Lecture 6: Chapter 3, part 2: Equivalence Relations, Partitions, Orderings

3.4. Equivalence relations and partitions

Using properties of relations we can consider some important classes of relations, of which *equivalence relations* are probably the most important.

3.3.1. Equivalence relation.

An equivalence relation is a relation which is reflexive, symmetric and transitive. For every equivalence relation there is a natural way to divide the set on which it is defined into mutually exclusive (disjoint) subsets which are called *equivalence classes*. We write $[[x]]$ for the set of all y such that $<x,y> \in R$. Thus, when R is an equivalence relation, $[[x]]$ is the equivalence class which contains x.

[Optional note: not in text, will not be in homeworks or quizzes: The set $A/R =_{\text{def}} \{[[x]] \mid x \in A\}$ is called a *quotient set* of the set A by the equivalence R. A/R is a subset of $\wp(A)$. For every equivalence relation R, the function $\mathbf{nat}(R)$: $A \rightarrow A/R$ mapping every element $x \in A$ onto $[[x]]$ is called a *natural mapping* of A onto A/R.]

Examples. The relations "has the same hair color as" or "is the same age as" in the set of people are equivalence relations. The equivalence classes under the relation "has the same hair color as" are the set of blond people, the set of red-haired people, etc.

Partitions. Given a non-empty set A, a *partition* of A is a collection of non-empty subsets of A such that (1) for any two distinct subsets X and Y, $X \cap Y = \varnothing$ and (2) the union of all the subsets in collection equals A. The subsets of A that are members of a partition of A are called *cells* of that partition.

There is a close correspondence between partitions and equivalence relations. Given a partition of set A, the relation $R = \{<x,y> \mid x$ and y are in the same cell of the partition of $A\}$ is an equivalence relation in A. Conversely, given an equivalence relation R in A, there exists a partition of A in which x and y are in the same cell iff $<x,y> \in R$..

Example. Consider the following set *Countries* = {Germany, England, India, China, USA, Canada}. One possible partition on *Countries* is one that classifies them according to the continents they are in. We get the following partition P_C: {{Germany, England}, {India, China}, {USA, Canada}}. (Note that a partition is always a set of sets.)

What is the corresponding equivalence relation?

3.5. Orderings.

An *order* is a binary relation which is transitive and in addition either (i) reflexive and antisymmetric or else (ii) irreflexive and asymmetric. The former are *weak* orders; the latter are *strict* (or *strong*). [This is where the property "antisymmetric" becomes important.]

Examples: (There are more on pp. 48-49 in the book.)

Let $A = \{a,b,c,d\}$. Here is a *weak order* on A.

(3-21) $R_1 = \{ \langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle, \langle a,a\rangle, \langle b,b\rangle, \langle c,c\rangle, \langle d,d\rangle\}$

We'll draw the diagram for R_1, and check that it is reflexive, antisymmetric, and transitive.

There is a *strong* (or *strict*) *order* that is a very close counterpart to R_1, which we'll call S_1:

(3-22) $S_1 = \{ \langle a,b\rangle, \langle a,c\rangle, \langle a,d\rangle, \langle b,c\rangle\}$

What did we do to turn R_1 into S_1? You can always turn a weak order into a strong order that way, and vice versa.

Other good examples:

- "subset of" (weak order) and "proper subset of" (strong order), on some set of sets.
- "greater than" (strong order) on the set of positive integers. What's the corresponding weak order?

Some terminology: if R is an order, either weak or strict, and $\langle x,y\rangle \in R$, we say that *x precedes y*, *x is a predecessor of y*, *y succeeds* (or *follows*) *x*, or *y is a successor of x*. If x precedes y and $x \neq y$, then we say that *x immediately precedes y* if and only if there is no element z distinct from both x and y such that x precedes z and z precedes y. In other words, there is no other element between x and y in the order. **Immediate predecessor diagrams.** See pp 49-50; we'll illustrate on the board.

There is also a useful set of terms for elements which stand at the extremes of an order. Given an order R in a set A,

1) an element x in A is *minimal* iff there is no other element in A which precedes x
2) an element x in A is *least* iff x precedes every other element in A
3) an element x in A is *maximal* iff there is no other element in A which follows x
4) an element x in A is *greatest* iff x follows every other element in A.

Note that greatest (least) element is maximal (minimal) but the opposite is not always the case.

If an order, strict or weak, is also connected, then it is said to be a *total* or *linear* order. Often orders in general are called ***partial orders*** or *partially ordered sets.* This terminology has the unfortunate consequence that a partial order may be total; we then often say "only a partial order" when we mean a partial order that is NOT total.

A relation R in A is *dense* if for every $\langle x,y\rangle \in R$ such that $x \neq y$, there exists a member $z \in A$ distinct from both x and y such that $\langle x,z\rangle \in R$ and $\langle z,y\rangle \in R$. The relation 'is greater than' is not dense on the natural numbers but is dense on the real numbers.

Examples. Relations $\geq$ and $=$ on the set $\mathbf{N}$ of natural numbers are examples of weak order, as are relations $\supseteq$ and $=$ on subsets of any set. The relations $>$ and $\supset$ are examples of strict orders on the corresponding sets. The relations $\geq$ and $>$ are linear orders. The relations $\supseteq$ and $\supset$ are partial orders that are not linear orders.