HANDOUT LECTURE 4

EXERCISE 1 ON QUOTIENT ALGEBRAS.

Consider the signature Ω_{Num} = < Op, a >, where Op = { +, x } and a = { <+,2>, <x,2> }

Consider the Ω_{Num}–algebra **Mod4** with carrier Mod4 = {0,1,2,3}, and the operations of addition and multiplication as defined in page 3 in the handout for lecture 4.

Consider also the Ω_{Num}–algebra **Parity**, with carrier Parity = {even, odd}, and the operations of addition and multiplication defined in the usual way (*odd + odd = odd*, *odd x odd = odd…*)

Consider now the function f: Mod4 → Parity = {<0,even>, <1,odd>, <2,even>, <3,odd>}

f is a homomorphism, since it can be verified that

(i) $\forall$x,y $\in$ Mod4, f (x $+_{Mod4}$ y) = f(x) $+_{Parity}$ f (y)[1]

(ii) $\forall$x,y $\in$ Mod4, f (x x_{Mod4} y) = f(x) x_{Parity} f (y)

Given f, **ker** f = { <0,0>, <0,2>, <2,0>, <2,2>, <1,1>, <1,3>, <3,1>, <3,3>}. **ker** f is an equivalence relation, since it is reflexive, symmetric and transitive. It induces the following partition in Mod4: Mod4/**ker** f = { {0,2}, {1,3} }.

There exists a Ω_{Num} –algebra on Mod4/**ker** f such that the natural mapping Mod4 → Mod4/**ker** f is a homomorphism. Let's call that algebra (the quotient algebra of **Mod4** by **ker** f) **Mod4/ker** f. The operations on **Mod4/ker** f are defined as follows:

(Read horizontal x, vertical y)

[1] For ease of exposition, I intend expressions like '$+_{Mod4}$' to mean the operation '+', as defined in the algebra **Mod4.**

+	{0,2}	{1,3}
{0,2}	{0,2}	{1,3}
{1,3}	{1,3}	{0,2}

X	{0,2}	{1,3}
{0,2}	{0,2}	{0,2}
{1,3}	{0,2}	{1,3}

We have the function nat (**ker** f) : Mod4 → Mod4/**ker** f ('the natural mapping of Mod4 onto Mod4/**ker** f ') = {<0, {0,2}>, <1, {1,3}>, <2, {0,2}>, <3, {1,3}>}

nat (**ker** f) is the natural homomorphism **Mod4** → **Mod4/ker** f, since it can be verified that

(i) $\forall$x,y $\in$ Mod 4 nat (x $+_{Mod4}$ y) = nat (x) $+_{Mod4/kerf}$ nat (y)

For any pair <x,y> $\in$ Mod4,

If x is even and y is odd, we will have x $+_{Mod4}$ y = z, where z is odd, hence nat (z) = {1,3}. Then, nat (x) = {0,2} and nat (y) = {1,3}, hence nat (x) $+_{Mod4/kerf}$ nat (y) = {1,3}. Similarly for x odd and y even.

If both x and y are even, we will have x $+_{Mod4}$ y = z, where z is even, hence nat (z) = {0,2}. Then, nat (x) = {0,2} and nat (y) = {0,2}, hence nat (x) $+_{Mod4/kerf}$ nat (y) = {0,2}.

If both x and y are odd, we will have x $+_{Mod4}$ y = z, where z is even, hence nat (z) = {0,2}. Then, nat (x) = {0,2} and nat (y) = {0,2}, hence nat (x) $+_{Mod4/kerf}$ nat (y) = {0,2}.

(ii) $\forall$x,y $\in$ Mod 4 nat (x $_{Mod4}$ y) = nat (x) $x_{Mod4/kerf}$ nat (y)

For any pair <x,y> $\in$ Mod4,

If x is even and y is odd, we will have x x_{Mod4} y = z, where z is even, hence nat (z) = {0,2}. Then, nat (x) = {0,2} and nat (y) = {1,3}, hence nat (x) $+_{Mod4/kerf}$ nat (y) = {0,2}. Similarly for x odd and y even.

If both x and y are even, we will have x $+_{Mod4}$ y = z, where z is even, hence nat (z) = {0,2}. Then, nat (x) = {0,2} and nat (y) = {0,2}, hence nat (x) $+_{Mod4/kerf}$ nat (y) = {0,2}.

If both x and y are odd, we will have x $+_{Mod4}$ y = z, where z is odd, hence nat (z) = {1,3}. Then, nat (x) = {1,3} and nat (y) = {1,3}, hence nat (x) $+_{Mod4/kerf}$ nat (y) = {1,3}.

There exists a monomorphism **Mod4/ker** f → **Parity**, namely f_1 = {<{0,2}, even>, <{1,3}, odd> } . f_1 maps every equivalence class in Mod4/**ker** f onto Parity. If we start from **Mod4**, nat (**ker** f) maps every x ∈ Mod4 onto a member of Mod4/**ker** f, the quotient set of Mod4 by **ker** f. Then f_1 maps every member of Mod4/**ker** f onto Parity.

Notice that f = f_1 • **nat** (**ker** f).

Recall that f = {<0,even>, <1,odd>, <2,even>, <3,odd>}

f_1 = {<{0,2}, even>, <{1,3}, odd> }

nat (**ker** f) = {<0, {0,2}>, <1, {1,3}>, <2, {0,2}>, <3, {1,3}>}

hence f_1 • **nat** (**ker** f) = f = {<0,even>, <1,odd>, <2,even>, <3,odd>}.

The corresponding commutative diagram represents the relation between **Mod4**, its quotient algebra and **Parity**: