Lecture 3. Properties of Relations.

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Reading: Chapter 3 of Partee, ter Meulen, andWall (PtMW).

1. Properties of Relations

1.1. Reflexivity, symmetry, transitivity, and connectedness

We consider here certain properties of binary relations. All these properties apply only to relations *in* (*on*) a (single) set, i.e., in $A \times A$ for example.

Reflexivity. Given a set *A* and a relation *R* in *A*, *R* is *reflexive* iff all the ordered pairs of the form <*x,x*> are in *R* for every *x* in *A*. A relation which fails to be reflexive is called *nonreflexive*, but if it contains no ordered pair <*x,x*>, it said to be *irreflexive*.

Another way to state the definitions above is to use the identity relation id_A. Relation *R* is reflexive iff $id_A \subseteq R$, it is nonreflexive iff $id_A \not\subseteq R$, and it is irreflexive iff $id_A \cap \mathrm{R} = \varnothing$.

Examples. Relations "= " and "≥" on the set **N** of natural numbers and relations "⊇" and "⊆" between sets are reflexive. Relations "≠" and "<" on **N** are nonreflexive and irreflexive.

Remember that we always consider relations <u>in some set</u>. And a relation (considered as a set of ordered pairs) can have different properties in different sets. For example, the relation *R* = {<1,1>, <2,2>} is reflexive in the set A_1 = {1,2} and nonreflexive in A_2 = {1,2,3} since it lacks the pair <3,3> (and of course it nonreflexive in **N**).

Symmetry. Given a set *A* and a relation *R* in *A*, *R* is *symmetric* iff for every ordered pair <*x,y*>, if <*x,y*> is in *R,* then the pair <*y,x*> is also in *R*. If for some <*x,y*> in *R*, the pair <*y,x*> is not in *R*, then *R* is *nonsymmetric*. If it is never the case that for any <*x,y*> in *R*, the pair <*y,x*> is in *R* then the relation is called *asymmetric*. Note that an asymmetric relation must be irreflexive. [Why?] A relation *R* is *anti-symmetric* if whenever both <*x,y*> and <*y,x*> are in *R*, then *x* = *y*. Note that a relation need not be reflexive to be anti-symmetric. [Why?]

Examples. The relation "*brother of*" is nonsymmetric in the set of all people, but it can be symmetric in some set, say, in the set *A* = {John, Peter, Bill}, if John and Bill are brothers. [Can you think of a set in which it is asymmetric?]

Transitivity. A relation *R* is *transitive* iff for all ordered pairs <*x,y*> and <*y,z*> in *R*, the pair <*x,z*> is in *R*. If a relation fails to meet the definition of transitivity, it is *nontransitive*. If for no pairs $<x,y>$ and $<y,z>$ in R, the pair $<x,z>$ is in R, then the relation is *intransitive*.

Examples. Relations =, > and ≥ are transitive in the set of natural numbers. Is the relation "friend of" in the set of all people transitive?

Connectedness. A relation R in A is *connected* iff for every two *distinct* elements x and y in A, $<x,y> \in R$ or $<y,x> \in R$ (or both).

1.2. Diagrams of relations

It may be helpful to demonstrate the properties of relations representing them in relational diagrams. The members of the relevant set are represented by labeled points. If x is related to y, i.e. $<x,y> \in R$, an arrow connects the corresponding points. For example, Figure 3-1 represents the relation

$$R = \{<1,2>,<2,1>,<2,2>, <1,1>,<2,3>,<3,3>\}$$

[PtMW, p. 43, Fig.3-1]

It is apparent from the diagram that the relation is reflexive, since every point bears a loop. The relation is non-symmetric since there is no arrow from 3 to 2 (but there is one from 2 to 3). It cannot be called asymmetric or antisymmetric, since 1 is related to 2 and 2 is related to 1. It is not transitive since 1 is related to 2 and 2 to 3, but there is no arrow from 1 to 3.

1.3. Classes of relations

Using properties of relations we can consider some important classes of relations.

1.3.1. Equivalence relation.

An equivalence relation is a relation which is reflexive, symmetric and transitive. For every equivalence relation there is a natural way to divide the set on which it is defined into mutually exclusive (disjoint) subsets which are called *equivalence classes*. We write $[[x]]$ for the set of all y such that $<x,y> \in R$. Thus, when R is an equivalence relation, $[[x]]$ is the equivalence class which contains x.

The set $A/R =_{\text{def}} \{[[x]] \mid x \in A\}$ is called a *quotient set* of the set A by the equivalence R. A/R is a subset of $\wp(A)$. For every equivalence relation R, the function **nat**(R): $A \to A/R$ mapping every element $x \in A$ onto $[[x]]$ is called a *natural mapping* of A onto A/R.

