1. a. The homomorphism from **N** to **Parity** will map all even numbers (including 0) to *even*, and all odd numbers to *odd*. The kernel equivalence relation will therefore consist of all pairs of natural numbers that are either both odd or both even. More formally:

F: **N** → **Parity**
ker F = {⟨x,y⟩ | x,y ∈ N & (x and y are even) ∨ (x and y are odd)}

b. N/kerF = { {x| x is an even number}, {x| x is an odd number} }
This will be the carrier set of the algebra.

Given the signature {zero, one, +, -}, we would want to define the operations for the quotient algebra **N**/kerF as follows:

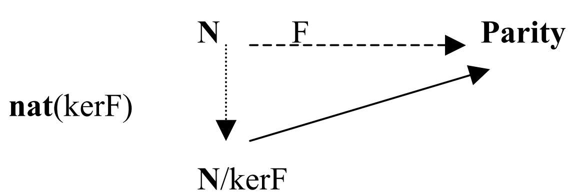
+

| | **{x\| x is an even number},** | **{x\| x is an odd number}** |
|---|---|---|
| **{x\| x is an even number},** | {x\| x is an even number}, | {x\| x is an odd number} |
| **{x\| x is an odd number}** | {x\| x is an odd number} | {x\| x is an even number}, |

×

| | **{x\| x is an even number},** | **{x\| x is an odd number}** |
|---|---|---|
| **{x\| x is an even number},** | {x\| x is an even number} | {x\| x is an even number} |
| **{x\| x is an odd number}** | {x\| x is an even number} | {x\| x is an odd number} |

c)




2. a. operation table:

| | I | R | R’ | R’’ | V | H | D | D’ |
|---|---|---|---|---|---|---|---|---|
| I | I | R | R’ | R’’ | V | H | D | D’ |
| R | R | R’ | R’’ | I | D’ | D | V | H |
| R’ | R’ | R’’ | I | R | H | V | D’ | D |
| R’’ | R’’ | I | R | R’ | D | D’ | H | V |
| V | V | D | H | D’ | I | R’ | R | R’’ |
| H | H | D’ | V | D | R’ | I | R’’ | R |
| D | D | H | D’ | V | R’’ | R | I | R’ |
| D’ | D’ | V | D | H | R | R’’ | R’ | I |

b. Subalgebras:

1.

| | I | R’ | V | H |
|---|---|---|---|---|
| I | I | R’ | V | H |
| R’ | R’ | I | H | V |
| V | V | H | I | R’ |
| H | H | V | R’ | I |

2.

| | I | R | R’ | R’’ |
|---|---|---|---|---|
| I | I | R | R’ | R’’ |
| R | R | R’ | R’’ | I |
| R’ | R’ | R’’ | I | R |
| R’’ | R’’ | I | R | R’ |

3.

| | I | R’ | D | D’ |
|---|---|---|---|---|
| I | I | R’ | D | D’ |
| R’ | R’ | I | D’ | D |
| D | D | D’ | I | R’ |
| D’ | D’ | D | R’ | I |

c.

1.

| | I | R’ |
|---|---|---|
| I | I | R’ |
| R’ | R’ | I |

2.

| | I | H |
|---|---|---|
| I | I | H |
| H | H | I |

3.

| | I | V |
|---|---|---|
| V | I | V |
| I | V | I |

4.

| | I | D |
|---|---|---|
| I | I | D |
| D | D | I |

5.

| | I | D’ |
|---|---|---|
| I | I | D’ |
| D’ | D’ | I |

d. The subalgebras 1 ({I, R, V, H} and 3 ({I, R’, D, D”}) are isomorphic. Consider the homomorphism f: {I, R, V, H} → {I, R’, D, D”}, with
f(I) = I
f(R) = R’
f(V) = D
f(H) = D’
This will satisfy the condition that for every pair ⟨x,y⟩ in {I, R, V, H},

ω(f(x), f(y)) = f(ω(x,y))
Where ω is any operator in the signature. We could show this by listing all the possibilities. Intuitively speaking, we can see that this is the case because you could replace every

element in the operation table (b1) with the value we get by applying f to that element, and the result would be identical to the operation table in (b3).

