HOMEWORK 10

1. Find three models for *L* other tan the one given in the text.

<u>Model 1</u>:

Let the set *D* consist of three persons, *María*, *Juan* and *Pedro*, associated with 1, 2 and 3, respectively. Let *R** be the relation "loves". And assume that *María* loves *Juan*, *Juan* loves *Pedro* and *Pedro* loves *María*, and that nobody else loves anybody else.

<u>Model 2:</u>

Let the set *D* consist of three persons, *María*, *Juan* and *Pedro*, associated with 1, 2 and 3, respectively. Let *R** be the relation "be the teacher of". And assume that *María* is the teacher of *Juan*, *Juan* is the teacher of *Pedro* and *Pedro* is the teacher of *María*, and that nobody else is the teacher of anybody else.

<u>Model 3:</u>

Let the set *D* = {a,b,c} , and let a be associated with 1, b associated with 2 and c associated with 3. Let *R** = {<a,b>, <b,c>, <c,a>}

2. If the deletion of a certain axiom from a formally complete system changes the system into one which is not formally complete, then that axiom is independent. Why?

[In what follows, I will assume that a system is formally complete iff the theory associated with it is formally complete. Is this right?] [*** BHP: yes.]

A theory Δ is formally complete iff Δ^c, the deductive closure of Δ is maximally consistent. Hence, saying that deleting the axiom from the system turns the system into a non-formally complete one amounts to saying that deleting the axiom from the system makes the deductive closure of the theory not maximally consistent. If the axiom were not independent, then, by definition, it could be derived from the other axioms in the system. If that were the case, then, even if we removed the axiom from our set of axioms, this would not alter the deductive closure of Δ, which would contain the axiom in question, now 'demoted' to the status of theorem. Since removing the axiom from our set of axioms alters the deductive closure of Δ, the axiom is not derivable from the other axioms of the system and hence it is not independent.

> [Question: We said that a theory is a set of axioms *together with all the theorems derivable from them*. But if this is the case, shouldn't Δ^c (the set of all the formulas derivable from the theory, together with the theory) be the same thing as Δ (the theory)]

[*** BHP: yes, this is the result of some ambivalence of what "theory" refers to. I should have said "A set of sentences/axioms Δ is formally complete iff Δ^c, the deductive closure of Δ, is maximally consistent.". OR "A theory Δ is formally complete iff it is maximally consistent". But the latter sounds kind of odd; why add a seeming synonym for 'maximally consistent'? And completeness is meant to be a property of a given choice of axiomatization. So I would choose "A set of axioms ...".]

3. If the deletion of a certain axiom changes a formal system from categorical to non-categorical, must that axiom be independent. Why?

A system is categorical iff all of its models are isomorphic. A non-categorical systeml allows for models that are not isomorphic to each other. Hence, the non-categorical system will have models that are not models of the categorical system. <u>If deleting an axiom changes a system from categorical to non-categorical, the axiom is independent</u>, since, semantically, an axiom is independent from the other axioms of a system if the system S' that results from deleting that axiom has models that are not models of the whole system S.

Consider, for instance, A5 in the formal system *L* (PMW, p. 204). *L* is a categorical system. Removing A5 from *L* gives us a non-categorical system, *L'* . *L'* allows for models that are not models of *L* (as in the model constructed by letting *Chang*, *Li* and *Yang* sit on the same side of a rectangular table with *Chang* at the far right and *Li* in the middle, associating them with 1, 2, and 3, respectively, and keeping R* as "sits immediately to the right of"). Hence, A5 is independent.

4. Find two models for axioms A2-A6 which are not isomorphic to the models of *L* nor to each other. What does this tell you about Axiom 1.

<u>Model 1</u>

Let D consist of two people, *Juan* and *Laura*, that are sitting opposite each other and that are associated with 1 and 2, respectively. Let R* be "sits opposite".

<u>Model 2</u>

Let D consist of four people, *Juan*, *Laura, Javier* and *Mario*, associated with 1, 2, 3, 4. Let R* be the relation "loves", where *Juan* loves *Laura*, *Laura* loves *Javier*, *Javier* loves *Mario* and *Mario* loves *Juan* and nobody else loves anybody else.