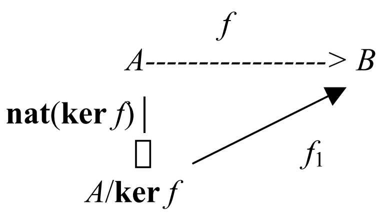
Examples. The relations "has the same hair color as" or "is the same age as" in the set of people are equivalence relations. The equivalence classes under the relation "has the same hair color as" are the set of blond people, the set of red-haired people, etc.

Partitions. Given a non-empty set A, a *partition* of A is a collection of non-empty subsets of A such that (1) for any two distinct subsets X and Y, $X \cap Y = \varnothing$ and (2) the union of all the subsets in collection equals A. The subsets of A that are members of a partition of A are called *cells* of that partition.

There is a close correspondence between partitions and equivalence relations. Given a partition of set A, the relation $R = \{\langle x,y\rangle \mid x$ and y are in the same cell of the partition of $A\}$ is an equivalence relation in A. Conversely, given an equivalence relation R in A, there exists a partition of A in which x and y are in the same cell iff $\langle x,y\rangle \in R$..

Functions and equivalences. Every function $f: A \to B$ determines an equivalence relation $f^{-1} \circ f$ on the set A. This equivalence relation is called the *kernel* of f, written **ker** f; so **ker** $f =_{\text{def}} f^{-1} \circ f$. For the quotient set of this equivalence we have the one-to-one mapping f_1: A/**ker** $f \to B$ such that $f_1([[x]]) = f(x)$. We can show that the following equation holds: $f = f_1 \circ$ **nat** (**ker** f).

This equation can be pictured in the form of a *commutative diagram*:



Example. Consider two sets *Countries* = {Germany, England, India, China}, *Continents* = {Europe, Asia, Australia, North America} and a function f: *Countries* $\to$ *Continents* mapping every country to its continent. Then
The equivalence relation **ker** f = {<Germany, Germany>, <Germany, England>, <England, England>, <England, Germany>, <India, India>, <India, China>, <China, China>, <China, India>},
Countries/**ker** f = {{Germany, England}, {India, China}},
f_1({Germany, England}) = Europe, f_1({India, China}) = Asia.

1.3.2. Tolerance.

A relation R in A is called a *tolerance* (or a *tolerance relation*) if it is reflexive and symmetric. So tolerance is weaker than equivalence; it does not need to be transitive.

The notion of tolerance relation is an explication of *similarity* or *closeness*. Relations “neighbor of”, “friend of” can be considered as examples if we hold that every person is a neighbor and a friend to him(her)self.

As analogs of equivalence classes and partitions, here we have tolerance classes and coverings. A set $B \subseteq A$ is called a tolerance *preclass* if it holds that for all $x, y \in B$, x and y are tolerant, i.e. $\langle x,y\rangle \in R$. A maximum preclass is called a *tolerance class*. So two tolerance classes can have common elements.

Coverings. Given a non-empty set A, a collection (set) $\mathcal{C}$ of non-empty subsets of A such that $\bigcup_{B \in \mathcal{C}} B = A$ is called a *covering* of A. Given a tolerance relation in A, the collection of its *tolerance classes* forms a *covering* of A.

Every partition is a covering; not every covering is a partition.

1.3.3. Orderings.

An *order* is a binary relation which is transitive and in addition either (i) reflexive and antisymmetric or else (ii) irreflexive and asymmetric. The former are *weak* orders; the latter are *strict* (or *strong*).

Some terminology: if R is an order, either weak or strict, and $\langle x,y\rangle \in R$, we say that *x precedes y*, *x is a predecessor of y*, *y succeeds* (or *follows*) *x*, or *y is a successor of x*. If x precedes y and $x \neq y$, then we say that *x immediately precedes y* if and only if there is no element z distinct from both x and y such that x precedes z and z precedes y. In other words, there is no other element between x and y in the order.

There is also a useful set of terms for elements which stand at the extremes of an order. Given an order R in a set A,

1) an element x in A is *minimal* iff there is no other element in A which precedes x
2) an element x in A is *least* iff x precedes every other element in A
3) an element x in A is *maximal* iff there is no other element in A which follows x
4) an element x in A is *greatest* iff x follows every other element in A.

Note that greatest (least) element is maximal (minimal) but the opposite is not always the case.

If an order, strict or weak, is also connected, then it is said to be a *total* or *linear* order.

Examples. Relations $\geq$ and $=$ on the set **N** of natural numbers are examples of weak order, as are relations $\supseteq$ and $=$ on subsets of any set. The relations $>$ and are examples of strict orders on the corresponding sets. The relations $\geq$ and $>$ are linear orders.

Homework 3.

Chapter 3. pp. 51 – 53, all 5 problems.

6) (optional) Invent a function analogous to the function f: Countries $\rightarrow$ *Continents*. Write down: (a) an equivalence relation which is a kernel of this function, (b) its quotient set, and (c) the corresponding commutative diagram.