To show that we are in fact dealing with an isomorphism, we also need to show that thte inverse of f is a function and a homomorphism. The inverse of f is a function, since f is both onto and one-to-one. It is a homomorphism, because the above condition holds; we can apply the same intuitive reasoning as above: every item in the operation table (b3) could be replaced by the element that it is mapped onto in {I, R, V, H}, and the result woul d be identicacl to (b1).

e.

f: {I, R, V, H} → {I, R, V, H}, such that
f(I) = I
f(R) = R''
f(R') = R'
f(R'') = R

This will give us the 'counter-clockwise version' of the square, which will be a homomorphism, since for every pair ⟨x,y⟩ in {I, R, V, H}, ω(f(x), f(y)) = f(ω(x,y)).

f.
Let g be a function from {I, R, R', R''} to {I, R'}, and define it as follows:

g(I) = I
g(R) = R'
g(R') = I
g(R'') = R'

This is similar to the homomorphism from Modulo4 to Modulo2. We get the following mapping equivalences, which show that g is a homomorphism:

| | |
|---|---|
| g(I°I) = g(I)°g(I) = I | g(R'°I) = g(R')°g(I) = I |
| g(I°R) = g(I)°g(R) = R' | g(R'°R) = g(R')°g(R) = R' |
| g(I°R') = g(I)°g(R') = I | g(R'°R') = g(R')°g(R') = I |
| g(I°R'') = g(I)°g(R'') = R' | g(R'°R'') = g(R')°g(R'') = R' |
| g(R°I) = g(R)°g(I) = R' | g(R''°I) = g(R'')°g(I) = R' |
| g(R°R) = g(R)°g(R) = I | g(R''°R) = g(R'')°g(R) = I |
| g(R°R') = g(R)°g(R') = R' | g(R''°R') = g(R'')°g(R') = R' |
| g(R°R'') = g(R)°g(R'') = I | g(R''°R'') = g(R'')°g(R'') = I |

Since all of the other subalgebras in (c) are isomorphic, the same homomorphism could be applied to them.

3.

**The $A^0$ question.**

[On my first attempt, I came up with something close to the homework version you have posted from 2001 (the one where you try to define N in terms of sets). But I got stuck, and looked at that solution. Although that one seemed convincing, I started to get curious whether there was another way. Below is my attempt to get the result in another way, which doesn't seem to be successful in the end. But it was fun trying…]

As is described on p.2 of the handout (lecture 4.1), n-ary operators will name n-ary operations of the kind f: $A^n \rightarrow A$. A unary operator therefore will map each element of A onto an element of A. A binary operator maps a pair of elements of A onto an element of A. In set talk, an n-ary operation is equivalent to a set of n+1-tuples: A unary operation is equivalent to a set of pairs, a binary one to a set of triples, etc.

One way to approach the problem of $A^0$, then, might be to think about how we get the domain of these functions. Cartesian products immediately come to mind. $A^2$ will be the set of all pairs formed from elements of A. $A^3$ will be the set of all triples made up of elements of A. So maybe we can define a generalized version of cartesian product that will give us $A^0 = \{\varnothing\}$.

Cartesian Products are defined as follows:

For any two sets A and B, the cartesian product $A \times B = \{\langle x,y\rangle | \, x \in A \ \& \ y \in B\}$

This defines × as a binary operator. We could define × as a more general notion that applies to n-tuples of sets:

$\times \langle A_1, A_2, \ldots, A_n\rangle = \{\langle a_1, a_2, \ldots, a_n\rangle | \, a1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$

We then define $A^n = \times \langle A_1, A_2, \ldots, A_n\rangle = \{\langle a_1, a_2, \ldots, a_n\rangle | \, a1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$

While the cases for n > 2 are unproblematic, we need to ask ourselves what happens for n =1 and n = 0:

$A^1 = \times \langle A_1\rangle = ?$

What this means depends on our definition of the pair brackets:

Let's try the following definition:

$\langle a_1, a_2, \ldots, a_n\rangle =$

$\{\{a_1\} \cap \{a_1, a_2, \ldots, a_n\}, (\{a_1\} \cup \{a_2\}) \cap \{a_1, a_2, \ldots, a_n\}, \ldots, (\{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_n\}) \cap \{a_1, a_2, \ldots, a_n\}\}$

then

$A^1 = \times \langle A_1\rangle = \{\langle a_1\rangle | a_1 \in A_1\} = \{A_1 \cap A_1\} = \{A_1\}$, and

$A^0 = \times\langle\,\rangle = \{\varnothing \cap \varnothing\} = \{\varnothing\}$.

I'm not too happy about this attempt. First of all, the definition of the pair brackets needs to be formalized more rigourosly to ensure the result for $A^0$. Though this should be possible, another problem is that $A^1 = \{A\}$. This is problematic for the unary operators, because they would have to be functions from $\{A\}$ to A, which is not what we want. I'm not sure how to fix this problem…