Lecture 2. Functions and Relations.

Reading: Chapter 2 of Partee, ter Meulen, and Wall (PtMW).

1. Relations and Functions

1.1. Ordered pairs and Cartesian products

As we see, there is no order imposed on the elements of a set. To describe functions and relations we will need the notion of an *ordered pair*, written $<a,b>$, for example, in which a is considered the *first member (element)* and b is the *second member (element)* of the pair. The ordered pair can be defined as follows:

$<a,b> =_{def} \{\{a\}, \{a,b\}\}$

So, in general, $<a,b> \neq < b,a >$.

If we have two sets A and B, we can form ordered pairs from them by taking an element of A as the first member of the pair and an element of B as the second member. The *Cartesian product* of A and B, written $A \times B$, is the set consisting of all such pairs. The predicate notation defines it as

$A \times B =_{def} \{<x,y> \mid x \in A \text{ and } y \in B\}$

Note that according to the definition, if either A or B is $\varnothing$, then $A \times B = \varnothing$.

Here are some examples of Cartesian products:

Let $K = \{a,b,c\}$ *and* $L = \{1,2\}$, then

$K \times L = \{<a,1>,<a,2>,<b,1>,<b,2>,<c,1>,<c,2>\}$
$L \times K = \{<1,a>,<2,a>,<1,b>,<2,b>,<1,c>,<2,c>\}$
$L \times L = \{<1,1>,<1,2>,<2,1>,<2,2>\}$

The definition of ordered pairs can be extended to ordered triples and in general to ordered n-tuples for any natural n. For example, ordered triples are defined as

$<a,b,c> =_{def} <<a,b>,c>$

And for three sets A, B and C the Cartesian product can be defined as
$A \times B \times C =_{def} ((A \times B) \times C)$

In the case when $A = B = C = \dots$ a special notation is used: $A \times A = A^2$, $A \times A \times A = A^3$, etc. And we put $A^1 =_{\text{def}} A$.

1.2. Relations

In natural language *relations* are a kind of links existing between objects. Examples: 'mother of', 'neighbor of', "part of" etc. These are binary relations. Formally we will define relations between elements of sets.

If A and B are any sets and $R \subseteq A \times B$, we call R a *binary relation from A to B* or a *binary relation between A and B*. A relation $R \subseteq A \times A$ is called a relation *in* or *on A*. The set **dom** $R = \{a \mid \langle a,b \rangle \in R$ for some $b\}$ is called the *domain* of the relation R and the set **range** $R = \{b \mid \langle a,b \rangle \in R$ for some $a\}$ is called the *range* of the relation R.

We may visually represent a relation R between two sets A and B by arrows in a diagram displaying the members of both sets. In Figure 2-1, $A = \{a.b\}$, $B = \{c,d,e\}$ and the arrows represent a set-theoretic relation $R = \{\langle a,d \rangle, \langle a,e \rangle, \langle b,c \rangle\}$.

[PtMW, p. 29, Fig.2-1]

Let us consider some operations on relations. The *complement* of a relation $R \subseteq A \times B$ is defined as

$$R' =_{\text{def}} (A \times B) - R$$

The *inverse* of a relation $R \subseteq A \times B$ is defined as the relation $R^{-1} \subseteq B \times A$, $R^{-1} =_{\text{def}} \{\langle b,a \rangle \mid \langle a,b \rangle \in R\}$. Note that $(R^{-1})^{-1} = R$.

For example, for the relation R represented in Figure 2-1
$R' = \{\langle a,c \rangle, \langle b,d \rangle, \langle b,e \rangle\}$ and $R^{-1} = \{\langle d,a \rangle, \langle e,a \rangle, \langle c,b \rangle\}$.

We have focused so far on *binary* relations, i.e., sets of ordered pairs. In a similar way we could define *ternary*, *quaternary* or just *n-place* relations consisting respectively of ordered triples, quadruples or *n*-tuples. A unary relation R on a set A is just a subset of the set A.

1.3. Functions

A function is generally represented in set-theoretic terms as a special kind of relation. A relation F from A to B is called a function from A to B if and only if it meets both of the following conditions:

1. Each element in the domain of F is paired with just one element in the range, i.e., from $\langle a,b\rangle \in F$ and $\langle a,c\rangle \in F$ follows that $b = c$.

