Lecture 4. Algebra, continued
Section 2: Lattices and Boolean algebras

CONTENTS

Reading: Chapter 11: 11.1 – 11.2 of PtMW, pp. 275-281; Chapter 12: 12.1 – 12.2 of PtMW, pp.295 –299. Supplementary material on Boolean algebra in xeroxed extract from: Partee (1979) *Fundamentals of Mathematics for Linguists*. Stamford, CT: Greylock Publishers. Reprinted by D.Reidel, Dordrecht. (III.D. Boolean algebras. 127-136.)

Some aspects of what's covered in both readings will make more sense after we have looked at logic, but if you know a little logic already you can probably make sense of it. When we look at logic, we'll come back to this so that we can see how useful quotient algebras are for showing how propositional logic can be a Boolean algebra.

1. Lattices.

1.0. Why lattices?

There is a special class of algebraic structures, called *lattices*, which are used widely in many fields, and it is useful to be familiar with their basic properties. Lattices can be defined as algebras and they will be our first example of a specific kind of algebra. In this lecture we will also consider Boolean algebras, which are a special case of lattices. We will just be introducing the basic structure of lattices and Boolean algebras with some examples; there are many directions one can go from here, both in algebraic studies and in applications.

1.1. Posets.

In Lecture 1 we considered the relation of weak order (reflexive, anti-symmetric and transitive). Such a relation is also called a *partial order* (because it is not obligatorily a total order). Any set A on which a partial order $\leq$ is defined is called a *partially ordered set* or *poset* and write it as $<A, \leq>$ or just A assuming the intended order.

1.1.1. Upper and lower bounds. Duality.

Elements a and b of a poset A are called *comparable* if $a \leq b$ or $b \leq a$. If they are not comparable, they are called *incomparable*.

In an arbitrary poset A we define an *upper bound* of $B \subseteq A$ [note[1]] as an element $a \in A$, if it exists, such that for all $b \in B$, $b \leq a$. An upper bound a of B is *the least upper bound of B* or *the supremum of B* (abbreviated to sup B) if, for any upper bound c of B, we have $a \leq c$. We write $a = \sup B$, since by antisymmetry of the ordering $\leq$ we know that if B has a least upper bound, this is a unique least bound.

It is easy to see that the inverse relation of the partial order $\leq$ is also a partial order. We write $a \geq b$ for $b \leq a$, call the order $\geq$ the dual order to $\leq$ and call the poset $<A, \geq>$ with a dual order the *dual* poset to the poset $<A, \leq>$.

Many notions on posets and other structures will have their dual versions when we replace all the occurrences of the order $\leq$ in definitions on the order $\geq$. The first examples will be lower bound and the greatest lower bound.

We define the dual of an upper bound of $B \subseteq A$, called a lower bound of $B \subseteq A$, as an element $a \in A$, such that for all $b \in B$, $b \geq a$ which is equivalent to $a \leq b$. A lower bound a of B is *the greatest lower bound of B* or *the infimum of B* (abbreviated to inf B) if, for any lower bound c of B, we have $a \geq c$. We write $a = \inf B$.

1.1.2. Diagrams of posets.

Partial ordering may be represented visually by so called Hasse diagrams. The diagram of poset A represents the elements of the set A as points on the plane and the ordering relation pictured as a line reflecting the order from the bottom to top in the representation. Assuming reflexivity and transitivity of the order, only lines for immediate successors are pictured. See examples of diagrams below (examples are extracted from PtMW). In the Figure 11-1 below $0 \leq a$, $0 \leq b$, $a \leq 1$, $b \leq 1$ and corresponding "reflexive" and "transitive" pairs also belong to the order.

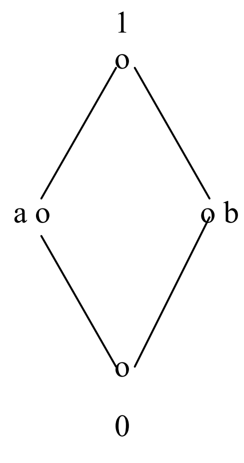

Figure 11-1: the diagram of a poset

$A = <\{0,a,b,1\}, \leq>$.

[1] A note on terminology. When we write "an *upper bound* of $B \subseteq A$", that is a shortcut way of writing "an *upper bound* of a subset B in a set A", i.e. an upper bound in A of some subset B of elements of A.

1.2. Lattices and semilattices.

1.2.1. Lattices.

There is a special class of posets, called *lattices*. On the other hand, lattices can also be defined as algebras. We consider both definitions.

Definition 1. A poset $<A, \leq>$ is a *lattice* if $\sup\{a,b\}$ and $\inf\{a,b\}$ exist for all $a,b \in A$.

