# Higher-Order Metaphysics in Frege and Russell

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## 1   Introduction

Second-order logic was first introduced by Gottlob Frege in his *Begriffsschrift* of 1879, and full higher-order logic was first popularized among analytic philosophers by Whitehead and Russell's *Principia Mathematica*, the first volume of which was published in 1910. Both Russell (who was mostly responsible for the philosophical foundations of his work with Whitehead) and Frege held related philosophical views on the nature of higher-order quantification which have been influential on the generations of analytic philosophers that followed them. Both developed these views in the context of their logicisms, i.e., in arguing that arithmetic is a branch of logic. Both understood higher-order quantification as quantification over a kind of function, but they differed in exactly how such functions were to be understood, and even over whether or not these functions could be considered entities in their own right. They also both had views that developed over time. Frege first understood functions as having to do with the analysis of the judgeable content of assertions, and later took functions simply

to be mappings from objects to other objects. He sharply divided functions from objects, but took the division between them to be one founded "deep in the nature of things" (Frege, 1984b [1891], 158). Early Russell took what he called propositional functions to be proposition-like entities containing variables, but later shied away from reifying variables or even propositional functions. The mature position of *Principia Mathematica* was that propositional functions were simply a device of language used when generalizing over possible expressions of a certain form. Both Frege and Russell encountered difficulties with versions of their views that postulated as many objects as functions, and both were led to views that seem to make their own metaphysical positions impossible to state from within their own strictures. It is worth examining the views of each in more detail.

## 2 Frege on Functions and Higher-Order Quantification

When Frege introduced second-order quantification, he did so in connection with his function/argument analysis of language. In his logical language, every proposition has the following form:

$$\vdash A$$

The vertical line at the far left is called the "judgment stroke", and is used to indicate that what follows is asserted as true. Without the judgment stroke:

$$- A$$

the remainder of the proposition beginning with the horizontal line is thought merely to represent the possible content of a judgment, or "the circumstance that $A$" as he puts it in his early work (Frege, 1972 [1879], §2). In his later work, he bifurcated the notion of content into *sense* and *reference*, and took a term of the form "$- A$" as a *name* of a truth-value, which was then asserted to be

the True if the judgment stroke was added. The horizontal line was itself considered as representing a function, mapping its argument to the True if the argument was itself the True, and mapping it to the False otherwise. Except for names of objects, Frege understood all simple signs in his logical language as representing functions. "Socrates is human" would be rendered:

$$\vdash H(s)$$

Here the "$H(\ )$" would be understood as representing a function, and "$H(s)$" a term. In his early work, the function would be understood as mapping Socrates to the circumstance that Socrates is human; in his later work, as mapping Socrates to the True just in case Socrates is human.

In his early *Begriffsschrift* he introduced the notion of function by consideration of the replaceability of one part of an expression with another:

> If, in an expression . . . , a simple or a complex symbol occurs in one or more places and we imagine it as replaceable by another (but the same one each time) at all or some of the places, then we call the part of the expression that shows itself invariant a function and the replaceable part its argument. (Frege, 1972 [1879], 127)

Context makes it clear that the expression with which we begin could be either a complex term for an object, or a symbol for the assertible content of a complete judgment—a "circumstance". So one might begin with a judgment like "Cato killed Cato". We might substitute for the name "Cato" the sign "$\xi$" at the places we regard it as replaceable. If we think of "Cato" as replaceable at its first instance, we divide the expression into the argument expression "Cato" and the function expression "$\xi$ killed Cato"; if we think of it as replaceable in its second occurrence we get "Cato" and "Cato killed $\xi$" . We may also think of "Cato" as replaceable at both instances, and then the function, Frege tells us, is "$\xi$ killed

$\xi$", or "killing oneself". A function with a single argument whose value is always a judgeable content (early on) or truth-value (later on), Frege calls a *concept*.

Technically, the above passage and others from Frege's early work identify functions with parts of expressions, pieces of language, but it is clear Frege thinks of the content of the expression as being what is important. He thinks of the content of a proposition as a function of the content of its replaceable parts. In his mature work, after the notion of content is further disambiguated into sense and reference, a function is described not as an expression, but as the *reference* of the remainder of a complex expression when another part is regarded as replaceable by other expressions (Frege, 1984b [1891], 2013 [1893–1902]).

Functions of multiple arguments can be obtained by regarding two different components of a judgment as replaceable. In "Jupiter is larger than Mars" we can regard both "Jupiter" and "Mars" as replaceable and end up with the two-argument function "$\xi$ is larger than $\zeta$". Frege dubs multi-argument functions, whose values are always judgeable contents or truth-values, "relations".

Frege describes an expression for a function as being in a sense "incomplete" or "unsaturated", since it contains a spot for the argument or arguments, and likewise, Frege tells us, is the function the expression refers to, insofar as it calls for an argument. He divides functions into various levels. A *first-level function* calls for an object to be supplied, so the reference of "$\xi$ killed Cato" would be such a function. A second-level function would be a function that calls for a first-level function as argument. Frege gives quantifiers as examples. If we regard "$\xi$ killed Cato" as replaceable by other first-level function expressions in "Something is such that it killed Cato" we get the expression "Something is such that . . . it . . . ", which refers to a second-level function. The kind of "incompleteness" exhibited by such a function is different in that what it is completed by must itself have a spot to receive the "it", i.e., must itself be incomplete. Similarly, in logical notation, a quantifier such as "$(\exists x) \ldots x \ldots$" must be completed by something such

as "$F(\ )$" that has a gap to receive the bound "$x$". A third-level function would be a function that calls for a second-level function as argument. These levels are strictly distinct. A function is never of the right level to take itself as argument. "$(\exists x)\ldots x\ldots$" cannot fit in its own argument spot.

