Supplement to answers to Homework 16: p.235: 12,13.

(The book has answers to problem 13 and problem 12c. The website answer page, which is HERE, has answers to problems 12a, b.)

More about problem 13.

13b. Axiom 2 gives you two points, call them p and q. Let’s consider p as an arbitrary point and show that it must be in at least two lines. By Axiom 3, we know there’s a line \( \{p,q\} \). By Axiom 4 there’s a point not in \( \{p,q\} \) – call it r.

Now use Axiom 3 again: since p and r are distinct points, there must be a line \( \{p,r\} \).

That shows that p is in at least two lines. QED. (Don’t bring Axiom 5 into the picture -- it’s irrelevant.)

13c. Why the empty set can’t be a “line” requires more reasoning. It’s not true that axioms 1 and 2 say that all lines must contain at least two points. Axiom 1 just says that a line is a set of points, but it doesn’t say how many – as far as Axiom 1 is concerned, there could be a line with zero points. And axiom 2 says there exist at least two distinct points, but doesn’t say anything about how many points are in a line. So try the following reasoning instead (this is not necessarily the only way to prove it; but this is one way that works):

Suppose \( \emptyset \) is indeed a line. Now consider an arbitrary point p not in \( \emptyset \). (Any point will be ‘not in \( \emptyset \)’, of course, since \( \emptyset \) is empty.) By the results of problem 13b, p is in at least two distinct lines – call them \( \{p,q\} \) and \( \{p,r\} \). Now since \( \emptyset \) is empty, \( \{p,q\} \) and \( \{p,r\} \) are both disjoint from \( \emptyset \). But that violates the parallel postulate, Axiom 5: for any line and any point p not in that line, there should be exactly ONE line containing p and disjoint from the given line.

13e. To show that it’s inconsistent to add Axiom 6, “Every line contains exactly one point”, it’s not enough to use Axiom 1. Axiom 1 says “Every line is a set of points”. But it doesn’t say how many – Axiom 1 is consistent with Axiom 6 and even with the possibility that the empty set is a line. The way to show that Axiom 6 is inconsistent with Axioms 1-5 is to use Axioms 2 and 3, which together require that there be at least one line containing two points.