1. Algebras N and Parity

I use the following notation, in accordance with the handout.

Ω_{Numb}: the signature defined in the handout, {**zero**, **one**, +, ×}.

N: the set of natural numbers, {0, 1, 2, …}.

N: the Ω_{Numb}-algebra on the set *N*.

Parity: the Ω_{Numb}-algebra on the set {*even*, *odd*}.

f: the homomorphism from **N** to **Parity**, which maps even natural numbers to *even* and odd natural numbers to *odd*.

(a) The kernel equivalence of the homomorphism *f*

ker(f) = $f^{-1} \circ f$ (by definition)

= {⟨*x*,*y*⟩| *x* and *y* are even natural numbers} ∪ {⟨*x*,*y*⟩| *x* and *y* are odd natural numbers}

(b) The corresponding quotient algebra N/ker(*f*)

The quotient set of *N* by the equivalence relation **ker**(f) is given by {[[0]], [[1]]}, where [[0]] denotes the set of even natural numbers and [[1]] the set of odd natural numbers. So the quotient algebra **N**/**ker**(f) is the Ω_{Numb}-algebra of the carrier {[[0]], [[1]]}, with the definitions of the operations as follows:

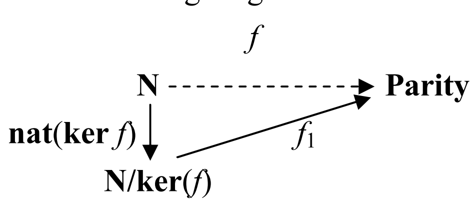
zero = [[0]], **one** = [[1]],

+([[0]],[[0]]) = [[0]], +([[0]],[[1]]) = [[1]], +([[1]],[[0]]) = [[1]], +([[1]],[[1]]) = [[0]],

×([[0]],[[0]]) = [[0]], ×([[0]],[[1]]) = [[0]], ×([[1]],[[0]]) = [[0]], ×([[1]],[[1]]) = [[1]].

(c) The commutative diagram

There exists a monomorphism f_1: **N**/**ker**(f) → **Parity**, which maps [[0]] to *even* and [[1]] to *odd*, so the following diagram is commutative.



2. Symmetries of the square

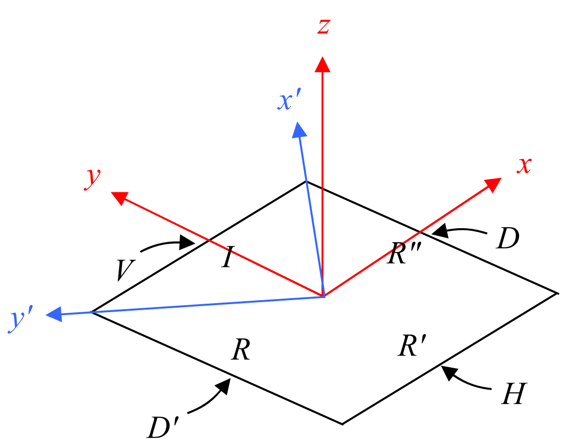
(This topic attracts physicists, too. For example, the algebra of this kind is extremely useful to study the structures of crystals or molecules. Also, in the case that the angle of rotations is not restricted to some discrete values, it has fundamental importance for theories of elementary forces such as gravity.)

(a) The group operation table

	I	R	R'	R''	H	V	D	D'
I	I	R	R'	R''	H	V	D	D'
R	R	R'	R''	I	D	D'	V	H
R'	R'	R''	I	R	V	H	D'	D
R''	R''	I	R	R'	D'	D	H	V
H	H	D'	V	D	I	R'	R''	R
V	V	D	H	D'	R'	I	R	R''
D	D	H	D'	V	R	R''	I	R'
D'	D'	V	D	H	R''	R	R'	I

(b) Three different subalgebras having exactly four elements

Let us call a line which penetrates the center of the square vertically *z-axis.* Then R' is 180° rotation around z-axis. Additionally I define (x, y)-axes and (x', y')-axes as shown in the figure below.



From this figure we see that H, V, D and D' are 180° rotations around x, y, x' and y'-axes, respectively.

We find one subalgebra having exactly four elements, which consists of I and 90°, 180° and 270° rotations around z-axis, and two subalgebras whose elements consist of I and 180° rotations around (x, y, z) or (x', y', z)-axes.