As we have just seen, Frege understands first-order quantification as the application of a second-level function to a first-level one. "$(\forall x)F(x)$" names the circumstance (early on), or the truth-value (later on), of the argument function $F(\ )$ yielding a truth (or the True) for all *its* arguments. He introduces second-order quantification as the application of a third-level function to a second-level function. Recall that a second-level function has a structure similar to "$(\exists x)\ldots x\ldots$", i.e., it must have a way of showing that it mutually saturates with its argument function. In this case, the bound variable "$x$" is placed in the argument spot of its argument function. Frege uses a structured variable $M_\beta(\ldots\beta\ldots)$ to indicate arbitrary second-level functions (Frege, 2013 [1893–1902], §25). The second-order quantification "$(\forall F)M_\beta(F(\beta))$" is then taken as asserting of the argument function $M_\beta(\ldots\beta\ldots)$ that it yields a truth for every first-level function as *its* argument. Similarly, we could quantify over relations, "$(\forall f)M_{\beta,\delta}(f(\beta,\delta))$", and here the argument function would be a second-level function taking relations as argument. It is thus understood as a third-level function taking this second-level function as argument. Frege does not introduce quantifiers of third or higher order, but it is clear that if he had he would have understood, e.g., a quantifier over second-level functions as a fourth-level function taking a third-level function as argument.

# 3 Frege's Analysis of Mathematical Induction and the Abundance of Concepts

Frege put higher-order quantification to work already in his early work in his analysis of the ancestrals of relations. If $F$ is a concept and $f$ a relation, we can say that $F$ is hereditary in the $f$-series, if whenever $x$ has $F$ and $f(x, y)$, then $y$ has $F$ as well. Partly modernizing Frege's notation, we may put this:

$$\text{Hered}_{\beta,\delta}(F(\beta), f(\beta, \delta)) =_{\text{df}} (\forall x)(F(x) \rightarrow (\forall y)(f(x, y) \rightarrow F(y)))$$

If the relation $f$ is taken as generating a series, Frege analyzes $y$ following $x$ in that series as the possession by $y$ of all hereditary concepts possessed by $f$-relata of $x$:

$$\text{follows}_f(x, y) =_{\text{df}} (\forall F)(\text{Hered}_{\beta,\delta}(F(\beta), f(\beta, \delta)) \rightarrow$$
$$((\forall z)(f(x, z) \rightarrow F(z)) \rightarrow F(y)))$$

The ancestral of relation $f$, which I write here as $f_*$, can be defined as the relation holding between $x$ and $y$ when $x = y$ or $y$ follows $x$ in the $f$-series:

$$f_*(x, y) =_{\text{df}} x = y \vee \text{follows}_f(x, y)$$

Frege argues that this analysis of following in a series shows that what might otherwise be taken as synthetic truths about the nature of series in fact turn out to be analytic consequences of these definitions.

Although he does not define numbers or the relation of successor between one number and another in his early work, it is clear that the principal application of these analyses is to the series of natural numbers. Once zero, and the relation "succ" between a number and its successor are defined, a natural number may be defined as anything in the ancestral of the successor relation beginning with 0:

$$\mathbb{N}(x) = \text{succ}_*(0, x)$$

The principle of mathematical induction—that all natural numbers have every succ-hereditary concept possessed by zero—becomes a logical consequence of these definitions.

It should be noted that in order for Frege's general analysis of following in a series, and with it his treatment of mathematical induction, to be plausible, the range of functions quantified over must be abundant. Put in modern terms, this means that a concept must exist for every possible subset of objects.[1] For if not, it might be accidentally true that an object has every hereditary concept possessed by zero, and still not be a natural number, if, for example, there simply isn't a concept that all successor-descendants of zero fall under but this object does not. Is it plausible to think of Fregean concepts as being abundant? Concepts are a species of function. In his early work, Frege's conception of a function is a bit too unclear to assess this point fully. Again, technically, he there defines functions as types of expression, and it is not very plausible to suppose that an open sentence exists for every subset of objects. This may have been a slip in presentation, however, as Frege clearly puts special emphasis on judgeable contents or possible circumstances when presenting his views on functions. If we think of a concept as a mapping from objects to circumstances about those objects, it becomes more plausible to think that one would exist for every subset of objects, but given the lack of clarity about what a "circumstance" is supposed to be, the point is still less than fully clear. Mature Frege thinks of concepts as mappings from objects to truth-values, and there it is natural to suppose that such a mapping exists for every subset of objects, i.e., one that maps precisely those objects to the True and everything else to the False. There could then be concepts not referred to by any

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[1]Frege of course did not himself employ or acknowledge the modern conception of set, but only a conception of extensions of concepts understood as value-ranges of functions. Here I allow myself to use modern terminology to make clear what is required even though Frege would not himself have expressed it this way. A similar point applies to our discussion of abundance for Russell in Section 6 below.

expression in language.

# 4   Treating Functions as Objects

As mentioned, Frege did not think that a function could take it-self as argument. There is a deep gulf between objects on the one hand and concepts and other functions on the other. A complete expression can only refer to an object. Frege held this view so strongly that he insisted that a complete expression that on its surface appears to refer to a concept, such as "the concept horse", in fact refers to an object. During the 1890s at least, he held that for every function, there was an object that goes proxy for that function when one attempts to refer to the function using a complete expression. If one writes "the concept horse is instantiated" one is asserting a first-level concept of a special object named "the concept horse" (Frege, 1984c [1892], 186).

   Similarly, in his mature logical work, Frege held that corresponding to every first-level function was an object, its value-range (*Werthverlauf*), considered as its complete mapping from arguments to values, sort of like its "graph", taken as an abstract object. Functions have the same value-range just in case they have the same value for every argument; this is Frege's infamous "Basic Law V". Using the notation $\grave{\alpha}F(\alpha)$ for the value-range of $F(\ )$, this could be expressed:

$$\vdash (\forall F)(\forall G)(\grave{\alpha}F(\alpha) = \grave{\alpha}G(\alpha) \leftrightarrow (\forall x)(F(x) = G(x)))$$

In the case of concepts, Frege identified their value-ranges with their extensions: concepts have the same extension just in case they map the same arguments to the True.