Model 1 and Model 2 are not isomorphic to each other, nor to the models of *L*. Since the domains of these models have different cardinalities, there cannot be a one-to-one correspondence between any two of them, and hence, there exists no isomorphism between any two of them.

In question 3, we showed that if deletion of an axiom changes a formal system from categorical to non-categorical, the axiom is independent. We have just seen that deleting

Axiom 1 from the set of axioms of *L* gives us a non-categorical system (i.e., a system that has models that are not isomorphic to each other). Hence, Axiom 1 is independent.

5. If A2 is replaced by A2': $\forall x\, Rxx$, is the resulting system consistent? If so, find a model for it. If not, deduce a contradiction from the new set of axioms.

The resulting system is not consistent: By axiom A2', we have R11. And by axiom A6, we have R12. Hence, by axiom A3, we have that $1 = 2$. But axiom A1 tells us that $1 \neq 2$. Therefore, we have both $1 = 2$ and $1 \neq 2$, i.e., a contradiction.

The formal proof would go as follows:

1.	$\forall x Rxx$	A2'
2.	$\forall x \forall y \forall z\ [(Rxy \,\&\, Rxz) \rightarrow y = z)$	A3
3.	R12	A6
4.	$1 \neq 2 \;\&\; 1 \neq 3 \;\&\; 2 \neq 3$	Simp., A1
5.	R11	1, U.I.
6.	R11 & R12	3, 5, Conj.
7.	$1 \neq 2$	4, Simp.
8.	$R11 \,\&\, R12 \rightarrow 1 = 2$	2, U.I.
9.	$1 = 2$	6, 8, MP
10.	$\perp$	7, 8

[*** BHP note: It's nice to see the formal proof, but it's normal to give the argument in clear but non-formal-proof form as you did preceding the formal proof.]

6. If A2 is replaced by A2' as above and A3 and A4 are deleted, is the resulting system consistent? Is the resulting system categorical? If not, find two non-isomorphic models for it.

The resulting system is consistent. The following is a model of the resulting system:

Model 1:

Let the set *D* consist of three persons, *María*, *Juan* and *Pedro*, associated with 1, 2 and 3, respectively. Let *R** be the relation "loves", where everybody loves him/herself and *María* loves *Pedro*.

> [***VB note: But the axioms specify that R(1,2) holds, so you must have Maria loving Juan. It's possible for her to love only John, or John and Pedro both; and John and Pedro have no restrictions on who they love (except that they must love themselves), so many different models are possible.]

The resulting system is <u>non-categorical</u>: Model 1 above and Model 2 below are not isomorphic (*R** ' holds between *María* and *Juan* in Model 2, but R* doesn't hold between the corresponding elements in Model 1)

<u>Model 2</u>

The domain is the same as that of Model 1. R*' is the "loves" relation, where everybody loves him or herself, *María* loves *Pedro* and *Maria* loves *Juan*.

7. What happens if we replace A5 by A5' $(\exists y)(\forall x)Rxy$? Is the resulting system consistent or inconsistent?

The resulting system <u>is inconsistent</u>: A3 and A4 together require R to be a one-to-one relation. But A5' requires R to be a many-to-one relation. Since no relation can be both one-to-one and many-to-one, there is no model that can satisfy A3, A4 and A5'.

Here is the syntactic proof:

1.	$\forall x \forall y \forall z\ (Ryx \,\&\, Rzx \rightarrow y = z)$	A3
2.	$\exists y \forall x\ Rxy$	A5'
3.	$1 \neq 2 \,\&\, 1 \neq 3 \,\&\, 2 \neq 3$	Simp., A1
4.	$\forall x Rx2$	2, EI

> [*** BHP note: It's not legitimate to do EI with a specific definite element. (With UI you can.) The existential premise just tells you that 'some element' has the property, it doesn't tell you which one. You have to instantiate to "an arbitrary element". This is done in various ways in various formalizations of logic. Suppose we pick a distinguished 'new' quasi-name like *w* (as in PtMW pp 155-156) so that line 4 is $\forall xRxw$; then what you can do is do UI from A1 to get '$w=1 \vee w=2 \vee w=3$", and from that and line 4 you can conclude $\forall xRx1 \vee \forall xRx2 \vee \forall xRx3$; the rest takes longer but you can just do for each of those what you did for $\forall xRx2$, eliminating the possibilities one at a time (by 'disjunctive syllogism) and ending up with a contradiction again.***]

