Lecture 1. Basic Concepts of Set Theory.

Reading: Chapter 1 of Partee, ter Meulen, and Wall (PtMW).

0. Goals

We presuppose that you are probably familiar with most of the notions and constructions we will discuss in this lecture. So our main goals here are to remind you of the main notions and definitions, prepare reference material for the next lectures and discuss the use of these notions in linguistics.

1. Basic Concepts of Set Theory.

1.1. Sets and elements

Set theory is a basis of modern mathematics, and notions of set theory are used in all formal descriptions. The notion of set is taken as “undefined”, “primitive”, or “basic”, so we don’t try to *define* what a set is, but we can give an informal description, describe important properties of sets, and give examples. All other notions of mathematics can be built up based on the notion of set.

Similar (but informal) words: collection, group, aggregate.
Description and terminology: a *set* is a collection of objects (entities) which are called the *members* or *elements* of that set. If we have a set we say that some objects *belong* (or *do not belong*) to this set, *are* (or *are not*) *in* the set. We say also that sets *consist* of their elements.

Examples: the set of students in this room; the English alphabet may be viewed as the set of letters of the English language; the set of even numbers; etc.

So sets can consist of elements of various natures: people, physical objects, numbers, signs, other sets, etc. (We will use the words *object* or *entity* in a very broad way to include all these different kinds of things.)

The membership criteria for a set must in principle be well-defined, and not vague. If we have a set and an object, it is possible that we do not know whether this object belongs to the set or not, because of our lack of information or knowledge. (E.g. “The set of people in this room over the age of 28”, if we don’t know everyone’s age.) But the answer should exist, at any rate in principle. It could be unknown, but it should not be vague. If the answer is vague for some putative set-description, we can not consider that a real description of a set. Another thing: If we have a set, then for every two elements of it, x and y, it should not be vague whether $x = y$, or they are different.

Sometimes we simply assume for the sake of examples that a description is not vague when perhaps for other purposes it would be vague – e.g., the set of all red objects.

Possible examples of non-sets: "the set of all words of a given language (Russian, English, etc)" [We will call it a non-set if we don't believe that the class is really well-defined]; the set of faces I saw in the street [similarly], the set of all sets. [This last one deserves separate discussion; see discussion of Russell's paradox on pp. 7-8] Because we often just implicitly "stipulate" that our predicates are unambiguously clearcut and not vague, it is hard to give clear examples of descriptions that definitely do not pick out sets.

Sets can be *finite* or *infinite*.
There is exactly one set, the *empty set*, which has no members at all.
A set with only one member is called a *singleton* or a *singleton set*.

Notation: *A, B, C, ...* for sets; *a, b, c, ...* or *x, y, z, ...* for members.
$b \in A$ if b belongs to A ($B \in A$ if both A and B are sets and B is a member of A)
and $c \notin A$, if c doesn't belong to A.
$\varnothing$ is used for the empty set.

1.2. Specification of sets

There are three main ways to specify a set:
(1) by listing all its members (*list notation*);
(2) by stating a property of its elements (*predicate notation*);
(3) by defining a set of rules which generates (defines) its members (*recursive rules*).

List notation. The first way of course is suitable only for finite sets. In this case we list names of elements of a set, separate them by commas and enclose them in braces:
Examples: {1, 12, 45}, {George Washington, Bill Clinton}, {a,b,d,m}.

Note that we do not care about the order of elements of the list, and elements can be listed several times. {1, 12, 45}, {12, 1, 45,1} and {45,12, 45,1} are different representations of the same set (see below the notion of identity of sets).

Predicate notation. Example:
$\{x \mid x$ is an even number greater than 3.$\}$
Read: "the set of all x such that x is an even number greater than 3"
So the second part of this notation is a property the members of the set share (a condition or a predicate which holds for members of this set).

Other examples:
$\{ x \mid x$ is a letter of Russian alphabet$\}$
$\{y \mid y$ is a student of UMass and y is older than 25 $\}$

General form:
$\{x \mid P(x)\}$, where P is some predicate (condition, property).