   Through the use of value-ranges, Frege held that second-level concepts could be reduced to first-level concepts. That is, for every second-level concept, there is a first-level concept true of the value-range of a function just in case the second-level concept applies to the first-level function of which it is the value-range. This, he

thought, led to a simplification in his logic, and obviated the need to introduce quantifiers above second-order.

Unfortunately, this attempt at simplification was unsuccessful, and it allowed the inconsistency due to Russell's paradox to enter the system, at least so long as the second-order quantifiers ranged over impredicative functions, i.e., those themselves defined in terms of quantification over functions. One such function is the concept $W$ of being the value-range of a function that does not map that value-range to the True, i.e.:

$$W(x) =_{\text{df}} (\exists F)(x = \dot{\alpha}F(\alpha) \wedge \neg F(x))$$

If we now consider the value-range of $W$ itself, viz., $\dot{\alpha}W(\alpha)$, and ask whether or not it falls under $W$, we get a contradiction. The proof of the contradiction uses the left-to-right direction of Law V, called Vb:

$$\vdash (\forall F)(\forall G)(\dot{\alpha}F(\alpha) = \dot{\alpha}G(\alpha) \rightarrow (\forall x)(F(x) = G(x))) \qquad \text{(Vb)}$$

Suppose $W(\dot{\alpha}W(\alpha))$. Then there is some $F$ such that $\dot{\alpha}W(\alpha) = \dot{\alpha}F(\alpha)$ and $\neg F(\dot{\alpha}W(\alpha))$. By Vb, $W(\dot{\alpha}W(\alpha)) = F(\dot{\alpha}W(\alpha))$, and so $\neg W(\dot{\alpha}W(\alpha))$, contradicting our supposition. So $\neg W(\dot{\alpha}W(\alpha))$, but then for every $F$, if $\dot{\alpha}W(\alpha) = \dot{\alpha}F(\alpha)$ then $F(\dot{\alpha}W(\alpha))$; this includes $W$ itself, and we have $\dot{\alpha}W(\alpha) = \dot{\alpha}W(\alpha)$, so $W(\dot{\alpha}W(\alpha))$, contradicting our earlier result.

Frege understood the value-range notation "$\dot{\alpha}(\ldots\alpha\ldots)$" as standing for a second-level function mapping its argument function to its value-range. In the appendix to the 1902 second volume of his *Grundgesetze der Arithmetik*, which he added after being informed of the contradiction by Russell, Frege goes on to prove that there can be no other interpretation of this functor that satisfies Vb, i.e., there can be no second-level function mapping first-level functions to objects that always yields a distinct object for functions that do not have the same value for every argument. Every such mapping from functions to objects must sometimes have the same object as value for two "different" functions; in short, there

cannot be as many objects as functions. The point is very much related to Cantor's powerclass theorem which proves that $2^n > n$ even when $n$ is infinite. Notice that there are $2^n$ possible mappings from objects to the two truth-values, the True and the False, i.e., $2^n$ possible concepts, for $n$ objects. The number of concepts must therefore be greater than the total number of objects. Frege puts the point in a 1925 letter thus: "We must set up a warning sign visible from afar: let no one imagine that he can transform a concept into an object" (Frege, 1980, 55). The point is instructive, as there are many intuitive arguments that would posit a distinct object for distinct concepts, not just the intuitive belief in value-ranges or extensions (or similar objects such as sets), but also belief in intensional entities such as senses, thoughts, propositions, properties, etc., comes to mind.[2]

Fortunately, value-ranges can be dropped from Frege's logic leaving a perfectly workable second-order system. Still, this requires thinking of the values of second-order quantifiers—functions—as distinct from objects, and not at all values of first-level quantifiers or as possible references of complete names. Arguably, this leaves the Fregean position with an awkward stance regarding its own metaphysics of functions. Consider such a claim as "functions are not objects". There appears to be no formal analogue of this truth in Frege's formal logic. If "functions" is treated as representing a first-level concept—applicable or not to objects—trivially, nothing falls under it. If it is treated instead as representing a second-level concept, then in saying that what satisfy this concept are not "objects", one must treat "objects" as a second-level concept as well, and again, it'll be trivial that nothing falls under it. Noting that his own wording of his own theories about the nature of functions and their difference from objects does not live up to the strictures of those very theories, Frege writes that "By a kind of necessity of language, my expressions, taken literally, sometimes miss my thought . . . I fully realize that

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[2]For more on the potential difficulties that might arise in Frege's philosophy from positing too many senses or thoughts, see Klement (2002, chaps. 5–7).

in such cases I was relying upon a reader who would be ready to meet me halfway—who does not begrudge a pinch of salt" (Frege, 1984c [1892], 193). Commentators have often found this to be one of the most puzzling aspects of Frege's views, and have often seen it as a forerunner of Wittgenstein's position in the *Tractatus* that there are things that can only be *shown* not said.[3]

# 5 The Development of Russell's Views on Propositional Functions

In his first letter to Frege, Russell writes that "On functions in particular..., I have been led independently to the same views even in detail" (Frege, 1980, 130). This is an exaggeration: there were important differences between Russell's view of what he called *propositional functions* and Frege's views on functions. What is true, however, is that prior to reading Frege, Russell too became convinced that a proposition—the objective content of a declarative sentence—could be understood as a function of its parts. He also introduced means for quantifying over these propositional functions.

