 \ldots \ker f_4\}$ contains those formulas that are true under $f_1$: the members of $[[\neg(p \& q)]]$ and those that are false under $f_1$, the members of $[[p \& q]]$

Similarly, for $f_2$ we get the members of $[[p \& q]]$ and the members of $[[\neg(p \& q)]]$.

For $f_3$ we get the members of $[[p \& q]]$ and the members of $[[p \& \neg q]]$ and for $f_4$ the members of $[[p \lor q]]$ and those of $[[p \lor \neg q]]$

Instructors’ note on the last part of the above. The answer above was going perfectly up until the last step. (Readers can compare other posted answers in part to see which is easier to read, a “list” that shows equivalences, as here, or “multiplication tables” with correspondences indicated by shading or the like, as in some of the other students’ answers. But those are questions of presentation, there are no substantive differences there.) But at the last step, when we get to the intersection, there’s an error.

If we understand you rightly, you’ve simply accidentally answered as if it were union, rather than intersection, in this last step. The result is a weird equivalence that isn’t of any “interest” that we can see. On the other hand, intersection gives a nice equivalence, namely logical equivalence. Each “$\ker f_i$” is analogous to a single row of a truth table; the intersection of all of them is analogous to the whole truth table – equivalence with respect to the intersection means having the same (last column in the) whole truth table, hence logical equivalence.