

Homework 10

I. Wang's system

1. **Model 1: The unloving family**

Imagine a family with three children, Tom, Dick, Harry. All of the children have good self-images, and they really like themselves. The brothers don't get along that well amongst each other. Each of the brothers gets along fine with one of his brothers, but really dislikes the other one. In fact, Tom dislikes Dick, Dick dislikes Harry, and Harry dislikes Tom.

Let D be the set with the three brothers, i.e. $D = \{\text{Tom, Dick, Harry}\}$.

Associate Tom with 1, Dick with 2 and Harry with 3.

Let R^* be the relation "dislikes" on the set D , and let R^* correspond to S .

This is a model for the L .

A1: There are exactly three distinct elements.

A2: Each one of the boys likes himself, i.e. no-one dislikes himself.

A3: Each brother dislikes only one of his brothers.

A4: Each brother is disliked by only brother.

A5: Each brother dislikes at least one of his brothers.

A6: Tom corresponds to 1, and Dick to 2. Tom dislikes Dick. Therefore, we have $R^*\text{Tom, Dick, and } R12$.

Model 2: The circle

Let C be a circle that is divided not into the usual 90Equarters, but into three sections of 120E. Call the three sections C_1 , C_2 , and C_3 , and define them as follows:

$$0E \leq C_1 < 120E$$

$$120E \leq C_2 < 240E$$

$$240E \leq C_3 < 360E$$

Now define the following relation on these three sections of the circle: "is the first section you encounter if you move clockwise from" = $\{<C_2, C_1>, <C_1, C_3>, <C_3, C_2>\}$.

If we associate C_1 with 2, C_2 with 1, and C_3 with 3, and let R be "is the first section you encounter if you move clockwise from", then this is a model for L :

A1: There are exactly three distinct elements from D associated with the elements of S .

- A2: You cannot move from one section into itself, without first moving into another section.
- A3: Moving clockwise from some section, you cannot move directly into two distinct sections – i.e. you can move directly into only section of you move clockwise.
- A4: You cannot move directly into the same section from the two distinct sections if you move only in a clockwise direction.
- A5: If you move clockwise from any section, you can move directly into at least one other section.
- A6: Moving in a clockwise direction from C_2 , the first section you encounter is C_1 .

Model 3: An arbitrary relation

Let: $D = \{a, b, c\}$

and $R^* = \{ \langle a,b \rangle, \langle b,c \rangle, \langle c,a \rangle \}$

Let a correspond to 1, b to 2, and c to 3, and R^* to R . Then we have a model for L .

- A1: There exactly three distinct elements.
- A2: Not one of a, b, c have the relation R^* to itself.
- A3: Each of a, b , and c bears the relation R^* to only one element.
- A4: There is only one element that bears the relation R^* to each of a, b , and c .
- A5: Each of a, b , and c bears relation R^* to at least one other element.
- A6: a corresponds to 1 and b to 2, and a stands in relation R^* to b .

2. Independent axioms and formally complete systems

Let Δ be the formally complete system of formulas $\{\varphi_1, \dots, \varphi_n\}$, and $\delta \in \Delta$, the axiom to be deleted.

Assume that δ is not independent.

Then $(\Delta - \delta) \mid \delta$ [By definition of (non)independence.]

Now, let φ be a sentence such that $\Delta \mid \varphi$.

This means that we can construct a finite sequence $\langle \psi_1, \dots, \psi_m \rangle$ of formulas from Δ where $\psi_m = \varphi$, and each ψ_i is (a) an axiom, (b) a premise, or (c) a formula inferred by means of one of the Rules of Inference from earlier formulas in the sequence.

If δ is not in the sequence $\langle \psi_1, \dots, \psi_m \rangle$ of formulas necessary to prove φ , then obviously we will have $(\Delta - \delta) \mid \varphi$.

Since under the assumption of δ 's non-independence, δ can be inferred from $(\Delta - \delta)$ by means of the Rules of Inference, it follows that even if δ is in the sequence $\langle \psi_1, \dots, \psi_m \rangle$ of formulas necessary to prove ϕ , we will still have $(\Delta - \delta) \mid \phi$. [By way of clause (c) in the definition of a proof.]

