Exercise 3, p. 272

(a) Draw the group operation table for the group of symmetries of the square {I, R, R', R'', H, V, D, D'}

	I	R	R'	R''	H	V	D	D'
I	I	R	R'	R''	H	V	D	D'
R	R	R'	R''	I	D	D'	V	H
R'	R'	R''	I	R	V	H	D'	D
R''	R''	I	R	R'	D'	D	H	V
H	H	D'	V	D	I	R'	R''	R
V	V	D	H	D'	R'	I	R	R''
D	D	H	D'	V	R	R''	I	R'
D'	D'	V	D	H	R''	R	R'	I

(b) There are different subgroups having exactly four elements. Find them and draw their group operation tables.

(1) <{I, R, R', R''}, °>

	I	R	R'	R''
I	I	R'	R'	R''
R	R	R'	R''	I
R'	R'	R''	I	R
R''	R''	I	R	R''

(2) <{I, R', V, H}, °>

	I	R'	V	H
I	I	R'	V	H
R'	R'	I	H	V
V	V	H	I	R'
H	H	V	R'	I

(3) <{I, R’, D, D’}, °>

	I	R’	D	D’
I	I	R’	D	D’
R’	R’	I	D’	D
D	D	D’	I	R’
D’	D’	D	R’	I

The following is an example of a *non*-algebra: The set {I, R, D, D’}is not closed under ° . For instance, D ° R = H, and H $\notin$ {I, R, D, D’}

	I	R	D	D’
I	I	R	D	D’
R	R	R’	V	H
D	D	H	I	R’
D’	D’	V	R’	I

(c) There are five different subgroups having exactly two members. Find them and draw their tables.

(4) {<I, R’>, °>

	I	R’
I	I	R’
R’	I	I

(5) {<I, D’}, °>

	I	D’
I	I	D’
D’	D’	I

(6) {<I,D>, °>

	I	D
I	I	D
D	I	I

(7) <{I, H>, °>

	I	H
I	I	H
H	H	H

(8) <{I, V>, °>

	I	V
I	I	V
V	V	I

The following is an example of a *non*-algebra, since R ° R = R' and R' ∉ {I, R'}

	I	R
I	I	R
R	R	R'

(d) Show which of the subgroups in (b) are isomorphic

The group in (2) and the group in (3) are isomorphic.

(2) **A =** <{I, R', H, V}, °>

(3) **B** = <{I, R', D, D'}, °>

The following function f: A → B is an isomorphism

f: {<I,I>, <R',R'>, <H,D>, <V,D'>}

a) f is a homomorphism: For any a, b in A, f(a°b) = f(a) ° f(b)

f(I ° I) = f (I) ° f (I)
f(I ° I) = f (I) = **I**
f (I) ° f (I) = I ° I = **I**

f(I ° R') = f (I) ° f (R')
f(I ° R') = f(R') = **R'**
f (I) ° f (R') = I ° R' = **R'**

f(I ° V) = f (I) ° f (V)
f(I ° V) = f(V) = **D'**
f (I) ° f (V) = I ° D' = **D'**

f(I ° H) = f (I) ° f (H)
f(I ° H) = f(H) = **D**
f (I) ° f (H) = I ° D = **D**

f(R'° I) = f (R') ° f (I)
f(R' ° I) = f(R') = **R'**
f (R') ° f (I) = R' ° I = **R'**

$f(R' \circ R') = f(R') \circ f(R')$

$f(R' \circ R') = f(I) = \mathbf{I}$
$f(R') \circ f(R') = R' \circ R' = \mathbf{I}$

$f(R' \circ V) = f(R') \circ f(V)$
$f(R' \circ V) = f(H) = \mathbf{D}$
$f(R') \circ f(V) = R' \circ D' = \mathbf{D}$

$f(R' \circ H) = f(R') \circ f(H)$
$f(R' \circ H) = f(V) = \mathbf{D'}$
$f(R') \circ f(H) = R' \circ D = \mathbf{D'}$

$f(V \circ I) = f(V) \circ f(I)$
$f(V \circ I) = f(V) = \mathbf{D'}$
$f(V) \circ f(I) = D' \circ I = \mathbf{D'}$

$f(V \circ R') = f(V) \circ f(R')$
$f(V \circ R') = f(H) = \mathbf{D}$
$f(V) \circ f(R') = D' \circ R' = \mathbf{D}$

$f(V \circ V) = f(V) \circ f(V)$
$f(V \circ V) = f(I) = \mathbf{I}$
$f(V) \circ f(V) = D' \circ D' = \mathbf{I}$

$f(V \circ H) = f(V) \circ f(H)$
$f(V \circ H) = f(R') = \mathbf{R'}$
$f(V) \circ f(H) = D' \circ D = \mathbf{R'}$