For early Russell a proposition was understood as a structured complex of constituents. For example, the proposition *Hypatia is wise* consists of Hypatia herself as well as the universal of Wisdom. His early notion of a propositional function was built on top of this notion of proposition. In his first major logical work, 1903's *The Principles of Mathematics*, Russell describes a propositional function as what is got from a proposition by replacing a constituent with a variable (or multiple constituents with multiple variables). He ontologized variables and understood them as a special kind of object: a variable is *any* individual, where the individual in question is not specified, though the same individual for each occurrence of the variable (Russell, 1931 [1903], chap. 8).

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[3]See, e.g., Geach (1976); for examples of those who believe Frege does not deserve his pinch of salt, see, e.g., Burgess (2005); Wright (1998).

The propositional function itself is therefore *any* proposition of a given form. The function "$x$ is human" denotes one, but no one in particular, of the various propositions "Socrates is human", "Xanthippe is human", etc.

These curious objects which are any thing but no one thing in particular were short-lived in Russell's philosophy, and in his 1903 manuscripts he reimagined the variable as a kind of denoting concept. The distinguishing feature of a denoting concept is that when a denoting concept occurs as a constituent in a proposition, the truth or falsity of the proposition is not determined by direct features of the concept but rather by those of some entity or entities to which it is related: its denotation, or those entities it is "about". For example, propositions in which the denoting concept "any human" occur are made true or false by the properties of humans, not by properties of the concept itself. Similarly, when a variable occurs in a proposition, the proposition is not about the variable itself but about the values of the variable. A propositional function is a kind of denoting complex. When one uses an open formula as part of a sentence expressing a proposition, the proposition expressed will not be made true or false by direct features of the function but rather by features of the propositions which are among its range of values. However, Russell did consider it possible for there also to exist propositions whose truth or falsity does depend on features of denoting concepts or denoting complexes themselves, i.e., propositions *about* them as entities in their own right. In his own writing, Russell signified the difference with notational variants: using inverted commas for simple denoting concepts and circumflected variable letters when speaking of denoting complexes containing variables:

> The circumflex has the same sort of effect as inverted commas have. E.g., we say
>> Any man is a biped;
>> "Any man" is a denoting concept.
> The difference between $p \supset q \mathbin{.\supset.} q$ and $\hat{p} \supset \hat{q} \mathbin{.\supset.} \hat{q}$ corresponds to the difference between any man and

"any man". (Russell, 1994a, 128–129)

This does not directly answer the question as to what the difference is, metaphysically, between the two kinds of propositions: those containing denoting concepts that are about the denotata, and those about the concepts themselves. Do the latter kind contain the denoting concepts themselves but occurring in an unusual way, or do they contain other denoting concepts that denote the denoting concepts they are about? This is an issue Russell struggled with and never settled on a stable view.

The notion of a propositional function was an important one to early Russell. He was of course aware of paradoxes about classes such as the one bearing his name, which made him doubt the existence of classes as entities separable from their members. As early as mid-1903, he had hoped to eliminate class-talk from his logic in favor of talk of functions, and quantification over functions.[4] For example, the cardinal number of $\phi$s might be defined as the function satisfied by (true of) all functions that can be put in 1–1 correspondence with $\phi$.

Early Russell took the variable $x$ to be absolutely unrestricted. Even a propositional function could be the value of its own variable. Russell soon realized that this meant one would have a propositional function satisfied by all and only propositional functions that do not satisfy themselves, and hence a contradiction stemming from asking whether or not this function satisfied itself. He summarized the problem in a 1906 letter this way:

> Then, in May 1903, I thought I had solved the whole thing by denying classes altogether; I still kept propositional functions, and made $\phi$ do duty for $\dot{z}(\phi z)$. I treated $\phi$ as an entity. All went well till I came to consider the function $W$, where
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> $$W(\phi) .\equiv. \sim\phi(\phi).$$

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[4]See, e.g., his May 1903 letter to Frege (Frege, 1980, 158–160).

> This brought back the contradiction, and showed that
> I had gained nothing by rejecting classes. (Grattan-
> Guinness, 1977, 78)

As is clear from his wording here, Russell became suspect of the idea that a propositional function could be considered an "entity" in its own right, distinct from the propositions that are its values, and the individuals that are its arguments. This suspicion was no doubt strengthened by his coming to the conclusion that it is incoherent to attempt to disambiguate between denoting concepts and what they denote,[5] leading him to abandon his former theory of denoting concepts in favor of 1905's theory of descriptions.

In late 1905, through most of 1907, Russell took the strategy of eschewing propositional functions as single entities in favor of what he called substitutional matrices, which consist of propositions and entities-to-be-substituted-for in them. That is, rather than considering "$x$ is human" to be an ambiguous proposition, or something that denotes the various propositions *Socrates is human*, *Xanthippe is human* and so on, Russell would consider the pair of entities, the proposition *Socrates is human*, and Socrates. The various values of what he formerly regarded as values of this function could instead be regarded as various results of substituting other things for Socrates in the proposition. He wrote "$p\frac{x}{a}!q$" for the four-place relation meaning that $x$ substituted for $a$ in $p$ yields $q$. A single result of substitution, $p\frac{x}{a}$, could be defined using his theory of descriptions as *the* $q$ such that $p\frac{x}{a}!q$. Then, instead of quantifying over functions, one could quantify over such pairs of entities. E.g., instead of writing:

$$(\forall\phi)(\phi a \supset \phi b)$$

to say that whatever is true of $a$ is also true of $b$, one could instead put:

$$(\forall p)(\forall x)(p\frac{a}{x} \supset p\frac{b}{x})$$

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[5]This is the main thrust of the infamous "Gray's Elegy" passage of Russell (1994b [1905]).

I.e., for every proposition $p$ and entity in it $x$, if substituting $a$ for $x$ yields a truth, so does substituting $b$. This replaces a single higher-order quantifier with two quantifiers for objects (propositions included among objects).

