

HW5 Ex5 Ans1

Question 5

About powersets

Let $\mathcal{P}(A)$ be an algebra as defined on p. 4 of the handout.

Let $B, C, D \in \mathcal{P}(A)$.

(B1) **Associative laws**

$$(B \cap C) \cap D = B \cap (C \cap D)$$

$$(B \cup C) \cup D = B \cup (C \cup D)$$

Associative law holds of set intersection and union [see PtMW p. 18]

(B2) **Commutative laws**

$$B \cap C = C \cap B$$

$$B \cup C = C \cup B$$

Commutative law holds of set intersection and union [see PtMW p. 18]

(B3) **Distributive laws**

$$B \cap (C \cup D) = (B \cap C) \cup (B \cap D)$$

$$B \cup (C \cap D) = (B \cup C) \cap (B \cup D)$$

Distributive law holds of set intersection and union [see PtMW p. 18]

(B4) **Top and Bottom Laws**

[1] $B \cap 1 = B$

[2] $B \cap A = B$

[3] And [2] is true.

This is what the law states.

Definition of algebra $\mathcal{P}(A)$ ($1 = A$)

Identity law for set intersection – PtMW p. 18.

[1] $B \cap 0 = 0$

[2] $B \cap \emptyset = \emptyset$

[3] And [2] is true.

Definition of algebra $\mathcal{P}(A)$ ($0 = \emptyset$)

Identity law for set intersection – PtMW p. 18.

[1] $B \cup 1 = 1$

[2] $B \cup A = A$

[3] And [2] is true.

Definition of algebra $\mathcal{P}(A)$ ($1 = A$)

Identity law for set union – PtMW p. 18.

[1] $B \cup 0 = B$

[2] $B \cup \emptyset = B$

[3] And [2] is true.

Definition of algebra $\mathcal{P}(A)$ ($0 = \emptyset$)

Identity law for set union – PtMW p. 18.

(B5) **Complementation Laws**

[1] $B \cap (\neg B) = 0$

[2] $B \cap B' = \emptyset$

[3] And [2] is true.

Definition of algebra $\mathcal{P}(A)$ ($\neg B = B'$)

($0 = \emptyset$)

Complement law for set intersection – PtMW p. 18.

$$[1] B \cup (\neg B) = 1$$

$$[2] B \cup B' = A$$

[3] And [2] is true.

Definition of algebra \mathcal{A} ($\neg B = B'$)
($1 = A$)

Complement law for set intersection –
PtMW p. 18.

Therefore, (B1) to (B5) holds of \mathcal{A} , and \mathcal{A} is therefore a Boolean algebra.

*** Instructors' note: so far, perfect, and probably can't be any shorter. ***

About Truth-Values

Let \mathbf{T} be an algebra as defined on p. 4 and 5 of the handout.

(B1) Associative laws

$$(1 \wedge 1) \wedge 1 = 1 \wedge 1 = 1$$

$$(1 \wedge 0) \wedge 1 = 0 \wedge 1 = 0$$

$$(0 \wedge 1) \wedge 1 = 0 \wedge 1 = 0$$

$$(0 \wedge 0) \wedge 1 = 0 \wedge 1 = 0$$

$$(1 \wedge 1) \wedge 0 = 1 \wedge 0 = 0$$

$$(1 \wedge 0) \wedge 0 = 0 \wedge 1 = 0$$

$$(0 \wedge 1) \wedge 0 = 0 \wedge 1 = 0$$

$$(0 \wedge 0) \wedge 0 = 0 \wedge 1 = 0$$

$$1 \wedge (1 \wedge 1) = 1 \wedge 1 = 1$$

$$1 \wedge (0 \wedge 1) = 1 \wedge 0 = 0$$

$$0 \wedge (1 \wedge 1) = 0 \wedge 1 = 0$$

$$0 \wedge (0 \wedge 1) = 0 \wedge 0 = 0$$

$$1 \wedge (1 \wedge 0) = 1 \wedge 0 = 0$$

$$1 \wedge (0 \wedge 0) = 1 \wedge 0 = 0$$

$$0 \wedge (1 \wedge 0) = 0 \wedge 0 = 0$$

$$0 \wedge (0 \wedge 0) = 0 \wedge 0 = 0$$

$$(1 \vee 1) \vee 1 = 1 \vee 1 = 1$$

$$(1 \vee 0) \vee 1 = 1 \vee 1 = 1$$

$$(0 \vee 1) \vee 1 = 1 \vee 1 = 1$$

$$(0 \vee 0) \vee 1 = 0 \vee 1 = 1$$

$$1 \vee (1 \vee 1) = 1 \vee 1 = 1$$

$$1 \vee (0 \vee 1) = 1 \vee 1 = 1$$

$$0 \vee (1 \vee 1) = 0 \vee 1 = 1$$

$$0 \vee (0 \vee 1) = 0 \vee 1 = 1$$

$$(1 \vee 1) \vee 0 = 1 \vee 0 = 1$$

$$(1 \vee 0) \vee 0 = 1 \vee 0 = 1$$

$$(0 \vee 1) \vee 0 = 1 \vee 0 = 1$$

$$(0 \vee 0) \vee 0 = 0 \vee 0 = 0$$

$$1 \vee (1 \vee 0) = 1 \vee 1 = 1$$

$$1 \vee (0 \vee 0) = 1 \vee 0 = 1$$

$$0 \vee (1 \vee 0) = 0 \vee 1 = 1$$

$$0 \vee (0 \vee 0) = 0 \vee 0 = 0$$

***Instructor's comment:** There should be a shorter way. But I (BHP) can remember sometimes being caught off guard by operations that I thought were associative but didn't check carefully enough and it turned out they weren't. So unless you can find an *argument* of some kind, I don't know any shortcuts, including with pictures.