5.	R12	4, UI

6.	$R32$	5, UI
7.	$R12 \;\&\; R32 \rightarrow 1 = 3$	1, UI
8.	$2 = 3$	5, 6, 7, MP
9.	$2 \neq 3$	3, Simp.
10.	$\perp$	8, 9

8. Let axioms A2-A6 be replaced by the single axiom A2'': $\forall x \forall y \forall z\ ((Rxy \;\&\; Rxz) \rightarrow y \neq z)$. Is this system consistent? Is it categorical?

The system <u>is consistent</u> (the following model is a model of this system)

<u>Model 1:</u>

Let the set *D* = {a,b,c}, where a is associated with 1, b is associated with 2 and c is associated with 3. Let *R** = {<a,b>, <b,c>, <c,a>} [Note that axiom A2'' is trivially true in this model].

The system is <u>non-categorical</u>: Both Model 1 above and Model 2 below are models of this system. Model 1 and Model 2 are not isomorphic, since *R** does not hold between a and c in Model 1, but *R**' holds between the corresponding elements in Model 2.

> [***BHP note 1: At first I was going to object to your presupposing that if there were any isomorphism it would have to involve the mapping of a to a, b to b, and c to c. But then I realized that since Axiom 1 mentions the elements 1,2, and 3, they must be in the signature of the algebra (a notion that's not included in PtMW), and therefore any isomorphism must map the 1 element onto the 1 element, the 2 onto the 2, the 3 onto the 3. So your argument is perfectly correct. It would also be possible to prove that there cannot be any isomorphism even if one were free to pick the mapping, but your way makes the proof shorter.***]

> [***BHP note 2: When I wrote note 1, I still hadn't (re-)discovered the "trap" in this question. But in fact any non-empty relation R will lead to a contradiction. Suppose we have R12. An instance of A2'' is: $R12 \;\&\; R12 \rightarrow 2 \neq 2$. So if we have R12 we derive the contradiction $2 \neq 2$. Hence only the empty relation is possible, and the system is (degenerately) categorical.]

<u>Model 2</u>

Let the set *D* = {a,b,c}, where a is associated with 1, b is associated with 2 and c is associated with 3. Let *R**' ={<a,b>, <b,c>, <a,c>, <c,a>}

9. Show that axiom A3 is not independent in the system *L*.

[I tried to construct the syntactic proof and couldn't. I think the following shows that a model that satisfies A1, A2, A4-A6 also satisfies A3. In other words, that there can be no model that satisfies the remaining axioms but not A3. Proving A3 from the other axioms should be easier, though. I'll keep trying]

> [***BHP: The reasoning below looks good to me. And it makes me realize that a syntactic proof that uses the method of *reductio ad absurdum*, i.e. assuming the negation of axiom A3 and deriving a contradiction, is not so very different from showing that there can't be any model, i.e. doing the semantic argument.***]

A3 says that R is not one-many. Let us assume that R is one-many.

By axiom 6, we know that R12. If R were one-many, we could also have R13 (R11 is out because of axiom A2, i.e., the relation is irreflexive). However, that would make R many-to-one, and, hence, it would make A4 false. By axiom 5, we know that every element of S has to bear R to at least one element of S. Given irreflexivity, we only have the possibilities in (a) through (d). It's easy to see that all of them (together with our assumption that R13 and with A6) would make A4 false.

(a) both 2 and 3 bear R to 1
(b) R21 and R32
(c) R23 and R31
(d) R21 and R32

The reasoning above was applied to two elements whose relationship is fixed by one of our axioms. Let us see what happens if we take two arbitrary elements, say 2 and 3. Let R23. If R were one-many, then we could also have R21. However, none of the resulting models will satisfy our axioms,

If R33, then A2 is false

If R31, then A4 is false (R is many-one, since we have both R21 and R31)

If R32, then, since we also have R12 (by A6), A4 is false.