Most of the things one would want to say about a "function" could be proxied in this way, and something vaguely like a hierarchy of levels or types of functions emerges. One could speak of a function whose argument itself is a function by talking of two substitutions within a proposition, substituting for the entire substitutional matrix of both a proposition and the entity in it. E.g., if $(\forall x)p\frac{x}{a}$ says of the matrix consisting of $p$ and $a$ that all its values are true, one can regard this as a matrix for other matrices by substituting other propositions for $p$ and other entities for $a$. Moreover, the method disallowed any kind of proxy for a function taking itself as argument. A substitutional matrix that does duty for a function of individuals is two entities, and there is no means in the theory for substituting a pair of things for a single thing. A substitutional matrix that does duty for a function of functions is three entities: an original proposition, a proposition in it to be substituted for, and an individual, but again, its "values" are gotten by substituting two entities, not three.

The difficulty with this approach was that it required taking propositions as entities, and quantifying over them. We have seen in the case of Frege that it must not be possible to generate a distinct object for every function. Russell's substitutional theory successfully avoided taking functions themselves to be objects on their own, but as first formulated, the theory allowed one to generate a distinct proposition for every substitutional matrix. For example, one could generate the proposition that all values of that matrix are true. Below, we use the notation $\ulcorner\langle A\rangle\urcorner$ for the proposition expressed by $A$. Diagonalizing then, one would be led to consider the matrix consisting of $p_0$ and $a_0$, where $a_0$ is an arbitrary thing, and $p_0$ defined thus:

$$p_0 =_{\mathrm{df}} \langle(\exists p)(\exists a)(a_0 = \langle(\forall q)(\forall x)(p\frac{x}{a}!q \supset q)\rangle \wedge \sim p\frac{a_0}{a})\rangle$$

That is, $p_0$ is the proposition that there is some matrix of $p$ and $a$ where $a_0$ is identical to the proposition that all values of this matrix are true, but $a_0$ itself does not satisfy this matrix. We then consider the proposition $r_0$ that all values of the matrix $p_0$ and $a_0$ are true:

$$r_0 =_{\text{df}} \langle (\forall q)(\forall x)(p_0 \frac{x}{a_0}!q \supset q) \rangle$$

Accepting Russellian assumptions about the identity conditions of propositions, we are led to a contradiction if we ask whether $r_0$ substituted for $a_0$ in $p_0$ yields a truth or not.[6] Suppose $r_0$ is substituted for $a_0$ in $p_0$: the result would be $\langle (\exists p)(\exists a)(r_0 = \langle (\forall q)(\forall x)(p \frac{x}{a}!q \supset q) \rangle \wedge \sim p \frac{r_0}{a}) \rangle$. For this to be true, there must be a $p$ and $a$ where $r_0 = \langle (\forall q)(\forall x)(p \frac{x}{a}!q \supset q) \rangle$ but when $r_0$ is substituted for $a$ in $p$, the result is false. Identical propositions have identical constituents. Since $r_0$ is identical to both $\langle (\forall q)(\forall x)(p_0 \frac{x}{a_0}!q \supset q) \rangle$ and $\langle (\forall q)(\forall x)(p \frac{x}{a}!q \supset q) \rangle$, it must be that $a_0 = a$ and $p_0 = p$. But this would mean that when $r_0$ is substituted for $a_0$ in $p_0$, the result is false. In short, in order for $r_0$ substituted for $a_0$ in $p_0$ to be true, it must be false. It is therefore false. The negation of what we get when we substitute $r_0$ for $a_0$ in $p_0$ is thereby true, i.e., $\sim (\exists p)(\exists a)(r_0 = \langle (\forall q)(\forall x)(p \frac{x}{a}!q \supset q) \rangle \wedge \sim p \frac{r_0}{a})$. This is equivalent to $(\forall p)(\forall a)(r_0 = \langle (\forall q)(\forall x)(p \frac{x}{a}!q \supset q) \rangle \supset p \frac{r_0}{a})$. Instantiating to $p_0$ and $a_0$ we get $r_0 = \langle (\forall q)(\forall x)(p_0 \frac{x}{a_0}!q \supset q) \rangle \supset p_0 \frac{r_0}{a_0}$. By the definition of $r_0$, then $p_0 \frac{r_0}{a_0}$, i.e., the result of substituting $r_0$ for $a_0$ in $p_0$ is true. This contradicts our earlier result.

Obtaining the contradiction requires the possibility of the $(\exists p)$ quantifier in $p_0$ taking $p_0$ itself as a value, i.e., for a proposition to be able to quantify over a range that includes that very proposition. In his manuscripts and works of the period,[7] Russell tried various solutions, including denying the existence of quantified propositions, and also dividing quantified propositions into various orders depending on what kinds of propositions they quanti-

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[6]See Landini (1998) for more details as well as further discussion of this period of Russell's thought and the problems he encountered.

[7]See the papers on the substitutional theory gathered in Russell (2014).

fied over. In his well-known "Mathematical Logic as Based on the Theory of Types" (Russell, 1956b [1908], 77), written in 1907 and published in 1908, Russell reintroduced a notation for quantifying over functions, but explained why functions must be divided into various types, so that a function cannot take itself as argument, by citing the notion of substitution, so that function-quantification was not to be taken as fundamental. Functions would also be divided into various orders, depending on the types of quantification that exist in their values, in keeping with the idea that the matrices in virtue of which they're defined are divided by the order of the proposition into which substitutions are made in their values. The result was the first version of a so-called "ramified" theory of types.