In order to develop the second definition, we introduce two new operations on A:

(Op) $a \wedge b = \inf\{a,b\}$ and $a \vee b = \sup\{a,b\}$,

calling $a \wedge b$ the *meet* and $a \vee b$ the *join* of a and b. In lattices these operations are binary. So we can consider a lattice as an algebra in a signature $\{\wedge, \vee\}$. But not every algebra in that signature is a lattice. To be a lattice, its operations should have some properties. It is easy to see that operations defined by (Op) have the properties (L1) – (L4):

(L1)	$a \wedge a = a$, $a \vee a = a$	idempotent law
(L2)	$a \wedge b = b \wedge a$, $a \vee b = b \vee a$	commutative law
(L3)	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	associative law
	$(a \vee b) \vee c = a \vee (b \vee c)$	
(L4)	$a \wedge (a \vee b) = a$	absorption law
	$a \vee (a \wedge b) = a$	

With the help of these properties we can give another definition of lattices:

Definition 2. An algebra **A** in the signature $\{\wedge, \vee\}$ on the carrier A is a lattice if properties (L1) – (L4) hold for its operations.

It is not very difficult to show that these two definitions are equivalent. Indeed, as we have seen for the lattice A with the order $\leq$ from definition 1, the operations $\wedge$ and $\vee$ given by (Op) define on A the algebra **A** and have the properties (L1) – (L4). If we consider the order $\leq_l$ defined by these operations by the condition

(Ord) $a \leq_l b$ iff $a \wedge b = a$

we will get the same order that we have in the lattice A.

On the other hand, if we begin from definition 2 and define on the carrier A of the algebra **A** the order corresponding to the condition (Ord), the conditions of the definition 1 will be satisfied. The operations defined by this order by the conditions (Op) will coincide with the operations of algebra **A**.

You can try to figure out the full proof by yourself, or you can look in PtMW, pp. 280-281.

Examples.

1) It is easy to verify that the poset A in the Figure 11-1 above is a lattice.
2) For every nonempty set A the set $\wp(A)$ of all its subsets is a lattice with the order $\leq$ defined as the relation $\subseteq$ on subsets and the operations $\wedge$ and $\vee$ defined as intersection and union.

1.2.2. Semilattices.

We can weaken our definitions and get the notion of semilattice.
A poset $<A, \leq>$ is a *join semilattice* if $\sup\{a,b\}$ exists for all $a,b \in A$.

A poset $<A, \leq>$ is a *meet semilattice* if $\inf\{a,b\}$ exists for all $a,b \in A$.

The notion of *meet semilattice* was used by Godehard Link to model the semantics of plurals and mass nouns in natural language.

2. Boolean algebras.

Boolean algebras are an important and well-known example of algebras closely related to logic. In a way, they were introduced by John Boole to investigate Statement Logic.

Boolean algebras are a special case of lattices but we define them here "from scratch". Let us consider the signature $\Omega_{BA} = \{0, 1, \neg, \vee, \wedge\}$ where 0 and 1 are 0-ary symbols (constants), $\neg$ is a unary one[2], $\vee$ and $\wedge$ are binary.

Definition 1. An algebra in a signature Ω_{BA} is called a *Boolean algebra* if properties (B1) – (B5) hold for its operations.

(B1)	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$ $(a \vee b) \vee c = a \vee (b \vee c)$	associative laws
(B2)	$a \wedge b = b \wedge a$ and $a \vee b = b \vee a$	commutative laws
(B3)	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	distributive laws
(B4)	$a \wedge 1 = a$ and $a \wedge 0 = 0$ $a \vee 0 = a$ and $a \vee 1 = 1$	top and bottom laws
(B5)	$a \wedge (\neg a) = 0$ and $a \vee (\neg a) = 1$	complementation laws

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Examples.

1) Let A be a nonempty set. We know that the set $\wp(A)$ of all its subsets is a lattice with the order $\leq$ defined as the relation $\subseteq$ on subsets and the operations $\wedge$ and $\vee$ defined as intersection and union. But $\wp(A)$ is a specific lattice. It is a classical example of Boolean algebra with constants 1 and 0 defined as the set A and the empty set $\varnothing$ respectively and the operation $\neg$ defined as a complement to A. It is not difficult to verify that properties (B1) – (B5) hold for operations defined in this way.
2) The algebra **T** defined on the set of truth-values $\{0,1\}$. Constants 0 and 1 represent truth- values 0 and 1 (False and True; we can consider constants as names of truth-

[2] The book PtMW uses * for the complement operation rather than $\neg$. The book Partee 1979 uses $'$, $\cup$, $\cap$, rather than $\neg$, $\vee$, $\wedge$.

values or can identify constants with truth-values). Other operations are given by well known tables:

$\neg 0 = 1$	$1 \wedge 1 = 1$	$1 \vee 1 = 1$
$\neg 1 = 0$	$1 \wedge 0 = 0$	$1 \vee 0 = 1$
	$0 \wedge 1 = 0$	$0 \vee 1 = 1$
	$0 \wedge 0 = 0$	$0 \vee 0 = 0$

Homework 5.