The language to describe these predicates is not usually fixed in a strict way. But it is known that unrestricted language can result in *paradoxes*. Example: $\{x \mid x \notin x\}$.
(Russell's paradox: see PtMW pp 7-8. Type theory is one way to guarantee avoidance of this kind of paradox: require sets always to be of higher type than their members.)

Recursive rules. (Always safe.) Example – the set E of even numbers greater than 3:
a) $4 \in E$
b) if $x \in E$, then $x + 2 \in E$
c) nothing else belongs to E.

The first rule is the basis of recursion, the second one generates new elements from the elements defined before and the third rule restricts the defined set to the elements generated by rules a and b. (The third rule should always be there; sometimes in practice it is left implicit.)

1.3. Identity and cardinality

Two sets are identical if and only if they have exactly the same members. So $A = B$ iff for every x, $x \in A \leftrightarrow x \in B$.

For example, $\{0,2,4\} = \{x \mid x$ is an even natural number less than $5\}$

From the definition of identity it follows that there exists only one empty set; its identity is fully determined by its absence of members. Note that empty list notation {} is not usually used for the empty set, we have a special symbol $\varnothing$ for it.

The number of elements in a set A is called the *cardinality* of A, written $|A|$. The cardinality of a finite set is a natural number. Infinite sets also have cardinalities but they are not natural numbers. We will not discuss cardinalities of infinite sets here.

1.4. Subsets

A set A is a *subset of a set B* iff every element of A is also an element of B. Such a relation between sets is denoted by $A \subseteq B$. If $A \subseteq B$ and $A \neq B$ we call A a *proper subset* of B and write $A \subset B$. (Caution: sometimes $\subset$ is used the way we are using $\subseteq$.)

Both signs can be negated using the slash / through the sign.

Examples:
$\{a,b\} \subseteq \{d,a,b,e\}$ and $\{a,b\} \subset \{d,a,b,e\}$, $\{a,b\} \subseteq \{a,b\}$, but $\{a,b\} \not\subset \{a,b\}$.

Note that the empty set is a subset of every set. $\varnothing \subseteq A$ for every set A. Why?

1.5. Power sets

The set of all subsets of a set A is called the *power set* of A and denoted as $\wp(A)$ or sometimes as 2^A.

For example, if $A = \{a,b\}$, $\wp(A) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

From the example above: $a \in A$; $\{a\} \subseteq A$; $\{a\} \in \wp(A)$
$\varnothing \subseteq A$; $\varnothing \notin A$; $\varnothing \subseteq \wp(A)$; $\varnothing \in \wp(A)$

1.6 and 1.7. Operations on sets: union, intersection, difference, complement

We define several operations on sets. Let *A* and *B* be arbitrary sets.

The *union* of *A* and *B*, written $A \cup B$, is the set whose elements are just the elements of *A* or *B* or of both. In the predicate notation the definition is

$$A \cup B =_{\text{def}} \{ x \mid x \in A \text{ or } x \in B\}$$

Examples. Let $K = \{a,b\}$, $L = \{c,d\}$ and $M = \{b,d\}$, then

$K \cup L = \{a,b,c,d\}$
$K \cup M = \{a,b,d\}$
$L \cup M = \{b,c,d\}$
$(K \cup L) \cup M = K \cup (L \cup M) = \{a,b,c,d\}$
$K \cup K = K$
$K \cup \varnothing = \varnothing \cup K = K = \{a,b\}$.

There is a nice method for visually representing sets and set-theoretic operations, called *Venn diagrams*. Each set is drawn as a circle and its members represented by points within it. The diagrams for two arbitrarily chosen sets are represented as partially intersecting – the most general case – as in Figure 1–1 below. The region designated ‘1’ contains elements which are members of *A* but not of *B*; region 2, those members in *B* but not in *A*; and region 3, members of both *B* and *A*. Points in region 4 outside the diagram represent elements in neither set.