However, this precise metaphysical picture of the underpinnings of ramified type-theory was short-lived. Russell had been convinced as early as *The Principles of Mathematics* that all genuine entities form a logical type—that of an individual, or something that can be counted as one (Russell, 1931 [1903], §47). Having to divide propositions into various orders depending on what they quantify over, and not having a single variable encompassing all entities whatsoever, appears to be something with which he was uncomfortable. As a result, he reconsidered his entire metaphysics of propositions as the contents of judgments and assertions. Whereas he had previously understood judgment as a relation between a subject and a single thing, a proposition considered as a single unit, he now adopted what is known as the multiple-relation theory of judgment (Russell, 1992b [1910], 1984). On this view, *S* judging that Hypatia is wise is not a binary relation between *S* and a proposition taken as a single unit but a three-place relation between *S*, Hypatia, and Wisdom as separate entities. Giving up his older view of propositions required giving a rather different account of how to understand quantification over functions.

# 6 The Mature View of *Principia Mathematica*

The metaphysics behind the higher-order logic of *Principia Mathematica*, and its countenancing of individuals and propositional functions of various order-types, are often seen as obscure. Geach claimed that Russell's notion of propositional function is "hopelessly confused and inconsistent" (Geach, 1972), and Cartwright agrees adding that "attempts to say what exactly a Russellian propositional function is, or is supposed to be, are bound to end in frustration" (Cartwright, 2005, 915).

I believe we can get a better handle on his view by considering its origins, as well as the account of truth and falsity sketched briefly in the introduction to *Principia Mathematica*. The truth of a higher-order quantified proposition is still understood in terms of substitutions within propositions, but the notion of a "proposition" is de-ontologized and mainly understood linguistically. Around the same time he adopted the metaphysical view of logical atomism. According to this view, the world consists of simple entities holding simple properties and standing in simple relations. Simple facts of this form were called atomic facts, and serve as direct truth-makers for the elementary propositions expressing them. However, all truths, even more complicated ones, ultimately rest on the collection of atomic facts. Part and parcel of this involved explaining the truth or falsity of quantified propositions in terms of their non-quantified instances. In 1940, he wrote that "in the language of the second order, variables denote symbols, not what is symbolized" (Russell, 1940, 192). This does not mean that higher-order quantification is to be understood as objectual quantification over expressions. It is rather to be understood as claiming that the truth or falsity of second (and higher) order formulas is to be defined substitutionally in terms of the truth or falsity of the lower order formulas that are their instances, ultimately resolving into the basic truths upon which all others rest.

A quantifier-free formula in the language of *Principia Mathe-*

*matica* is thought of as expressing an elementary proposition. It records a judgment a subject makes to the effect that a certain relation holds between certain relata.[8] This judgment is a multiple relation between a subject, that relation, and those relata, and is true if the relation in fact holds between those relata, i.e., there is a complex or fact of this form. The truth or falsity of elementary propositions is the basis for all other notions of truth, which Russell tells us, form a hierarchy. Functions are like propositions except containing variables—only now, both propositions and variables, and thereby functions—are understood linguistically. The values of a function, Russell also tells us, are always prior to the function itself, and when a variable is used essentially—as in a quantified proposition—the truth or falsity of the resulting proposition is defined in terms of the truth or falsity of its instances:

> ...the meaning of *truth* which is applicable to this [a first-order quantified] proposition is not the same as the meaning of *truth* which is applicable to "$x$ is a man" or to "$x$ is mortal". And generally, in any judgment $(x).\phi x$, the sense in which this judgment is or may be true is not the same as to which $\phi x$ may be true. If $\phi x$ is an elementary judgment, it is true when it *points* to a corresponding complex. But $(x).\phi x$ does not point to a single corresponding complex: the corresponding complexes are as numerous as the possible values of $x$. (Whitehead and Russell, 1925–1927 [1910–1914], 46)

The quotation is a bit difficult in that Russell allows himself to use a free variable in his informal discussion to speak of *arbitrary* variable-free propositions. I.e., when he speaks of the kind of truth applying to "$x$ is a man", he means the kind of truth applicable to all elementary judgments such as "$a$ is a man", "$b$ is a man", etc., alike. These do not contain variables. The facts that make

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[8]For an explanation of how even molecular quantifier-free judgments can be understood as expressing a relation between relata, see Klement (2015, §5).

these particular judgments true are ultimately what make quantified propositions—those that do use variables essentially—true. A quantified proposition is not made true or false by a single fact. Instead, a quantified proposition has a different kind of truth or falsity that depends on whether its instances are true or false. The truth or falsity of those instances therefore cannot depend on anything only definable utilizing the function contained in the quantified proposition, for fear of circularity.

Once truth or falsity for first-order quantified propositions is defined, we can go on to define a notion of truth or falsity for second-order quantified propositions which depends on whether or not all instances of first-order propositions of a certain form are true. For example, "$(\forall\phi)\phi!a$" could be taken to possess its appropriate variety of truth iff all first-order propositions containing "$a$" have their appropriate variety of truth. Order restrictions come from the place in the hierarchy of senses of truth and falsity a given formula holds. Russell adds the shriek "!" to bound function variables to indicate that they must be predicative, i.e., they are of the lowest order possible for their type and the notion of truth applicable to their instances must always be lower in the hierarchy than the notion of truth applicable to the propositions in which they are used. Although this was not well-understood by the first generation of commentators on *Principia Mathematica*, all bound function variables in *Principia Mathematica* are predicative.[9] For a second-order quantifier, this means, for example, that the instances of $\phi!a$ in virtue of which the truth of "$(\forall\phi)\phi!a$" is defined cannot make use of function quantifiers.

