Lecture 7.
Logic III. Axiomatic description of properties of relations.
Algebra III. Homomorphisms, congruences and quotient algebras.

CONTENTS

Reading: Chapter 8: 8.5, Chapter 9: 9.1 – 9.4 of PMW, pp. 206 –211, 247- 253.

1. Axiomatic description of properties and classes of relations

We can use PL to define properties and classes of relations which we studied in Lectures 1-3. Below we will write axioms for different relations using the same binary predicate symbol ***R***. Models in which these axioms (or sets of axioms) will be true will represent corresponding relations. In all cases below, the domain of quantification is the set A on which the relation is defined.

1.1. Properties of relations

1.1.1. Reflexivity, etc.

Reflexivity: $\forall x R(x, x)$

It is easy to see that in every model in which this axiom holds the relation ***R*** will be reflexive.

Nonreflexivity: $\exists x \neg R(x, x)$

Irreflexivity: $\forall x \neg R(x, x)$

1.1.2. Symmetry, etc.

Symmetry: $\forall x \forall y (R(x, y) \rightarrow R(y, x))$

Nonsymmetry: $\exists x \exists y (R(x, y) \ \& \ \neg R(y, x))$

Asymmetry: $\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$

Antisymmetry: $\forall x \forall y((R(x, y) \;\&\; R(y, x)) \rightarrow (x = y))$

Note. In Lecture 6 we considered Predicate Logic without equality (we did not introduce the predicate "="). To deal with it we need additional syntactic and semantic rules. The syntactic rule simply introduces formulas of the type $(x = y)$ for any pair of variables $x, y \in$ Var. The natural infix notation is used. The corresponding semantic rule is also natural:

$[\![(x = y)]\!]^{M,g} = 1$ iff $g(x) = g(y)$

(or in the terms of operations on D_t, $[\![(x = y)]\!]^{M,g} = [\![x]\!]^{M,g} \Box [\![y]\!]^{M,g}$).

1.1.3. Transitivity, etc.

Transitivity: $\forall x \forall y \forall z((R(x, y) \;\&\; R(y, z)) \rightarrow R(x, z))$

Nontransitivity: $\exists x \exists y \exists z(R(x, y) \;\&\; R(y, z) \;\&\; \neg R(x, z))$

Intransitivity: $\forall x \forall y \forall z((R(x, y) \;\&\; R(y, z)) \rightarrow \neg R(x, z))$

1.1.4. Connectedness:

$\forall x \forall y(R(x, y) \vee R(y, x))$

1.2. Classes of relations

1.2.1. Equivalence relations

We know that this property of relations needs three axioms together: reflexivity, symmetry and transitivity:

1) $\forall x R(x, x)$
2) $\forall x \forall y(R(x, y) \rightarrow R(y, x))$
3) $\forall x \forall y \forall z((R(x, y) \;\&\; R(y, z)) \rightarrow R(x, z))$

1.2.2. Tolerance relations

Here we need only two first ones from equivalence: reflexivity and symmetry:

1) $\forall x R(x, x)$
2) $\forall x \forall y(R(x, y) \rightarrow R(y, x))$

1.2.3. Orderings

A relation R is a **weak order** iff it is transitive, reflexive and antisymmetric**:**

1) $\forall x \forall y \forall z((R(x, y) \;\&\; R(y, z)) \rightarrow R(x, z))$
2) $\forall x R(x, x)$
3) $\forall x \forall y((R(x, y) \;\&\; R(y, x)) \rightarrow (x = y))$

A **strict order** is transitive, irreflexive and asymmetric**:**

1) $\forall x \forall y \forall z((\boldsymbol{R}(x, y) \;\&\; \boldsymbol{R}(y, z)) \rightarrow \boldsymbol{R}(x, z))$

2) $\forall x \neg R(x, x)$

3) $\forall x \forall y(\boldsymbol{R}(x, y) \rightarrow \neg \boldsymbol{R}(y, x))$

For a **linear ordering** (strict or weak) we need the connectedness axiom to be added, so, for example, **linear weak order** is defined by the following four axioms:

1) $\forall x \forall y \forall z((\boldsymbol{R}(x, y) \;\&\; \boldsymbol{R}(y, z)) \rightarrow \boldsymbol{R}(x, z))$

2) $\forall x R(x, x)$

3) $\forall x \forall y((\boldsymbol{R}(x, y) \;\&\; \boldsymbol{R}(y, x)) \rightarrow (x = y))$

4) $\forall x \forall y(\boldsymbol{R}(x, y) \vee \boldsymbol{R}(y, x))$

A **preorder** or **quasi-order** is a relation which is not an order because it violates antisymmetry. For instance, "is at least as old as" is a weak order on the domain of *ages*, but it is not a weak order on the domain of *people*, because "John as at least as old as Peter" and "Peter is at least as old as John" do not together entail that John = Peter, only that John and Peter are the same age. (We can transform this preorder into an order by grouping people of the same age into equivalence classes; the relation "is at least as old as" on these equivalence classes is a weak linear order.)

We can derive the axiomatic definition of weak **preorder** from the definition of weak order by throwing out the condition of antisymmetry. So we have:

1) $\forall x \forall y \forall z((\boldsymbol{R}(x, y) \;\&\; \boldsymbol{R}(y, z)) \rightarrow \boldsymbol{R}(x, z))$

2) $\forall x R(x, x)$

Thought question: Why isn't "is older than" on the set of people a "strong preorder"? What kind of relation is it?

2. Algebra, continued. Section 3: Homomorphisms, congruences and quotient algebras.

[Repetition, from Lecture 4, sections 2.2 (here 2.1) and 2.3 (here 2.2), with added examples]

2.1. Congruences.

We will say that any equivalence relation on the <u>carrier</u> of an Σ-algebra holds on the <u>algebra</u>. Some equivalence relations on Σ-algebras are in close relations with homomorphisms.

