Lecture 1. Basic Concepts of Set Theory, Functions and Relations

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0. Preliminaries

- Aims of the course. Emphasis on "math appreciation", aiming to be USEFUL, interesting, may or may not be fun (hope so.). Everything we cover should be doable if you work conscientiously. Creativity is important for really BEING a mathematician, and we'll include some optional problems that ask for creativity, but we'll try to avoid any questions that demand mathematical creativity.
- Course description: Introduction to some basic mathematical concepts and techniques central to linguistic theory. Set theory, logic and formal systems, modern algebra, automata theory, and model theory. Applications to syntax, phonology, semantics. No prior mathematics assumed. Not open to math majors. Prerequisite: LING 201 or 401 or consent of instructor. [See Syllabus for more.]
- Textbook: available from instructor, cost $26.00 (Partee's cost direct from publisher with author's discount.) Payment due by next Wednesday, September 14. (Cash or check to Barbara H. Partee.) Sign for book on attendance sheet. Florian or I will give you a receipt when I receive payment.
- Student questionnaire, to be collected at end of class or next time.
- Introductions. Florian Schwarz will give some guest lectures when I'm away.

Reading: Chapter 1 (1.1 – 1.7) of Partee-ter Meulen-Wall (MMiL), pp.3—17. Also "Preliminaries" from Partee 1979, *Fundamentals of Mathematics for Linguistics* (xeroxed).

1. Basic Concepts of Set Theory.

1.1. Sets and elements

Set theory is a basis of modern mathematics, and notions of set theory are used in all formal descriptions. The notion of set is taken as "undefined", "primitive", or "basic", so we don't try to *define* what a set is, but we can give an informal description, describe important properties of sets, and give examples. All other notions of mathematics can be built up based on the notion of set.

Similar (but informal) words: collection, group, aggregate.
Description: a *set* is a collection of objects which are called the *members* or *elements* of that set. If we have a set we say that some objects *belong* (or *do not belong*) to this set, *are* (or *are not*) *in* the set. We say also that sets *consist* of their elements.

Examples: the set of students in this room; the English alphabet may be viewed as the set of letters of the English language; the set of natural numbers[1]; etc.

So sets can consist of elements of various natures: people, physical objects, numbers, signs, other sets, etc. (We will use the words *object* or *entity* in a very broad way to include all these different kinds of things.)

A set is an ABSTRACT object; its members do not have to be physically collected together for them to constitute a set.

The membership criteria for a set must in principle be well-defined, and not vague. If we have a set and an object, it is possible that we do not know whether this object belongs to the set or not, because of our lack of information or knowledge. (E.g. "The set of students in this room over the age of 21": a well-defined set but we may not know who is in it.) But the answer should exist, at any rate in principle. It could be unknown, but it should not be vague. If the answer is vague for some collection, we cannot consider that collection as a set. Another thing: If we have a set, then for any two elements of it, x and y, it should not be vague whether $x = y$, or they are different. (If they are identical, then they are not actually "two" elements of it; the issue really arises when we have two *descriptions* of elements, and we want to know whether those descriptions describe the same element, or two different elements.)

For example: is the letter q the same thing as the letter Q? Well, it depends on what set we are considering. If we take the set of the 26 letters of the English alphabet, then q and Q are the same element. If we take the set of 52 upper-case and lower-case letters of the English alphabet, then q and Q are two distinct elements. Either is possible, but we have to make it clear what set we are talking about, so that we know whether or not q = Q.

Sometimes we simply assume for the sake of examples that a description is not vague when perhaps for other purposes it would be vague – e.g., the set of all red objects.

Sets can be *finite* or *infinite*.

There is exactly one set, the *empty set*, or *null set,* which has no members at all.

A set with only one member is called a *singleton* or a *singleton set*. ("*Singleton of a"*)

Notation: $A, B, C, \ldots$ for sets; $a, b, c, \ldots$ or $x, y, z, \ldots$ for members.
$b \in A$ if b belongs to A ($B \in A$ if both A and B are sets and B is a member of A)
and $c \notin A$, if c doesn't belong to A.

$\varnothing$ is used for the empty set.

1.2. Specification of sets

There are three main ways to specify a set:
(1) by listing all its members (*list notation*);
(2) by stating a property of its elements (*predicate notation*);
(3) by defining a set of rules which generates (defines) its members (*recursive rules*).