This restriction to quantification over predicative functions poses certain problems for the mathematical project of *Principia Mathematica*. The primary use of higher-order quantification in *Principia Mathematica* is as a part of its treatment of classes, the so-

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[9]For a defense of this claim and a comparison of different interpretations, see Landini (1998, chap. 10). For further explanation of how to understand the non-predicative function variables that *apparently* occur in *Principia Mathematica*, see Klement (2013).

called "no classes theory of classes". On this approach, talk about classes or sets is not taken as fundamental but defined in terms of higher-order quantification. To say something about $\{x|Ax\}$, the class of all $x$ such that $Ax$, is really to say that there is a function—a predicative function—$\phi!$, such that for all $x$, $\phi!x$ iff $Ax$ and that something holds of $\phi!$. If the expression $Ax$ contains function variables, it is not obvious that there will always be such a function. Moreover, Russell treats mathematical induction in a manner similar to Frege, using a kind of quantification over classes that is resolvable into quantifying over functions. If there isn't a predicative function coextensive with arbitrary open sentences regardless of the orders of their quantifiers, mathematical induction would be limited to those instances in which there was such a function, crippling the system. To get around these limitations, Russell assumes something called the *axiom of reducibility*, which for the lowest type can be stated schematically as follows:

$$(\exists\phi)(\forall x)(\phi!x \leftrightarrow Ax)$$

where $Ax$ is any open sentence, even one containing higher-order quantifiers. *Semantically*, the truth or falsity of a formula containing a predicative second-order variable $\phi!$ is not defined in terms of non-predicative instances, but there must be a predicative instance coextensive with any arbitrary open sentence. The inclusion of the axiom of reducibility makes Russell's logic as powerful as, and easily mutually interpretable with, full impredicative higher-order logic.

It is natural to ask whether or not the axiom of reducibility is a plausible assumption, and whether or not it can be considered a basic principle of logic. This is a difficult question. Let us focus again for the moment on the lowest type, and remember that a formula using higher-order quantification is to be understood as getting its truth or falsity from the formulas that are its instances. The axiom of reducibility requires that for every formula containing a variable $Ax$, even one containing higher-order quantifiers, there is a value of the second-order variable $\phi!$ such that $\phi!x$ is

satisfied if and only if $Ax$ is, or crudely, that $\phi!x$ is coextensive with $Ax$. This is plausible if, similar to the case for Frege, predicative first-order functions are abundant, i.e, for every subset of the set of individuals, there is a predicative function satisfied by all and only members of that subset. Given the substitutional semantics described above, the values of the predicative variable $\phi!$ are determined by the otherwise closed predicative formulas that can be substituted for it, i.e., formulas not containing any second or higher order quantifiers. Taking predicative first-order functions to be abundant therefore is tantamount to holding that for every subset of the domain there is an open sentence containing $x$ without higher-order quantifiers satisfied by exactly that subset.

Russell argues (Russell, 1919, 192–193) that this is plausible if we accept a generalized version of Leibniz's principle of the identity of indiscernibles. On this version, it is assumed that the entirety of a given individual's monadic properties and relations to other individuals is unique to it. For a given individual $a$, let $B^a x$ be the conjunction of the form $P^1 x \wedge P^2 x \wedge \ldots \wedge R^1 x b \ldots$, containing all of $a$'s properties and relations. Now let $Bx$ be the disjunction of such conjunctions, $B^a x \vee B^b x \vee \ldots$, etc., for any arbitrary subset of the domain containing $a, b, \ldots$, etc. The resulting open sentence $Bx$ does not contain higher-order quantifiers and thus *is* one in terms of which the truth or falsity of which the relevant instance of the axiom of reducibility is defined, and arguably, such a $Bx$ exists coextensive with any arbitrary $Ax$. Russell seems unperturbed that such disjunctions of conjunctions might be infinitely complex, both in terms of the properties and relations of a given individual, and in terms of the number of individuals making up the subset. This suggests that Russell believes the lower-order instances in virtue of which the truth of a higher-order quantified proposition is defined can include infinitely long or complex instances. Russell confirms that this is his view, writing later that our human inability to produce infinitely complex formulas for defining arbitrary classes is "is an empirical fact which logicians

should ignore" (Russell, 1958, 91).[10]

Although Russell employs quantifiers for what he calls "propositional functions", it is clear that this kind of quantification is not to be understood as introducing a new kind of entity into his metaphysics. The truth or falsity of a formula that involves such quantification resolves into the truth or falsity of its instances, which, if they contain further (lower-order) quantifiers, are similarly resolved into the truth or falsity of their instances, and so on, until we reach elementary propositions. The entities and complexes making elementary propositions true or false indirectly make all other orders of propositions true or false as well. Hence we find Russell writing that a propositional function "in itself is nothing; it is merely a schema" (Russell, 1956c [1918], 234), or "is nothing but an expression" (Russell, 1958, 53). As Russell himself claims, this means that "we do not need to ask, or attempt to answer, the question 'What is a propositional function?'" (Russell, 1919, 157)—so long as we can understand the semantics of higher-order quantified formulas, there is nothing more that needs explaining, or so Russell believed during this period. Perhaps commentators such as Cartwright and Geach are reading too much into the notion.

# 7    Logical Form and General Facts

While Russell did not have a realist metaphysics of "propositional functions", and therefore did not have one that could not be expressed according to its own strictures, there are puzzles in Russell's view that do not have a clear solution. As we have seen, a universally quantified proposition is true just in case all the instances of a more basic logical form are true. But what is a logical form? One possible answer to the question *what is the form of "Socrates loves Xanthippe"?* would be the propositional function $\hat{x}\hat{R}\hat{y}$, where

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[10]For further discussion of this and related points, see Klement (2010, 655–659).

all the constituents are replaced by variables. If we are eschewing propositional functions as metaphysical entities, this could only define a form as a kind of expression containing variables. As we have seen, expressions containing variables are used to speak about all or some of the expressions that are their instances. Is talk about a "form" really just a way of talking about a certain class or pattern of propositions, considered linguistically?

Part of the problem with this is that it doesn't seem to adequately capture what propositions of the same form really have in common. It is natural to think that what all binary relational propositions have in common is not merely something linguistic but something reflected in reality, something in the facts corresponding to them. Isn't what we're really interested in the thing in reality reflected by these linguistic patterns? Russell himself often spoke about creating a logically perfect language: in such a language, there would be a simple expression for every simple entity, and complex things would be represented by complex expressions. But this seems to suggest that there is an objectivity that goes beyond language to what is simple, and what is complex, and that a logically ideal language is one that "gets the true logical form" correct in its own syntax, suggesting there is a non-linguistic criterion for what counts as correct. Without this, it is unclear what could make one language more "logically perfect" than another.

