HW5 Ex3 Ans1

3. Define the notion of sublattice, using the algebraic definition of lattice and give examples.

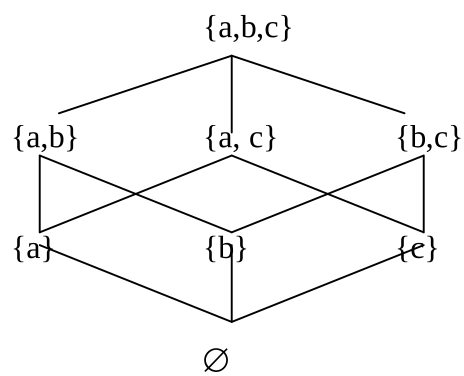
Any algebra **A** in the signature $\{\wedge,\vee\}$ on the carrier A is a lattice if the operations $\wedge_A$ $\vee_A$ denoted by $\boldsymbol{\wedge},\boldsymbol{\vee}$ respectively are idempotent, commutative, associative and obey the absorption law.

Let $\Omega_L = \{\wedge,\vee\}$, given two Ω_L -lattices **L1** on the carrier L1 with operations $\wedge_{L1}$, $\vee_{L1}$ and **L2** on the carrier L2 with operations $\wedge_{L2}$ and $\vee_{L2}$, we say that **L2** is a sublattice of **L1** if:

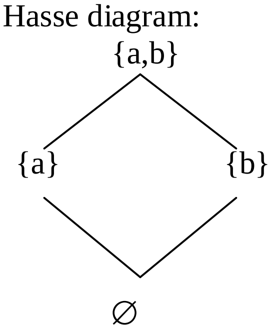
1) $L2 \subseteq L1$
2) L2 is closed with respect to both $\wedge_{L1}$ and $\vee_{L1}$
3) $\wedge_{L1} \upharpoonright L2 = \wedge_{L2}$ and $\vee_{L1} \upharpoonright L2 = \vee_{L2}$

<u>Example 1</u>

Consider the Ω_L -lattice **Pow** on the carrier $\wp(\{a,b,c\})$ with operations $\wedge$ and $\vee$ defined as set-theoretic intersection and union. We represent **Pow** by means of the following Hasse diagram:



The Ω_L -lattice **Pow2** on the carrier $\wp(\{a,b\})$ with operations $\wedge$ and $\vee$ defined as set-theoretic intersection and union is a sublattice of **Pow**. We represent **Pow2** by means of the following Hasse diagram:



Pow2 is a subalgebra of **Pow**, because

1) $\wp(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$ is a subset of $\wp(\{a,b,c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\} \}$
2) $\wp(\{a,b\})$ is closed with respect to intersection and union.

3) Intersection and union as defined in **Pow** restricted to $\wp(\{a,b\})$ equal intersection and union as defined in **Pow2**.

<u>Example 2</u>

Consider the Ω_L -lattice **L** on the carrier Three = {1,2,3}. We define the operations in Ω_L as follows:

($\wedge$) $\forall x,y, z \in$ Three [$x \wedge y =$ inf $\{x,y\}$]

($\vee$) $\forall x,y, z \in$ Three [$x \vee y =$ sup $\{x,y\}$]

We represent **L** by means of the following Hasse diagram:

$$\begin{array}{c} 3 \\ | \\ 2 \\ | \\ 1 \end{array}$$

Then the Ω_L -lattice **L2** on the carrier Two= {1,2} and operations defined as above is a sublattice of **L**, since:

(1) Two $\subseteq$ Three

(2) Two is closed under $\wedge_L$ and $\vee_L$

(3) $\wedge_L \mid$ Two = $\wedge_{L2}$ and $\vee_L \mid$ Two = $\vee_{L2}$

We could define the notion of sublattice in order theoretic terms, as follows:

<u>Def</u>. For any posets P1 = <P, $\leq$> and P2 = <P', $\leq$> , if P1 is a lattice P2 is a sublattice of P1 iff P' $\subseteq$ P and P1 is a lattice.

But then,the notions of sublattice as a subalgebra and as a posets are not equivalent. Consider the poset Pow3 = <{{a,b, c}, {a}, {b} $\emptyset$}, $\leq$}. Pow3 is a lattice as a poset, but it is not a subalgebra of **Pow**, since in **Pow** {a} $\wedge$ [b} = {a,b}