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[1] Natural numbers: 0,1,2,3,4,5,... . No notion of positive or negative. The numbers used for "counting". Integers: positive, negative, and 0. See xeroxed section "Preliminaries" from Partee 1979.

List notation. The first way is suitable only for finite sets. In this case we list names of elements of a set, separate them by commas and enclose them in braces:
Examples: {1, 12, 45}, {George Washington, Bill Clinton}, {a,b,d,m}.
"Three-dot abbreviation": {1,2, ..., 100}. (See xeroxed "preliminaries", pp xxii-xxiii)

{1,2,3,4,…} – this is not a real list notation, it is not a finite list, but it's common practice as long as the continuation is clear.
Note that we do not care about the order of elements of the list, and elements can be listed several times. {1, 12, 45}, {12, 1, 45,1} and {45,12, 45,1} are different representations of the same set (see below the notion of identity of sets).

Predicate notation. Example:
$\{x \mid x$ is a natural number and $x < 8\}$
Reading: "the set of all x such that x is a natural number and is less than 8"
So the second part of this notation is a property the members of the set share (a condition or a predicate which holds for members of this set).

Other examples:
$\{ x \mid x$ is a letter of Russian alphabet$\}$
$\{y \mid y$ is a student of UMass and y is older than $25\}$

General form:
$\{ x \mid \mathrm{P}(x)\}$, where P is some predicate (condition, property).

The language to describe these predicates is not usually fixed in a strict way. But it is known that unrestricted language can result in *paradoxes*. Example: $\{ x \mid x \notin x\}$. ("Russell's paradox") -- see the historical notes about it on pp 7-8. The moral: not everything that looks on the surface like a predicate can actually be considered to be a good defining condition for a set. Solutions – type theory, other solutions; we won't go into them. (If you're interested, see Chapter 8, Sec 2.)

Recursive rules. (Always safe.) Example – the set E of even numbers greater than 3:
a) $4 \in E$
b) if $x \in E$, then $x + 2 \in E$
c) nothing else belongs to E.

The first rule is the basis of recursion, the second one generates new elements from the elements defined before and the third rule restricts the defined set to the elements generated by rules a and b. (The third rule should always be there; sometimes in practice it is left implicit. It's best when you're a beginner to make it explicit.)

1.3. Identity and cardinality

Two sets are identical *if and only if*[2] they have exactly the same members. So $A = B$ iff for every x, $x \in A \Leftrightarrow x \in B$.
For example, $\{0,2,4\} = \{x \mid x$ is an even natural number less than $5\}$

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[2] Be careful about "if and only if"; its abbreviation is *iff.* See Preliminaries, p. xxiii.

From the definition of identity follows that there exists only one empty set; its identity is fully determined by its absence of members. Note that empty list notation {} is not usually used for the empty set, we have a special symbol $\varnothing$ for it.
The number of elements in a set A is called the *cardinality* of A, written $|A|$. The cardinality of a finite set is a natural number. Infinite sets also have cardinalities but they are not natural numbers. We will discuss cardinalities of infinite sets a little later (Chapter 4).

1.4. Subsets

A set A is a *subset of a set B* iff every element of A is also an element of B. Such a relation between sets is denoted by $A \subseteq B$. If $A \subseteq B$ and $A \neq B$ we call A a proper subset of B and write $A \subset B$. (Caution: sometimes $\subset$ is used the way we are using $\subseteq$.)
Both signs can be negated using the slash / through the sign.
Examples:
$\{a,b\} \subseteq \{d,a,b,e\}$ and $\{a,b\} \subset \{d,a,b,e\}$, $\{a,b\} \subseteq \{a,b\}$, but $\{a,b\} \not\subset \{a,b\}$.

Note that the empty set is a subset of every set. $\varnothing \subseteq A$ for every set A. Why?

Be careful about the difference between "member of" and "subset of"; the homework gives lots of chance for practice.

1.5. Power sets

The set of all subsets of a set A is called the *power set* of A and denoted as $\wp(A)$ or sometimes as 2^A.
For example, if $A = \{a,b\}$, $\wp(A) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$.

From the example above: $a \in A$; $\{a\} \subseteq A$; $\{a\} \in \wp(A)$
$\varnothing \subseteq A$; $\varnothing \notin A$; $\varnothing \in \wp(A)$; $\varnothing \subseteq \wp(A)$

Be careful with exercise 4 in Homework 1! Pay close attention to the definitions and it should come out all right. But if you *don't* pay close attention to the definitions, it's easy to make mistakes. Make sure you understand these examples before you try it. (But do try; and if you don't get something right the first time, we give you a chance to redo it.)

