Lecture 4: Chapter 3, part 1. Properties of Relations

Homework notes: See end of handout.

Reading: pp 39-45 (Sections 3.1 – 3.3)

3.1. Reflexivity, symmetry, transitivity, and connectedness

We consider here certain properties of binary relations. All these properties apply only to relations *in* (*on*) a (single) set, i.e., in A □ A for example.

Reflexivity. Given a set A and a relation R in A, R is *reflexive* iff all the ordered pairs of the form <*x,x*> are in R for every x in A. A relation which fails to be reflexive is called *nonreflexive*, but if it contains no ordered pair <*x,x*>, it said to be *irreflexive*.

Another way to state the definitions above is to use the identity relation id_A. Relation R is reflexive iff id_A □ R, it is nonreflexive iff id_A □ R, and it is irreflexive iff id_A □ R = ∅.

Note on "strong and weak negative properties". "Irreflexive" is a "strong negative": NO pairs <*x,x*> are in the relation. "Nonreflexive" is a weaker negative property, equivalent to "not reflexive". It requires only that there is at least one pair <x,x> that is not in the relation. For all three of the properties *reflexive, symmetric, transitive*, there will be two such negations. The *non-* form always simply means 'not', and the stronger negation is always expressed with a Latin prefix: *irreflexive, asymmetric, intransitive.*

As long as the set A is not empty, any irreflexive relation will also be nonreflexive.

Examples. Relations "= " and "≥" on the set **N** of natural numbers and relations "⊇" and "□" between sets are reflexive. Relations "≠" and "<" on **N** are nonreflexive and irreflexive.

Remember that we always consider relations in some set. And a relation (considered as a set of ordered pairs) can have different properties in different sets. For example, the relation R = {<1,1>, <2,2>} is reflexive in the set A_1 = {1,2} and nonreflexive in A_2 = {1,2,3} since it lacks the pair <3,3> (and of course it nonreflexive in **N**).

More examples: Classify these relations on the set of UMass students and their parents: *Has the same birthday as; Pays the tuition bills of; Has at least as many friends as; Lives at least 10 miles away from.*

Symmetry. Given a set A and a relation R in A, R is *symmetric* iff for every ordered pair <*x,y*>, if <*x,y*> is in *R,* then the pair <*y,x*> is also in R.

If for some <*x,y*> in R, the pair <*y,x*> is not in R, then R is *nonsymmetric*.

If it is never the case that for any <*x,y*> in R, the pair <*y,x*> is in R then the relation is called *asymmetric*.

Note that an asymmetric relation must be irreflexive. Why? ... This illustrates an important point about **variables and distinctness**. When we consider “every ordered pair <*x, y*> of elements of *A*”, we have to include cases where the variables *x* and *y* happen to get the same value. Distinct variables can happen to get the same value, and to consider all possibilities, you have to remember to include those.

A relation *R* is *antisymmetric* if whenever both <*x,y*> and <*y,x*> are in *R*, then *x* = *y*. Note that a relation need not be reflexive to be antisymmetric. [Why?]

[The property *antisymmetric* is a special one which will be useful when we define *orderings* in the next lecture. It doesn’t fit the usual pattern of weak/strong negatives.]

Examples. The relation “*brother of*” is nonsymmetric in the set of all people. Why? But it can be symmetric in some set, say, in the set *A* = {John, Peter, Bill}, if John and Bill are brothers. Can you think of a set in which it is asymmetric?

-- Let’s think of more examples together, of relations which are symmetric, asymmetric, nonsymmetric. (Let’s postpone thinking about antisymmetric. One typical example is ≥.)

Transitivity. A relation *R* is *transitive* on set *A* iff for all elements *x,y,z* of *A,* whenever the ordered pairs <*x,y*> and <*y,z*> are in *R*, the pair <*x,z*> is also in *R*. If a relation fails to meet the definition of transitivity, it is *nontransitive*. If for no pairs <*x,y*> and <*y,z*> in *R*, the pair <*x,z*> is in *R*, then the relation is *intransitive*.

Examples. The relations =, > and ≥ are transitive in the set of natural numbers. Is the relation “friend of” in the set of all people transitive?

What about the relation {<2,2>}

Here’s one that’s non-reflexive; can you see why? {<2,2>, <2,3>, <3,2>}

Connectedness. A relation *R* in *A* is *connected* (sometimes called *connex*) iff for every two *distinct* elements *x* and *y* in *A*, <*x,y*> ∈ *R* or <*y,x*> ∈ *R* (or both). (Note that this time we put the word *distinct* into the condition. So this time we <u>don’t</u> look at pairs like <2,2>.

Examples: What are the properties (considering all four sorts) of the following two relations? $\mathbf{R_f}$ = “is the father of” on the set H of all humans

$\mathbf{R_g}$ = “is greater than” on the set $\mathbb{Z}^+$ of positive integers.

3.2. Diagrams of relations

It may be helpful to demonstrate the properties of relations representing them in relational diagrams. The members of the relevant set are represented by labeled points. If *x* is related to *y*, i.e. <*x,y*> ∈ *R*, an arrow connects the corresponding points. For example, Figure 3-1 represents the relation

$R = \{<1,2>,<2,1>,<2,2>,<1,1>,<2,3>,<3,3>\}$

[see PtMW, p. 43, Fig.3-1]

It is apparent from the diagram that the relation is reflexive, since every point bears a loop. The relation is non-symmetric since there is an arrow from 2 to 3 but there is no arrow from 3 to 2. It cannot be called asymmetric (nor antisymmetric), since 1 is related to 2 and 2 is related to 1. It is not transitive since 1 is related to 2 and 2 to 3, but there is no arrow from 1 to 3. Is it intransitive or just non-transitive?

3.3 Properties of inverses and complements: general principles

For example: If we know that R is reflexive on set A, what can we conclude about its inverse R^{-1}?
What can we conclude about its complement R'?

See more such principles in section 3.3 in the book.

Next time: the rest of Chapter 3. Equivalence relations, partitions, and orderings.

Homework notes from HW 2 and in general: (i) Problem 9a in homework 2 was a killer and I apologize – I shouldn't have assigned it. I had discovered that earlier but had forgotten. There is a good solution posted on the course website.

(ii) When a problem asks you to show that something is true for any sets A, B, C or the like, as in questions 9 and 11 on pp. 25-26, don't pick specific sets to show it for. Use the set theoretic equalities to show that it's true in general. Write out a proof step by step and tell at each step what justifies that step. Use the examples in the book on pp. 22-23 as models of what an answer should look like.

(iii) There are occasional errors in the book and even in the answer pages. An errata sheet will be posted on the website sometime Wednesday. Be sure to check it. If you find any other errata or suspected errata, please let us know.