Therefore, if δ is not independent, then deletion of δ from formally complete system Δ will still leave a formally complete system $(\Delta - \delta)$.

And therefore, the only way in which deletion of δ from formally complete system Δ can result in a formally incomplete system $(\Delta - \delta)$, is when δ is indeed independent.

3. Independent axioms and categorical formal systems

- [1] Let Δ be a categorical formal system.
- [2] Then all models of Δ are isomorphic. Def. of categorical systems.
- [3] For all models M of Δ , it is true that for all $\gamma \in \Delta$, $M \checkmark \gamma$. Definition of Model.
- [4] Let $\delta \in \Delta$ be a non-independent axiom of Δ .
- [5] $(\Delta - \delta) \subset \Delta$, and therefore for all $\beta \in (\Delta - \delta)$, $\beta \in \Delta$.
- [6] For every model M of Δ , we have that for every $\beta \in (\Delta - \delta)$, $M \checkmark \beta$. [3] and [5].
- [7] Therefore every model of Δ is also a model of $(\Delta - \delta)$.
- [8] $(\Delta - \delta) \mid \delta$ [4] and by def. of (non)independence.
- [9] Therefore in every model M of $(\Delta - \delta)$, it must be that $M \checkmark \delta$. [8] and under assumption of soundness.
- [10] Therefore, in every model M of $(\Delta - \delta)$, for all $\beta \in (\Delta - \delta) \cup \delta$, $M \checkmark \beta$. Def. of model, and [9].
- [11] $\Delta = (\Delta - \delta) \cup \delta$.
- [12] Every model of $(\Delta - \delta)$ is a model of Δ . [10], [11] and definition of model.
- [13] Every model of $(\Delta - \delta)$ is a model of Δ , and *vice versa*. [7] and [12]
- [14] Therefore all models of $(\Delta - \delta)$ are isomorphic to each other. [2] and [13]

[15] Therefore $(\Delta - \delta)$ is categorical.

Def. of categorical system.

[16] Therefore, deleting a non-independent axiom from a categorical system Δ still leaves a categorical system.

[17] Therefore, if deleting an axiom from a categorical system results in a non-categorical system, then that axiom must be independent.

4. Models for A2-A6.

Model 1:

Let $D = \{a, b, c, d\}$ and $R^* = \{\langle a,b \rangle, \langle b,c \rangle, \langle c,d \rangle, \langle d,a \rangle\}$.

And let a correspond to 1, b to 2, and R^* to R .

Model 2:

Let $D = \{a, b, c, d, e\}$ and $R^* = \{\langle a,b \rangle, \langle b,c \rangle, \langle c,d \rangle, \langle d,e \rangle, \langle e,a \rangle\}$.

And let a correspond to 1, b to 2, and R^* to R .

Since A2-A6 can have distinct non-isomorphic models, it follows that A2-A6 is not a categorical system. Therefore, removing A1 from L, changes the categorical L to a non-categorical system. The result in Question 3 therefore shows that A1 is an independent Axiom of L.

5. Making L reflexive

When we replace A2 with A2', the new system is inconsistent. To show this we will derive a contradiction:

[1] Let $x = 1$ and $y = 2$.

[2] $x \neq y$.

A1

[3] Rxy

[1] and A6

[4] $(\forall z)(Rzy \rightarrow z = x)$

[3] and A4

[5] Ryy

A2'

[6] $y = x$

[4]

And [2] and [6] contradict each other.

6. Making L non-categorical

Replacing A2 with A2', and deleting A3 and A4, leaves a consistent system. We can show this by finding a model for the new system. This new system is, however, not categorical anymore. We can show by finding two non-isomorphic models for the new system. I discuss two such models below:

Model 1:

Let $D = \{a, b, c\}$

Let $R^* = \{\langle a,a \rangle, \langle b,b \rangle, \langle c,c \rangle, \langle a,b \rangle\}$

Let a correspond to 1, b to 2 and c to 3, and let R^* correspond to R .