(B2) Commutative laws

$$1 \wedge 1 = 1$$

$$1 \wedge 0 = 0$$

$$0 \wedge 1 = 0$$

$$0 \wedge 0 = 0$$

$$1 \wedge 1 = 1$$

$$0 \wedge 1 = 0$$

$$1 \wedge 0 = 0$$

$$0 \wedge 0 = 0$$

$$1 \vee 1 = 1$$

$$1 \vee 0 = 1$$

$$0 \vee 1 = 1$$

$$0 \vee 0 = 0$$

$$1 \vee 1 = 1$$

$$0 \vee 1 = 1$$

$$1 \vee 0 = 1$$

$$0 \vee 0 = 0$$

*** **Instructor's note.** This I do know how to verify with a picture. Draw the operation table, and observe that it's symmetrical in the diagonal axis that goes from top left to bottom right. If it is, it must be commutative.

(B3) **Distributive laws**

$$1 \wedge (1 \vee 1) = 1 \wedge 1 = 1$$

$$1 \wedge (1 \vee 0) = 1 \wedge 1 = 1$$

$$1 \wedge (0 \vee 1) = 1 \wedge 1 = 1$$

$$1 \wedge (0 \vee 0) = 1 \wedge 0 = 0$$

$$(1 \wedge 1) \vee (1 \wedge 1) = 1 \vee 1 = 1$$

$$(1 \wedge 1) \vee (1 \wedge 0) = 1 \vee 0 = 1$$

$$(1 \wedge 0) \vee (1 \wedge 1) = 0 \vee 1 = 1$$

$$(1 \wedge 0) \vee (1 \wedge 0) = 0 \vee 0 = 0$$

$$0 \wedge (1 \vee 1) = 0 \wedge 1 = 0$$

$$0 \wedge (1 \vee 0) = 0 \wedge 1 = 0$$

$$0 \wedge (0 \vee 1) = 0 \wedge 1 = 0$$

$$0 \wedge (0 \vee 0) = 0 \wedge 0 = 0$$

$$(0 \wedge 1) \vee (0 \wedge 1) = 0 \vee 0 = 0$$

$$(0 \wedge 1) \vee (0 \wedge 0) = 0 \vee 0 = 0$$

$$(0 \wedge 0) \vee (0 \wedge 1) = 0 \vee 0 = 0$$

$$(0 \wedge 0) \vee (0 \wedge 0) = 0 \vee 0 = 0$$

$$1 \vee (1 \wedge 1) = 1 \vee 1 = 1$$

$$1 \vee (1 \wedge 0) = 1 \vee 0 = 1$$

$$1 \vee (0 \wedge 1) = 1 \vee 0 = 1$$

$$1 \vee (0 \wedge 0) = 1 \vee 0 = 1$$

$$(1 \vee 1) \wedge (1 \vee 1) = 1 \wedge 1 = 1$$

$$(1 \vee 1) \wedge (1 \vee 0) = 1 \wedge 1 = 1$$

$$(1 \vee 0) \wedge (1 \vee 1) = 1 \wedge 1 = 1$$

$$(1 \vee 0) \wedge (1 \vee 0) = 1 \wedge 1 = 0$$

$$0 \vee (1 \wedge 1) = 0 \vee 1 = 1$$

$$0 \vee (1 \wedge 0) = 0 \vee 0 = 0$$

$$0 \vee (0 \wedge 1) = 0 \vee 0 = 0$$

$$0 \vee (0 \wedge 0) = 0 \vee 0 = 0$$

$$(0 \vee 1) \wedge (0 \vee 1) = 1 \wedge 1 = 0$$

$$(0 \vee 1) \wedge (0 \vee 0) = 1 \wedge 0 = 0$$

$$(0 \vee 0) \wedge (0 \vee 1) = 0 \wedge 1 = 0$$

$$(0 \vee 0) \wedge (0 \vee 0) = 0 \wedge 0 = 0$$

*** **Instructor's note.** Here's a shortcut for all of these in this particular case: \mathbf{T} is isomorphic to the powerset algebra $\mathcal{P}(\emptyset)$, whose members are \emptyset and $\{\emptyset\}$, with \cup for \vee , \cap for \wedge , \emptyset for 0, and $\{\emptyset\}$ for 1. And you just showed that $\mathcal{P}(A)$ is always a Boolean algebra. You can show this isomorphism with pictures. And then you can eliminate all of this separate proving of properties. But of course we can do that only because you already went through the proof for $\mathcal{P}(A)$.

(B4) **Top and bottom laws**

$$1 \wedge 1 = 1$$

$$0 \wedge 1 = 0$$

$$1 \vee 0 = 1$$

$$0 \vee 0 = 0$$

$$1 \wedge 0 = 0$$

$$0 \wedge 0 = 0$$

$$1 \vee 1 = 1$$

$$0 \vee 1 = 1$$

(B5) **Complementation laws**

$$1 \wedge (\neg 1) = 1 \wedge 0 = 0$$

$$1 \vee (\neg 1) = 1 \vee 0 = 1$$

$$0 \wedge (\neg 0) = 0 \wedge 1 = 0$$

$$0 \vee (\neg 0) = 0 \vee 1 = 1$$

Therefore (B1) to (B5) holds of \mathbf{T} , and therefore \mathbf{T} is a Boolean algebra.