2. The domain of F is equal to A, **dom**$F = A$.

For example, consider the sets $A = \{a.b\}$ and $B = \{1,2,3\}$. The following relations from A to B are functions from A to B:

$P = \{\langle a,1\rangle,\langle b,1\rangle\}$
$Q = \{\langle a,2\rangle,\langle b,3\rangle\}$
The following relations from A to B are not functions from A to B:

$S = \{\langle a,1\rangle\}$
$T = \{\langle a,2\rangle,\langle b,1\rangle,\langle b,3\rangle\}$

S does not satisfy the condition 2, and T fails to meet condition 1. S is a function on the smaller domain $\{a\}$; T is not a function at all.

Much of the terminology used in talking about functions is the same as that for relations. We say that a function with domain A and range a subset of B is a function *from A to B*, while one in $A \times A$ is said to be a function *in* or *on A*. The notation 'F: $A \to B$' is used for 'F is a function from A to B'. Elements of the domain of a function are called *arguments* and their correspondents in the range, *values*. If $\langle a,b\rangle \in F$, the familiar notation $F(a) = b$ is used. 'Map', 'mapping' are commonly used synonyms for 'function'. A function F: $A^n \to A$ is also called an n-ary *operation* in A.

Functions as processes. Sometimes functions are considered in a different way, as processes, something like devices or boxes with inputs and outputs. We put the argument in the input and get the value of the function in output. In this case the set of ordered pairs in our definition is called the *graph* of the function.

Sometimes *partial* functions are considered. In this case the condition 2 in our definition can fail.

Some terminology. Functions from A to B in the general case are said to be *into B.* If the range of the function equals B, then the function is *onto B* (or *surjection*). A function F: $A \to B$ is called *one-to-one* function (or *injection*) just in case no member of B is assigned to more than one member of A (so if $a \neq b$, then $F(a) \neq F(b)$). A function which is both one-to-one and onto is called a *one-to-one correspondence* (or *bijection*). It is easy to see that if a function F is one-to-one correspondence, then the relation F^{-1} is a function and one-to-one correspondence.

In Figure 2-2 three functions are indicated by the same sort of diagrams we introduced previously for relations. It is easy to see that functions F and G are onto but H is not.

[PtMW, p. 32, Fig.2-2]

One useful class of functions are *characteristic functions* of sets. The *characteristic function of a set S,* considered as a subset of some larger domain D, is defined as follows:

$$F_S : D \rightarrow \{0,1\} : \; F_S(x) = 1 \text{ iff } x \in S$$
$$F_S(x) = 0 \text{ otherwise}$$

There is a one-to-one correspondence between sets and their characteristic functions. In semantics, where it is common to follow Frege in viewing much of semantic composition as carried out by function-argument application, it is often convenient to work with the characteristic functions of sets rather than with sets directly. Characteristic functions are used in many other applications as well.

1.4. Compositions

Given two functions $F: A \rightarrow B$ and $G: B \rightarrow C$, we may form a new function from A to C, called the *composition* of F and G, written $G \circ F$. Function composition is defined as

$$G \circ F =_{\text{def}} \{\langle x,z \rangle \mid \text{for some } y, \langle x,y \rangle \in F \text{ and } \langle y,z \rangle \in G\}$$

Figure 2-3 shows two functions F and G and their composition.

[PtMW, p. 33, Fig.2-3]

The function $F: A \rightarrow A$ such that $F = \{\langle x,x \rangle \mid x \in A\}$ is called the *identity function on A*, written id_A (or 1_A). Given a function $F: A \rightarrow B$ that is a one-to-one correspondence, we have the following equations: $F^{-1} \circ F = id_A$, $F \circ F^{-1} = id_B$.

The definition of composition need not be restricted to functions but can be applied to relations in general. Given relations $R \subseteq A \times B$ and $S \subseteq B \times C$ the composite of R and S, written $S \circ R =_{\text{def}} \{\langle x,z \rangle \mid \text{for some } y, \langle x,y \rangle \in R \text{ and } \langle y,z \rangle \in S\}$

Homework 2.

Chapter 2, pp 36. ## 1(a)(i, iv, vi), (b)(ii, iii, v), (c); 2; 3; 4.