(Subalgebra **I**) Rotations of multiples of 90° around z-axis

	I	R	R'	R''
I	I	R	R'	R''
R	R	R'	R''	I
R'	R'	R''	I	R
R''	R''	I	R	R'

(Subalgebra **II**) 180° rotations around x, y, z-axes

	I	H	V	R'
I	I	H	V	R'
H	H	I	R'	V
V	V	R'	I	H
R'	R'	V	H	I

(Subalgebra **III**) 180° rotations around x', y', z-axes

	I	D	D'	R'
I	I	D	D'	R'
D	D	I	R'	D'
D'	D'	R'	I	D
R'	R'	D'	D	I

(c) Five different subalgebra having exactly two elements

It is easy to find these subalgebras. They are algebras which consist of I and one of 180° rotations around x, y, x', y' or z-axis.

(Subalgebra **Z**) 180° rotation around z-axis

	I	R'
I	I	R'
R'	R'	I

(Subalgebra **X**) 180° rotation around x-axis

	I	H
I	I	H
H	H	I

(Subalgebra **Y**) 180° rotation around y-axis

	I	V
I	I	V
V	V	I

(Subalgebra **X′**) 180° rotation around x'-axis

	I	D
I	I	D
D	D	I

(Subalgebra **Y′**) 180° rotation around y'-axis

	I	D'
I	I	D'
D'	D'	I

(d) Isomorphic subalgebras in (b)

(Here, I try to give physical meanings to the isomorphism, rather than giving the result of check of the relation $f(\omega(a_1, a_2)) = \omega(f(a_1), f(a_2))$ because *reading it* might be not so fun although *doing it* is important.) The algebra of 180° rotations around (x, y, z)-axes (Subalgebra **II**) and the algebra of 180° rotations around (x', y', z)-axes (Subalgebra **III**) are obviously isomorphic under the following mapping:

(x, y, z)-coordinate system		(x', y', z)-coordinate system
180° rotation around x-axis (H)	↔	180° rotation around x'-axis (D)
180° rotation around y-axis (V)	↔	180° rotation around y'-axis (D').

To be precise, the following homomorphism f from Subalgebra **II** to Subalgebra **III** and the homomorphism f^{-1} from Subalgebra **III** to Subalgebra **II** constitute an isomorphism between them:

$$f(I) = I,\quad f(H) = D,\quad f(V)= D',\quad f(R') = R',$$
$$f^{-1}(I) = I,\quad f^{-1}(D) = H,\quad f^{-1}(D') = V,\quad f^{-1}(R') = R'.$$

(e) A non-trivial automorphism in (b)

For example, take Subalgebra **I** (the algebra of 0°, 90°, 180° and 270° clockwise rotations around z-axis). Note that an algebra of 0°, 90°, 180° and 270° anti-clockwise rotations is obviously isomorphic to an algebra of 0°, 90°, 180° and 270° clockwise rotations. So we can obtain an automorphism on Subalgebra **I** by identifying the elements of these algebras as follows.

anti-clockwise rotation		clockwise rotation
0° rotation	=	0° rotation (I)
90° rotation	=	270° rotation (R'')
180° rotation	=	180° rotation (R')
270° rotation	=	90° rotation (R)

In other words, the following function f from Subalgebra **I** to Subalgebra **I**,

$$f(I) = I, \quad f(R) = R'', \quad f(R') = R', \quad f(R'') = R,$$

which maps 0°, 270°, 180°, and 90° clockwise rotations to 0°, 90°, 180°, and 270° anti-clockwise rotations, respectively, is an automorphism.

(f) A homomorphism of (b) with (c)

Let us take Subalgebra **II** (identity and 180° rotations around x, y, z-axes) with the carrier $\{I, H, V, R'\}$ and Subalgebra **X** (identity and 180° rotation around x-axis) with the carrier $\{I, H\}$ for instance. Consider the function f from Subalgebra **II** to Subalgebra **X** defined by

$$f(I) = f(R') = I, \quad f(H) = f(V) = H.$$

This is a homomorphism from Subalgebra **II** to Subalgebra **X**.

This is similar to the homomorphism from **Mod4** to **Mod2**, though **Mod *n*** is equipped with two binary operations + and ×, not one ∘.