9.1

$\equiv_{Mod4}$ partitions **N** into classes of natural numbers that are equivalent Mod4

N/$\equiv_{Mod4}$ = { [[4]], [[5]], [[6]], [[7]] }

We are given the operators in Ω_{Numb} = {**one, zero, +, x**}. We define the operations named by the members of Ω_{Numb} as follows:

zero = [[4]]

one = [[5]]

+ = For any x ∈ Nat/$\equiv_{Mod4}$, x + [[4]] =x + [[4]] = x

[[5]] + [[5]] = [[6]]

[[6]] + [[6]] = [[4]]

[[7]] + [[7]] = [[5]]

[[5]] + [[6]] = [[6]] + [[5]] = [[7]]

[[5]] + [[7]] = [[7]] + [[5]] = [[4]]

[[6]] + [[7]] = [[7]] + [[6]] = [[5]]

x = For any x ∈ Nat/$\equiv_{Mod4}$, [[4]] times x = [[4]]

For any x ∈ Nat/$\equiv_{Mod4}$, [[5]] times x= [[5]]

[[6]] x [[6]] = [[4]]

[[7]] x [[7]] = [[5]]

[[5]] x [[6]] = [[6]] x [[5]] = [[6]]

[[5]] x [[7]] = [[7]] x [[5]] = [[7]]

[[6]] x [[7]] = [[7]] x [[6]] = [[5]]

N/$\equiv_{Mod4}$ is the quotient algebra of **N** by the congruence $\equiv_{Mod4}$ since the natural mapping **N** → Nat/$\equiv_{Mod4}$ is a homomorphism.

f_{nat} (4 + 5) = f_{nat} (4) $+_{Nat/\equiv Mod4}$ f (5), i.e. [[5]] = [[5]]

f_{nat} (10 x 30) = f_{nat} (10) $+_{Nat/\equiv Mod4}$ f (30), i.e. [[4]] = [[6]] $x_{Nat/\equiv Mod4}$ [[6]]

…

2. Show that **N**/$\equiv_{Mod4}$ is isomorphic to **Mod4**

We show that there is a homomorphism f: N/$\equiv_{Mod4}$ → Mod4 such that f^{-1} is a function and a homorphism.

Let f: $N/\equiv_{Mod4} \rightarrow$ Mod4 = { <[[4]],0>, <[[5]],1>, <[[6]], 2>, <[[7]],3>}

a) f is a homomorphism:

zero

$f(\mathbf{zero}_{\mathbf{Nat}/\equiv Mod4}) = f([[4]]) = 0$

one

$f(\mathbf{one}_{\mathbf{Nat}/\equiv Mod4}) = f([[5]]) = 1$

+

$f([[4]] +_{\mathbf{Nat}/\equiv Mod4} [[4]]) = f([[4]]) +_{Mod4} f([[4]]) = 0$

$f([[4]] +_{\mathbf{Nat}/\equiv Mod4} [[5]]) = f([[4]]) +_{Mod4} f([[5]]) = 1$

$f([[4]] +_{\mathbf{Nat}/\equiv Mod4} [[6]]) = f([[4]]) +_{Mod4} f([[6]]) = 2$

$f([[4]] +_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[4]]) +_{Mod4} f([[7]]) = 3$

$f([[5]] +_{\mathbf{Nat}/\equiv Mod4} [[5]]) = f([[5]]) +_{Mod4} f([[5]]) = 2$

$f([[5]] +_{\mathbf{Nat}/\equiv Mod4} [[6]]) = f([[5]]) +_{Mod4} f([[6]]) = 3$

$f([[5]] +_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[5]]) +_{Mod4} f([[7]]) = 0$

$f([[6]] +_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[6]]) +_{Mod4} f([[7]]) = 1$

$f([[7]] +_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[7]]) +_{Mod4} f([[7]]) = 1$[1]

x

$f([[4]]\ x_{\mathbf{Nat}/\equiv Mod4} [[4]]) = f([[4]])\ x_{Mod4} f([[4]]) = 0$

$f([[4]]\ x_{\mathbf{Nat}/\equiv Mod4} [[5]]) = f([[4]])\ x_{Mod4} f([[5]]) = 0$

$f([[4]]\ x_{\mathbf{Nat}/\equiv Mod4} [[6]]) = f([[4]])\ x_{Mod4} f([[6]]) = 0$

$f([[4]]\ x_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[4]])\ x_{Mod4} f([[7]]) = 0$