$f(H \circ I) = f(H) \circ f(I)$
$f(H \circ I) = f(H) = \mathbf{D}$
$f(H) \circ f(I) = D \circ I = \mathbf{D}$

$f(H \circ R') = f(H) \circ f(R')$
$f(H \circ R') = f(V) = \mathbf{D'}$
$f(H) \circ f(R') = D \circ R' = \mathbf{D'}$

$f(H \circ V) = f(H) \circ f(V)$
$f(H \circ V) = f(R') = \mathbf{R'}$
$f(H) \circ f(V) = D \circ D' = \mathbf{R'}$

f(H° H) = f (H)° f (H)

f(H ° H) = f(I) = **I**

f (H) ° f (H) = D ° D = **I**

b) f is an isomorphism: it is a homomorphism (see above) and its inverse f^{-1} : {<I,I>, <R',R'>, <D, H>, <D', V>} is also a homomorphism. That is, for any a, b in B, f (a° b) = f (a)° f (b).

For instance,

f(R'° D) = f (R')° f (D)

f(R' ° D) = f(D') = **V**

f (R') ° f (D) = R' ° H = **V**

(the same obtains for the remaining elements of B)

[<u>Question</u>: is there any way to *show* that the algebras in (1) and (3), on the one hand, and the algebras in (1) and (2), on the other hand, are not isomorphic? Saying that I could not find any isomorphism between the corresponding carriers does not entitle me to conclude so, since I basically used a trial-and-error-method. So I guess my question is: is there any general method of finding out how many homomorphisms there are between the carriers of two algebras?]

*** Answers (BHP). In the particular case, yes, and the *strategy* is general, though there isn't any algorithm for carrying it out, as far as I know. Find some properties that some elements in one algebra have that distinguish it from the other. In this case: in the algebras (2) and (3), which are isomorphic to each other, each element is its own inverse. In the algebra (1), two elements are their own inverses and the other two are inverses of each other. Given such differences, the two algebras can't be isomorphic.

What properties will be the crucial ones can vary from case to case, but if they're not isomorphic there will always be some "structural difference". ***

(e) Show a non-trivial automorphism for one of the subgroups of (b)

The following function is an automorphism of the <{I, R', V, H}, °> onto itself

f: {<I,I>, <R',V>, <V, H>, <H,R'>}

f(I ° I) = f (I) ° f (I)

f(I ° I) = f (I) = **I**

f (I) ° f (I) = I ° I = **I**

$f(I \circ R') = f(I) \circ f(R')$

$f(I \circ R') = f(R') = \mathbf{V}$

$f(I) \circ f(R') = I \circ V = \mathbf{V}$

$f(I \circ V) = f(I) \circ f(V)$

$f(I \circ V) = f(V) = \mathbf{H}$

$f(I) \circ f(V) = I \circ H = \mathbf{H}$

$f(I \circ H) = f(I) \circ f(H)$

$f(I \circ H) = f(H) = \mathbf{R'}$

$f(I) \circ f(H) = I \circ R' = \mathbf{R'}$

$f(R' \circ I) = f(R') \circ f(I)$

$f(R' \circ I) = f(R') = \mathbf{V}$

$f(R') \circ f(I) = V \circ I = \mathbf{V}$

$f(R' \circ R') = f(R') \circ f(R')$

$f(R' \circ R') = f(I) = \mathbf{I}$

$f(R') \circ f(R') = V \circ V = \mathbf{I}$

$f(R' \circ V) = f(R') \circ f(V)$

$f(R' \circ V) = f(H) = \mathbf{R'}$

$f(R') \circ f(V) = V \circ H = \mathbf{R'}$

$f(R' \circ H) = f(R') \circ f(H)$

$f(R' \circ H) = f(V) = \mathbf{H}$

$f(R') \circ f(H) = V \circ R' = \mathbf{H}$

$f(V \circ I) = f(V) \circ f(I)$

$f(V \circ I) = f(V) = \mathbf{H}$

$f(V) \circ f(I) = H \circ I = \mathbf{H}$

$f(V \circ R') = f(V) \circ f(R')$

$f(V \circ R') = f(H) = \mathbf{R'}$

$f(V) \circ f(R') = H \circ V = \mathbf{R'}$

$f(V \circ V) = f(V) \circ f(V)$

$f(V \circ V) = f(I) = \mathbf{I}$
$f(V) \circ f(V) = H \circ H = \mathbf{I}$

$f(V \circ H) = f(V) \circ f(H)$
$f(V \circ H) = f(R') = \mathbf{V}$
$f(V) \circ f(H) = H \circ R' = \mathbf{V}$