Suggested priorities: Do at least one "Claim" and one "Example" from question 1, and all of questions 2 and 3, and 5. If you have time, do 4 or part of it. Number 8 (in PtMW) would be good if you'd like more examples of posets and practice seeing which posets are lattices. If you have more time, try one or two of the ones listed as "6". Number 7 is completely optional. We certainly don't expect you to do all, or even most, of these.

1) Show that the claims given in connection with Definitions 1 and 2 of *lattices* in section 1.2.1 above hold. [For each of these, you need its context in the text of 1.2.1]

Claim 1: The operations $\wedge$ and $\vee$ defined by (Op) have the properties (L1)-(L4).

Claim 2: If we consider the order $\leq_1$ defined by these operations by the condition
(Ord) $a \leq_1 b$ iff $a \wedge b = a$
we will get the same order that in we have in the lattice *A*.

Claim 3: On the other hand, if we begin from definition 2 and define on the carrier *A* of the algebra **A** the order corresponding to the condition (Ord), the conditions of the definition 1 will be satisfied.

Claim 4: The operations defined by this order by the conditions (Op) will coincide with the operations of algebra A.

Example 1: Show that the Poset in Figure 11-1 is a lattice.

Example 2: Verify that the power set algebra given in Example 2 is always a lattice. [Optional addition: Can't the claim in Example 2 be strengthened by dropping the restriction to starting with a nonempty set *A*? Doesn't the power set of the empty set also produce a lattice?]

2) Define algebras corresponding to join and meet semilattices. [answer in text p.282]

3) Define the notion of sublattice, using the algebraic definition of lattice (not the poset definition) and give examples.

4) [optional] a. The notion of "subposet" is not defined in the text or in these notes, although it is used in the text on p. 281 and in Fig. 11-4 on p. 282. Let us define it as follows: $<B, \leq_B>$ is a subposet of a poset $<A, \leq_A>$ iff (i) $B \subseteq A$ and (ii) $\leq_B = \leq_A|B$.

Given that definition, let's look at the question about whether the interdefinability of lattices as a kind of poset and lattices as a kind of algebra extends also to the "sub" concepts. I.e. are subposets as defined above equivalent to sublattices on the algebraic definition? The answer turns out to be "no".

To show it, can you find an example of a subposet of a lattice such that the subposet fits the poset definition of a lattice, but which is not a sublattice of the original lattice according to the algebraic definition? (There is an example given in the text of PtMW, p. 281, diagram p. 282. If you look it up, then find a different one.) NB These are good examples of something that came up in class on Thursday Sept 20: namely that the third condition in the definition of subalgebra is not redundant!

Note: this result does not threaten the equivalence of the poset definition and the algebraic definition of lattices, but it does underscore the fact that posets are not algebras (why?), so the "sub-X" definitions that apply uniformly to algebras don't apply in the same way to posets.

b. Under Figure 11-4 on p.282 it says "The subposet **L'** = <{a,c,d,e}, ≤ } is a lattice and a subalgebra, but not a sublattice." I (BHP) think that it's false that it's a subalgebra. Yes, VB confirms it. Show that the given subposet, which is indeed a lattice, is neither a subalgebra nor a sublattice. (In fact I suspect, without having tried to prove it, that it's impossible for a subset of a lattice together with the original lattice operation to be a lattice and a subalgebra without being a sublattice. Is that right?)

5) Prove that $\wp(A)$ and **T** are Boolean algebras (verify that properties (B1) – (B5) hold). What subalgebras do these algebras have?

6) and more: See homework problems for the xeroxed handout IIID of Partee (1979), and try some of them. Postpone the ones about the propositional logic model (1,2, and 7; in fact don't read section 5 about the propositional logic model itself – we'll return to it later. I didn't know then that what I presented there has a name, the "Lindenbaum algebra" corresponding to propositional logic). But you can try any of 1 (any one or more of 1a-d; these are good to do), 4 (highly recommended), and the 'starred problem' (= more challenging) 5 (similar to 4 but harder). You can do 6 but it's less interesting. Number 7 is interesting but should be postponed.

7) We defined Boolean algebras "from scratch". How might the definition be shortened if we defined Boolean algebras in terms of lattices plus additional properties? (The answer is given in a possibly cryptic form at the beginning of 12.1 in PtMW; cryptic because we haven't asked you to read 11.5, where complemented and distributive lattices are defined. But you can get an idea from that paragraph what axioms you would have to write down in addition to saying that $<B, \vee, \wedge>$ forms a lattice.)

8) Try part or all of exercise 2 in PtMW, p. 293. Those are good examples for playing with posets and seeing which posets are and aren't lattices.