Given an Σ-algebra **A** with the carrier *A*, we say that an equivalence relation Q on *A* *agrees* with the operation $\sigma \in \Sigma(n)$ when for any *n*-tuples

$\langle a_1, ..., a_n \rangle , \langle b_1, ..., b_n \rangle \in A^n$ we have

$\langle \sigma(a_1, ..., a_n), \sigma(b_1, ..., b_n)\rangle \in Q$ if $\langle a_i, b_i \rangle \in Q$ for $i = 1,\ldots, n$.

<u>Definition</u>. We say that equivalence relation Q on Σ-algebra **A** is a *congruence* on **A** if it agrees with every operation $\sigma \in \Sigma(n)$.

In Lecture 3 we considered notions of a quotient set of the equivalence relations, natural mapping, kernel of a mapping, etc. We will use these notions now.

Theorem1. The kernel of a homomorphism is a congruence.

Examples. Algebras in the signature Σ_{Numb} = { zero, one, +, $\times$}

Let us return to algebras the signature Σ_{Numb} = { **zero**, **one**, +, $\times$}: **Nat**, **Parity**, **Mod4, Mod2** and their homomorphisms that we considered in the Lecture 4.

(1) The algebra **Nat** with the carrier **N** = {0,1,2,3,…}. In this algebra binary operators + and $\times$ define respectively operations of addition and multiplication in the usual way and 0-ary operators **zero** and **one** "mark" 0 and 1.

(2) The algebra **Parity** with the set {*even*, *odd*} as a carrier, 0-ary operators **zero** and **one** "marking" *even* and *odd* respectively (**zero** = *even* and **one** = *odd*) and binary operations + and $\times$ with obvious tables, that is *even* + *even* = *even* and so on.

(3) The homomorphism *h*: **Nat** → **Parity** (the function *h*: **N** → {*even*, *odd*} mapping even numbers to *even* and odd numbers to *odd*). The quotient set **ker** *h* of its kernel equivalence consists of two classes, which we will call "Even" and "Odd": Even = [[0]] = [[36]] (etc) ={0,2,4,…} and Odd = [[17]] = [[1]] ={1,3,5,…}.

It is easy to verify that **ker** *h* is a congruence on the algebra **Nat**, i.e. it agrees with operations of the signature Σ_{Numb}. For example, if ***x, y*** $\in$ Even and ***z,w*** $\in$ Odd then both **(*x* + *z*)** and **(*y* + *w*)** are odd.

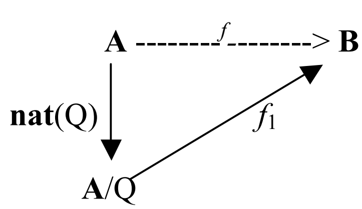
(4) The algebras **Mod4** and **Mod2** and the homomorphism *f*: **Mod4** $\to$ **Mod2** considered in Lecture 4. The quotient set Q = **ker** *f* of its kernel equivalence consists of two classes: {0,2} and {1,3}. It is easy to verify that **ker** *h* is a congruence on the algebra **Mod4.**

2.2. Quotient algebras.

Theorem 2. Let **A** be an Σ-algebra and Q a congruence on **A**. Then there exists a unique Σ-algebra on the quotient set *A*/Q of the carrier *A* of **A** such that the natural mapping *A* $\to$ *A*/Q is a homomorphism.

We will denote such an algebra as **A**/Q and will call it a *quotient algebra* of an algebra **A** by the congruence Q. In this case we have a natural homomorphism **A** $\to$ **A**/Q.

Given Σ-algebras **A** and **B** and a homomorphism *f*: **A** $\to$ **B**, consider the equivalence Q = **ker** *f*. By Theorem 1 the equivalence Q is a congruence on **A** and the natural mapping **nat**(Q): *A* $\to$ *A*/Q is the natural homomorphism **A** $\to$ **A**/Q. Consider the one-to-one mapping f_1: *A*/Q $\to$ *B* such that f_1([[*x*]]) = *f*(*x*). We can show that mapping f_1: **A**/Q $\to$ **B** is a monomorphism and the diagram below is commutative.



Example. Consider the homomorphism f: **Mod4** → **Mod2** from the earlier example. The quotient set Q = **ker** f of its kernel equivalence consists of two classes: {0,2} and {1,3}. The operations of the quotient algebra **Mod4**/Q are defined in a natural way:

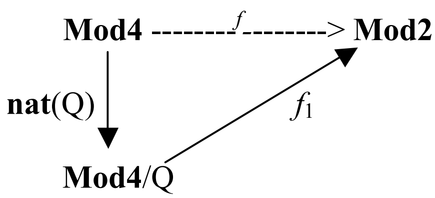
zero = {0,2}

one = {1,3}

{0,2}+ {0,2} = {0,2}, {0,2}+ {1,3} = {1,3}, …,{0,2} · {0,2} = {0,2}, {0,2} · {1,3} = {0,2}, etc.

And it is easy to see that there exists a monomorphism f_1: **Mod4**/Q → **Mod2**, such that

$f = f_1 \circ$ **nat**(Q), so the following diagram is commutative:



Homework 7

Do at least one of the following exercises.

1. Problem 1 from Homework 4 (Lecture 4), repeated here. Do it now if you didn't do it before.

Consider the homomorphism from the algebra on the set **N** of natural numbers with addition and multiplication to **Parity** (see example above). Write down (a) the kernel equivalence of this homomorphism, (b) the corresponding quotient algebra, and (c) the commutative diagram.

2. There are some questions interspersed in the handout; answer some of them (the more, the better).