

## HW5Ex1Ans2

### Homework 5

#### Question 1: Claims and examples

##### Claim 2

Consider  $\wedge$  only.

(1) **Reflexive.**

$$\begin{aligned} a \wedge a &= a && \text{(L1)} \\ \therefore a &\leq_1 a. && \text{Definition of (Ord)} \\ \therefore \leq_1 &\text{ is a reflexive order.} \end{aligned}$$

(2) **Anti-symmetric.**

$$\begin{aligned} \text{Iff } a \wedge b &= a, \text{ then } a \leq_1 b, \text{ and} \\ \text{Iff } b \wedge a &= b, \text{ then } b \leq_1 a && \text{Definition of (Ord)} \\ \text{But } a \wedge b &= b \wedge a && \text{(L2)} \\ \therefore a &= b \text{ whenever we have both } a \leq_1 b \text{ and } b \leq_1 a. \\ \therefore \leq_1 &\text{ is an anti-symmetric order.} \end{aligned}$$

(3) **Transitive.**

$$\begin{aligned} \text{Iff } a \wedge b &= a \text{ and } b \wedge c = b, \\ \text{then } a &\leq_1 b \text{ and } b \leq_1 c && \text{Definition of (Ord)} \\ \text{And } (a \wedge b) \wedge c &= a \wedge (b \wedge c) && \text{(L3)} \\ \therefore a \wedge c &= a \wedge b \\ \text{Therefore } a \wedge c &= a && a \wedge b = a \text{ (assumed)} \\ \text{And then } a &\leq_1 c && \text{Definition of (Ord)} \\ \text{Thus, when we have } a &\leq_1 b \text{ and } b \leq_1 c, \text{ we also have } a \leq_1 c. \\ \therefore \leq_1 &\text{ is a transitive order.} \end{aligned}$$

**Interim conclusion: The order defined by  $\leq_1$  is transitive, reflexive and anti-symmetric, i.e. it is partial order.**

(4) **That  $\vee$  and  $\wedge$  exists for any  $a, b$  in  $A$ .**

$$\begin{aligned} [1] \ a \wedge (a \vee b) &= a && \text{(L4)} \\ \therefore a &\leq_1 (a \vee b) && \text{Definition of (Ord)} \\ b \wedge (b \vee a) &= b && \text{(L4)} \\ \therefore b &\leq_1 (b \vee a) && \text{Definition of (Ord)} \\ \text{But } (b \vee a) &= (a \vee b) && \text{(L2)} \\ \therefore b &\leq_1 (a \vee b) \\ \therefore (a \vee b) &\text{ is an upperbound for both } a \text{ and } b. \\ [2] \ \text{Let } c &\text{ be any upperbound of } a \text{ and } b, \text{ i.e.:} \\ a &\leq_1 c \text{ and } b \leq_1 c \\ \text{Then } a \wedge c &= a \text{ and } b \wedge c = b && \text{Definition of (Ord)} \\ \text{Then:} \\ a \vee c &= (a \wedge c) \vee c && \text{Substituting for } a \\ (a \wedge c) \vee c &= c \vee (a \wedge c) && \text{(L2)} \\ c \vee (a \wedge c) &= c \vee (c \wedge a) && \text{(L2)} \\ c \vee (c \wedge a) &= c && \text{(L4)} \end{aligned}$$

$$\therefore a \vee c = c \qquad a \vee c = (a \wedge c) \vee c$$

And similarly, for  $b$ :

$$b \vee c = c$$

$$[3] (a \vee c) \vee (b \vee c) = c \vee c \qquad \text{Substituting for } c \text{ from [2]}$$

$$c \vee c = c \qquad (\text{L1})$$

$$\therefore (a \vee c) \vee (b \vee c) = c$$

$$(a \vee c) \vee (b \vee c) = (a \vee b) \vee (c \vee c) \qquad (\text{L2) and (L3)}$$

$$(a \vee b) \vee (c \vee c) = (a \vee b) \vee c \qquad (\text{L1})$$

$$\therefore (a \vee c) \vee (b \vee c) = (a \vee b) \vee c \qquad (\text{L2) and (L3)}$$

$$\therefore (a \vee b) \vee c = c$$

$$[4] (a \vee b) \wedge c = (a \vee b) \wedge ((a \vee b) \vee c) \qquad \text{Substituting for } c \text{ from [3]}$$

$$(a \vee b) \wedge ((a \vee b) \vee c) = (a \vee b) \qquad (\text{L4})$$

$$\therefore (a \vee b) \wedge c = (a \vee b)$$

$$\therefore (a \vee b) \leq_1 c$$

[5] Therefore in [1] we have seen that for any  $a$  and  $b$  in  $A$ , an upperbound exists, namely  $a \vee b$ . And we have also seen that  $a \vee b \leq_1 c$  for any  $c$  an upperbound of  $a$  and  $b$ , and therefore that  $a \vee b$  is a supremum for  $a$  and  $b$ . We therefore know that a least upper bound must exist for any two members  $a$  and  $b$  of  $A$ .

[7] By duality the same holds for an greatest lowerbound for any two members  $a$  and  $b$  of  $A$  – that is we can show similarly that the greatest lower bound  $a \wedge b$  does exist for any two  $a$  and  $b$  of  $A$ .

**Final conclusion: We have seen that  $\leq_1$  defines a partial order on the set  $A$ , and that set  $A$  with the order  $\leq_1$  imposed on it, has a least upperbound and a greatest lowerbound for any two members of  $A$ . Therefore,  $A$  is a lattice in terms of Definition 1.**

Instructors' note: But have you shown that the derived order  $\leq_1$  is the *same* as the partial order  $\leq$  that we started with? *That* was claim 2. I think all the ingredients for that are here, too. And you may have proved claim 3 here too, in addition or instead?