$f(H \circ I) = f(H) \circ f(I)$
$f(H \circ I) = f(H) = \mathbf{R'}$
$f(H) \circ f(I) = R' \circ I = \mathbf{R'}$

$f(H \circ R') = f(H) \circ f(R')$
$f(H \circ R') = f(V) = \mathbf{H}$
$f(H) \circ f(R') = R' \circ V = \mathbf{H}$

$f(H \circ V) = f(H) \circ f(V)$
$f(H \circ V) = f(R') = \mathbf{V}$
$f(H) \circ f(V) = R' \circ H = \mathbf{V}$

$f(H \circ H) = f(H) \circ f(H)$
$f(H \circ H) = f(I) = \mathbf{I}$
$f(H) \circ f(H) = R' \circ R' = \mathbf{I}$

(f) Show a homomorphism of one of the groups of (b) with one of the groups of (c)

The following function is a homomorphism from {I, R',V, H} to {I,R'}

f: {<I,I>, <R',R'>, <V,I>, <H,R'>}

*** Note from BHP. This method of proving that it's a homomorphism (below) is completely explicit, sound, and general. But of course tedious to write and to read. Here's one alternative. Draw the two tables, and use some typographical annotation to "show" the mapping. E.g. in the big table, make I, V red (or boldface) and R', H black (or italic), and in the little table make I red (bold) and R' black (italic). Then you can say "Look!". I.e. you can "see" that red times red = red etc in both. ***

$f(I \circ I) = f(I) \circ f(I)$

$f(I \circ I) = f(I) = \mathbf{I}$

$f(I) \circ f(I) = I \circ I = \mathbf{I}$

$f(I \circ R') = f(I) \circ f(R')$

$f(I \circ R') = f(R') = \mathbf{R'}$

$f(I) \circ f(R') = I \circ R' = \mathbf{R'}$

$f(I \circ V) = f(I) \circ f(V)$

$f(I \circ V) = f(V) = \mathbf{I}$

$f(I) \circ f(V) = I \circ I = \mathbf{I}$

$f(I \circ H) = f(I) \circ f(H)$

$f(I \circ H) = f(H) = \mathbf{R'}$

$f(I) \circ f(H) = I \circ R' = \mathbf{R'}$

$f(R' \circ I) = f(R') \circ f(I)$

$f(R' \circ I) = f(R') = \mathbf{R'}$

$f(R') \circ f(I) = R' \circ I = \mathbf{R'}$

$f(R' \circ R') = f(R') \circ f(R')$

$f(R' \circ R') = f(I) = \mathbf{I}$

$f(R') \circ f(R') = R' \circ R' = \mathbf{I}$

$f(R' \circ V) = f(R') \circ f(V)$

$f(R' \circ V) = f(H) = \mathbf{R'}$

$f(R') \circ f(V) = R' \circ I = \mathbf{R'}$

$f(R' \circ H) = f(R') \circ f(H)$

$f(R' \circ H) = f(V) = \mathbf{I}$

$f(R') \circ f(H) = R' \circ R' = \mathbf{I}$

$f(V \circ I) = f(V) \circ f(I)$

$f(V \circ I) = f(V) = \mathbf{I}$

$f(V) \circ f(I) = I \circ I = \mathbf{I}$

$f(V \circ R') = f(V) \circ f(R')$

$f(V \circ R') = f(H) = \mathbf{R'}$

$f(V) \circ f(R') = I \circ R' = \mathbf{R'}$

$f(V \circ V) = f(V) \circ f(V)$

$f(V \circ V) = f(I) = \mathbf{I}$

$f(V) \circ f(V) = I \circ I = \mathbf{I}$

$f(V \circ H) = f(V) \circ f(H)$

$f(V \circ H) = f(R') = \mathbf{R'}$

$f(V) \circ f(H) = I \circ R' = \mathbf{R'}$

$f(H \circ I) = f(H) \circ f(I)$

$f(H \circ I) = f(H) = \mathbf{R'}$

$f(H) \circ f(I) = R' \circ I = \mathbf{R'}$

$f(H \circ R') = f(H) \circ f(R')$

$f(H \circ R') = f(V) = \mathbf{I}$

$f(H) \circ f(R') = R' \circ R' = \mathbf{I}$

$f(H \circ V) = f(H) \circ f(V)$

$f(H \circ V) = f(R') = \mathbf{R'}$

$f(H) \circ f(V) = R' \circ I = \mathbf{R'}$

$f(H \circ H) = f(H) \circ f(H)$

$f(H \circ H) = f(I) = \mathbf{I}$

$f(H) \circ f(H) = R' \circ R' = \mathbf{I}$