1.6. Operations on sets: union, intersection.

We define several operations on sets. Let A and B be arbitrary sets.
The *union* of A and B, written $A \cup B$, is the set whose elements are just the elements of A or B or of both. In the predicate notation the definition is

$$A \cup B =_{def} \{ x \mid x \in A \text{ or } x \in B\}$$

Examples. Let $K = \{a,b\}$, $L = \{c,d\}$ and $M = \{b,d\}$, then

$$K \cup L = \{a,b,c,d\}$$
$$K \cup M = \{a,b,d\}$$
$$L \cup M = \{b,c,d\}$$

$(K \cup L) \cup M = K \cup (L \cup M) = \{a,b,c,d\}$
$K \cup K = K$
$K \cup \varnothing = \varnothing \cup K = K = \{a,b\}$.

There is a nice method for visually representing sets and set-theoretic operations, called *Venn diagrams*. Each set is drawn as a circle and its members represented by points within it. The diagrams for two arbitrarily chosen sets are represented as partially intersecting – the most general case – as in Figure 1–1 p.13. The region designated '1' contains elements which are members of *A* but not of *B*; region 2, those members in *B* but not in *A*; and region 3, members of both *B* and *A*. Points in region 4 outside the diagram represent elements in neither set.

The Venn diagram for the union of *A* and *B* is shown in Figure 1–2, p.13 The results of operations in this and other diagrams are shown by shading areas.

The *intersection* of *A* and *B*, written $A \cap B$, is the set whose elements are just the elements of both *A* and *B*. In the predicate notation the definition is

$A \cap B =_{\text{def}} \{ x \mid x \in A \text{ and } x \in B\}$

Examples:

$K \cap L = \varnothing$
$K \cap M = \{b\}$
$L \cap M = \{d\}$
$(K \cap L) \cap M = K \cap (L \cap M) = \varnothing$
$K \cap K = K$
$K \cap \varnothing = \varnothing \cap K = \varnothing$.

The general case of intersection of arbitrary sets A and B is represented by the Venn diagram of Figure 1–3, p.14. The intersection of three arbitrary sets A,B and C is shown in the Venn diagram of Figure 1–4, p.15

1.7 More operations on sets: difference, complement

Another binary operation on arbitrary sets is the *difference "A minus B"*, written $A - B$, which 'subtracts' from A all elements which are in B. [Also called *relative complement: the complement of B relative to A.*] The predicate notation defines this operation as follows:

$A - B =_{\text{def}} \{ x \mid x \in A \text{ and } x \notin B\}$

Examples: (using the previous *K, L, M*)

$K - L = \{a,b\}$
$K - M = \{a\}$
$L - M = \{c\}$
$K - K = \varnothing$
$K - \varnothing = K$
$\varnothing - K = \varnothing$.

The Venn diagram for the set-theoretic difference is shown in Figure 1–5, p.16.

$A - B$ is also called the *relative complement* of B relative to A. This operation is to be distinguished from the *complement* of a set A, written A', which is the set consisting of everything not in A. In predicate notation

$A' =_{def} \{ x \mid x \notin A\}$

It is natural to ask, where do these objects come from which do not belong to A? In this case it is presupposed that there exists a *universe of discourse* and all other sets are subsets of this set. The universe of discourse is conventionally denoted by the symbol U. Then we have

$A' =_{def} U - A$

The Venn diagram with a shaded section for the complement of A is shown in Figure1–6, p.16

Homework 1 notes.

1. Be careful with problem 4!

2. Strategy for problem 5e, p. 25. In general, when any problem is confusing or difficult, try to break it into smaller steps and SHOW the STEPS. In fact more generally, if it CAN be broken into steps, it usually SHOULD be. So to compute Pow(Pow({a,b})), FIRST compute Pow({a,b}). (It should have 4 members.) Then figure out some systematic way to write down all the subsets of Pow({a,b}), which will be the members of Pow(Pow({a,b})). I think it works best to group them by numbers of members: all the 0-member subsets, all the 1-member subsets, all the 2-member subsets, then 3, then 4. (Hint: there should be 16 in all, including 6 2-member subsets.)