Then this is a model for the new system:

- A1: There are exactly three distinct elements.
- A2': Each of the three elements bears relation R^* towards itself.
- A5: This follows directly the fact that the relation R^* is reflexive.
- A6: R^*ab .

Model 2:

Let $D = \{a, b, c\}$

Let $R^* = \{\langle a,a \rangle, \langle b,b \rangle, \langle c,c \rangle, \langle a,b \rangle, \langle b,a \rangle\}$

Let a correspond to 1, b to 2 and c to 3, and let R^* correspond to R .

Then this is a model for the new system:

This model is exactly the same as Model 1, except that R^*ba in Model 2 but not in Model 1. There is no axiom that prohibits symmetry, but also no axiom that requires it. Both models with and without symmetry are therefore possible.

Since the relation R^* in the two models are different, these models are not isomorphic.

7. Another inconsistent system

Replacing A5 with A5' changes the system into an inconsistent system. Again I will show this by deriving a contradiction from the axioms:

- [1] Let $a = 1, b = 2, c = 3$.
- [2] $a \neq b \neq c$ A1
- [3] At least one of the following must be true: A5'
 Rab, Rac, Raa OR Rba, Rbc, Rbb OR Rcb, Rca, Rcc
- [4] $a = b = c$ [3] and A3

And [2] and [4] contradict each other.

8. Yet another inconsistent system?

This was my first idea: Replacing A2-A6 with A2'' changes the system into an inconsistent system. Again I will show this by deriving a contradiction from the axioms:

- [1] Let $a = 1, b = 2$.
- [2] Let $R12$, therefore Rab
- [3] $(Rab \ \& \ Rab)$ is true
- [4] $b = b$ is obviously true
- [5] $(Rab \ \& \ Rab) \rightarrow b \neq b$ A2''
- [6] $(Rab \ \& \ Rab)$ [3]
- [7] $b \neq b$ [6] and M.P.

But [7] and [4] contradict each other.

But then I had another idea: What if we define as model the set $D = \{a, b, c\}$ and define the relation R^* as an empty relation, i.e. a, b, c are not related to themselves or to each other by R^* . If we then let a correspond to 1, b to 2, and c to 3, and R^* to R , we do have a model for this system:

A1: There are indeed exactly three elements, and they are distinct.

A2'': Since no two elements (either identical or distinct) are related in terms of R^* , the antecedent of this axiom is always false. When the antecedent is false, the consequent is true. Therefore, we will never be able to find an example to falsify this axiom.

Then we have a model for this system. Therefore this system is consistent. Furthermore, it is then also categorical. Any model for this system must have the same structure: (i) In order to satisfy A1 it must have a set with three distinct elements. (ii) In order to satisfy A2'', it must have some relation that doesn't actually establish a relation between any two of the elements. *Which answer is correct? [See below.]*

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Instructors' notes. Beautiful thinking. The second answer is correct. The relation you have described is *the empty relation*. It is perfectly well-defined; it is the empty set of ordered pairs. I.e. for all $x,y, \sim R(x,y)$. And indeed, that is the only relation possible in this axiomatization, so the system is indeed categorical.

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9. Proving that A3 is not independent

- [1] Let $a = 1, b = 2, c = 3$.
- [2] $a \neq b \neq c$. [1] and A1
- [3] Rab [1] and A6
- [4] For c , at least: A5
- Rcc . But this is impossible. A2
- or Rcb . But this is impossible. [2], [3] and A4
- or Rca
- Therefore, Rca .
- [5] For b , at least: A5
- Rbb But this is impossible. A2
- or Rba But this is impossible. [2], [4] and A4
- or Rbc
- Therefore Rbc
- [6] For a , at least: A5
- Raa But this is impossible. A2
- or Rac But this is impossible. [2], [5] and A4
- or Rab
- Therefore Rab Also [3]
- [7] But now we have:
- $Rca, \sim Rcc, \sim Rcb$ [4]
- $Rbc, \sim Rbb, \sim Rba$ [5]
- $Rab, \sim Raa, \sim Rac$ [6]

From which follows that $\forall x \forall y \forall z ((Rxy \ \& \ Rxz) \rightarrow y=z)$.

But this is A3. Therefore A3 can be derived from the other axioms.