$f([[5]]\ x_{\mathbf{Nat}/\equiv Mod4} [[5]]) = f([[5]])\ x_{Mod4} f([[5]]) = 1$

$f([[5]]\ x_{\mathbf{Nat}/\equiv Mod4} [[6]]) = f([[5]])\ x_{Mod4} f([[6]]) = 2$

$f([[5]]\ x_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[5]])\ x_{Mod4} f([[7]]) = 3$

$f([[6]]\ x_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[6]])\ x_{Mod4} f([[7]]) = 1$

$f([[7]]\ x_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[6]])\ x_{Mod4} f([[7]]) = 0$

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[1] Since both $+_{\mathbf{Nat}/\equiv Mod4}$ and $+_{Mod4}$ are commutative $f([[6]] +_{\mathbf{Nat}/\equiv Mod4} [[7]]) = f([[7]] +_{\mathbf{Nat}/\equiv Mod4} [[6]])$ $f([[6]]) +_{Mod4} f([[7]]) = f([[7]]) +_{Mod4} f([[6]])$ …

It can be easily verified that f^{-1}: Mod4 → $Nat/\equiv_{Mod4}$= { <0,[[4]],>, <1,[[5]]>, <2, [[6]]>, <3, [[7]]>} is a homomorphism and a function:

(i) **zero**

f^{-1} (**zero**$_{Mod4}$) = **zero**$_{\mathbf{Nat}/\equiv Mod4}$, since [[4]] =[[4]]

(ii) **one**

f^{-1} (**one**$_{Mod4}$) = **one**$_{\mathbf{Nat}/\equiv Mod4}$, since [[5]] =[[5]]

(iii)**+**

$f^{-1}(0 +_{Mod4} 0) = f^{-1}(0) +_{Nat/\equiv Mod4} f^{-1}(0) = [[4]]$

$f^{-1}(0 +_{Mod4} 1) = f^{-1}(0) +_{Nat/\equiv Mod4} f^{-1}(1) = [[5]]$

$f^{-1}(0 +_{Mod4} 2) = f^{-1}(0) +_{Nat/\equiv Mod4} f^{-1}(2) = [[6]]$

$f^{-1}(0 +_{Mod4} 3) = f^{-1}(0) +_{Nat/\equiv Mod4} f^{-1}(3) = [[7]]$

$f^{-1}(1 +_{Mod4} 1) = f^{-1}(1) +_{Nat/\equiv Mod4} f^{-1}(1) = [[6]]$

$f^{-1}(1 +_{Mod4} 2) = f^{-1}(1) +_{Nat/\equiv Mod4} f^{-1}(2) = [[7]]$

$f^{-1}(1 +_{Mod4} 3) = f^{-1}(1) +_{Nat/\equiv Mod4} f^{-1}(3) = [[4]]$

$f^{-1}(2 +_{Mod4} 3) = f^{-1}(2) +_{Nat/\equiv Mod4} f^{-1}(3) = [[4]]$

$f^{-1}(3 +_{Mod4} 3) = f^{-1}(3) +_{Nat/\equiv Mod4} f^{-1}(3) = [[5]]$

(iii)**x**

$f^{-1}(0\ x_{Mod4}\ 0) = f^{-1}(0)\ x_{Nat/\equiv Mod4}\ f^{-1}(0) = [[4]]$

$f^{-1}(0\ x_{Mod4}\ 1) = f^{-1}(0)\ x_{Nat/\equiv Mod4}\ f^{-1}(1) = [[4]]$

$f^{-1}(0\ x_{Mod4}\ 2) = f^{-1}(0)\ x_{Nat/\equiv Mod4}\ f^{-1}(2) = [[4]]$

$f^{-1}(0\ x_{Mod4}\ 3) = f^{-1}(0)\ x_{Nat/\equiv Mod4}\ f^{-1}(3) = [[4]]$

$f^{-1}(1\ x_{Mod4}\ 1) = f^{-1}(1)\ x_{Nat/\equiv Mod4}\ f^{-1}(1) = [[5]]$

$f^{-1}(1\ x_{Mod4}\ 2) = f^{-1}(1)\ x_{Nat/\equiv Mod4}\ f^{-1}(2) = [[6]]$

$f^{-1}(1\ x_{Mod4}\ 3) = f^{-1}(1)\ x_{Nat/\equiv Mod4}\ f^{-1}(3) = [[7]]$

$f^{-1}(2\ x_{Mod4}\ 3) = f^{-1}(2)\ x_{Nat/\equiv Mod4}\ f^{-1}(3) = [[5]]$

$f^{-1}(3\ x_{Mod4}\ 3) = f^{-1}(3)\ x_{Nat/\equiv Mod4}\ f^{-1}(3) = [[4]]$