The Venn diagram for the union of *A* and *B* is shown in Figure 1–2. The results of operations in this and other diagrams are shown by shading areas.

[PtMW, p.13, Fig.1-1, 1-2]

The *intersection* of A and B, written $A \cap B$, is the set whose elements are just the elements of both A and B. In the predicate notation the definition is

$A \cap B =_{\text{def}} \{ x \mid x \in A \text{ and } x \in B\}$

Examples:

$K \cap L = \varnothing$
$K \cap M = \{b\}$
$L \cap M = \{d\}$
$(K \cap L) \cap M = K \cap (L \cap M) = \varnothing$
$K \cap K = K$
$K \cap \varnothing = \varnothing \cap K = \varnothing$.

The general case of intersection of arbitrary sets A and B is represented by the Venn diagram of Figure 1–3. The intersection of three arbitrary sets A,B and C is shown in the Venn diagram of Figure 1–4.

[PtMW, p. 14, Fig 1-3, p.15, Fig. 1-4]

Another binary operation on arbitrary sets is the *difference*, written $A - B$, which 'subtracts' from A all elements which are in B. The predicate notation defines this operation as follows:

$A - B =_{\text{def}} \{ x \mid x \in A \text{ and } x \notin B\}$

Examples:

$K - L = \{a,b\}$
$K - M = \{a\}$
$L - M = \{c\}$
$K - K = \varnothing$
$K - \varnothing = K$
$\varnothing - K = \varnothing$.

The Venn diagram for the set-theoretic difference is shown in Figure 1–5.

$A - B$ is also called the *relative complement* of B relative to A. This operation is to be distinguished from the *complement* of a set A, written A', which is the set consisting of everything not in A. In predicate notation

$A' =_{\text{def}} \{ x \mid x \notin A\}$

It is natural to ask, where do these objects come from which do not belong to A? In this case it is presupposed that there exists a *universe of discourse* and all other sets are subsets of this set. The universe of discourse is conventionally denoted by the symbol U. Then we have

$A' =_{\text{def}} U - A$

The Venn diagram with a shaded section for the complement of A is shown in Figure1–6.

[PtMW, p.16, Fig. 1-5, p.16, Fig. 1-6]

1.8. Set-theoretic equalities

There are a number of general laws about sets which follow from the definitions of set-theoretic operations, subsets, etc. A useful selection of these is shown below. They are grouped under their traditional names. These equations below hold for any sets X, Y, Z:

1. *Idempotent Laws*
   (a) $X \square X = X$ (b) $X \square X = X$

2. *Commutative Laws*
   (a) $X \square Y = Y \square X$ (b) $X \square Y = Y \square X$

3. *Associative Laws*
   (a) $(X \square Y) \square Z = X \square (Y \square Z)$ (b) $(X \square Y) \square Z = X \square (Y \square Z)$

4. *Distributive Laws*
   (a) $X \square (Y \square Z) = (X \square Y) \square (X \square Z)$ (b) $X \square (Y \square Z) = (X \square Y) \square (X \square Z)$

5. *Identity Laws*
   (a) $X \square \varnothing = X$ (c) $X \square \varnothing = \varnothing$
   (b) $X \square U = U$ (d) $X \square U = X$

6. *Complement Laws*
   (a) $X \square X' = U$ (c) $X \square X' = \varnothing$
   (b) $(X')' = X$ (d) $X - Y = X \square Y'$

7. *DeMorgan's Laws*
   (a) $(X \square Y)' = X' \square Y'$ (b) $(X \square Y)' = X' \square Y'$

8. Consistency Principle
   (a) $X \square Y$ iff $X \square Y = Y$ (b) $X \square Y$ iff $X \square Y = X$

We will see later that operations on subsets of a set form a *Boolean algebra*.

Homework 1.

1) Exercises from PtMW:

Chapter 1, pp. 23-26. ##2, 5, 6(a,g,m), 7(a,f,l), 8(b,c), 9b, 11a,c,d.