Moreover, even if we do focus entirely on linguistic patterns, we need an explanation for why certain expressions match a given pattern and others do not. Why is "Socrates loves Xanthippe" an instance of $\hat{x}\hat{R}\hat{y}$ but "the Earth rotates" not? There is something *structurally different* in the very expressions that is not adequately captured merely by pointing out that they are members of different sets of propositions. Not every arbitrary set of propositions qualifies as a distinct logical form, only those in which the members genuinely have something in common. What is this something, and isn't that what we're really interested in when we speak of logical form? We could try to spell it in out in terms of some notion

of "substitution" or "assignment" for variables, but we still need an explanation of this notion that explains why one could never obtain "the Earth rotates" by substituting or assigning values to the variables in $\hat{x}\hat{R}\hat{y}$. Why isn't "Earth" an appropriate substitute for $\hat{R}$?

In his 1913 *Theory of Knowledge* manuscript (Russell, 1984, 114), Russell defines a logical form of a proposition as the existentially quantified fact corresponding to the proposition got by replacing all its definite constituents with variables and existentially generalizing. Thus the form of "Socrates loves Xanthippe" would be the fact that $(\exists x)(\exists y)(\exists R)xRy$—but how are we to understand this? It is a deviation from the view of *Principia Mathematica* according to which, as we have seen, quantified judgments are not made true or false directly by single facts but through their instances, ultimately bottoming out only in elementary, non-quantified judgments. The fact that Socrates loves Xanthippe has Socrates, love, and Xanthippe as constituents, but what are the constituents of fully general facts such as he now defines logical forms? The official answer in 1913 seems to be that these facts have no constituents, as all the particular constituents have been removed. What are they then? What's left? Russell tells us no more.

In his 1918 lectures on "The Philosophy of Logical Atomism" (Russell, 1956c [1918], 234–235), Russell goes on to argue that in addition to the atomic facts that make atomic propositions true or false, there must be general and existence facts corresponding to quantified formulas. His argument for this proceeds as follows. Consider the quantified truth that all humans are mortal, or $(\forall x)(x$ is human $\supset x$ is mortal$)$. Even if we have an enumeration of every object there is, and have established for each, either that it is non-human, thus making the conditional true by making the antecedent false, or that it is mortal, thus making the conditional true by making the consequent true, this is not enough to guarantee the truth of the quantified formula. One needs additionally that those enumerated are all the objects there are—itself a gen-

eral fact. However, he does not say much about how to understand the nature of general facts, even admitting uncertainty:

> I do not profess to know what the right analysis of general facts is. It is an exceedingly difficult question, and one which I should very much like to see studied. I am sure that, although the convenient technical treatment is by means of propositional functions, that is not the whole of the right analysis. Beyond that I cannot go. (Russell, 1956c [1918], 236–237)

Russell also discusses *fully* general, or logical, truths, and claims not to fully understand what their constituents are, hinting that *forms* may be one possible answer, but without giving an account of what forms are supposed to be. However, if it is right that the correct *technical* treatment of generality/quantification involves propositional functions, and there are not just general formulas, but also facts corresponding to those formulas, this suggests very much that there must be something with external ontological status associated with propositional functions.

It is a natural worry that by introducing general facts into his metaphysics, Russell runs the risk of again postulating distinct entities for every propositional function. Can a general fact be about a generality that includes that fact itself? When functions and general facts aren't ontologized, it is natural to think that the truth conditions of a quantified formula must be specifiable in a non-circular way and that, therefore, the values of a propositional function must be prior to the function itself. But this is not so clear if general facts are understood as ontologically real.[11] Recall that Russell pushed for a view according to which all genuine entities, those that do not "disappear on analysis" (Whitehead and Russell, 1925–1927 [1910–1914], 51), can be considered individuals. Facts would appear to be such entities. As we have seen, Russell

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[11]A related point was made by Gödel in his remarks on Russell's logic; see Gödel (1944, 148).

recognized the paradoxical nature of postulating as many propositions as propositional functions thereof, a problem which plagued even the substitutional theory where functions were reimagined as substitutional matrices. Isn't he now in danger of postulating as many facts as functions on facts, and thereby facing a Cantorian paradox similar to those considered earlier?

Russell did not believe in "false" facts: there are just facts. Since he did early on believe in false propositions, arguably this would mean he is not postulating quite as many facts as he previously did propositions. However, in this period, he does believe in what he calls "negative" facts: facts which make the negation of a false proposition true. In the case of quantified propositions, the distinction is rather between general and existence facts: the negation of a false general proposition is made true by an existence fact, and the negation of a false existential proposition is made true by a general fact (Russell, 1956c [1918], 211–215, 236–238). So for every pair of proposition and its negation, there is a fact. Half of an infinite number is the same infinite number, so this does not lead to a meaningful reduction in his metaphysics. We can still assume that, for every formula $A$, there is a fact which either makes $A$ true *or* makes it false (by making $\sim A$ true).

Suppose we replace our earlier $\ulcorner \langle A \rangle \urcorner$ notation for propositions with $\ulcorner [A] \urcorner$ for the fact that makes it true or false that $A$, whichever is the case. Whether or not they are individuals, facts must be in some logical type; let us use Greek letters as variables for facts. There must also be properties of (propositional functions satisfied or not by) facts. For each property $\phi$ of facts, there is fact making it either true or false that every fact has $\phi$. Now there appear to be too many facts, and Cantorian paradoxes loom. Consider the function:

$$\psi\alpha =_{\mathrm{df}} (\exists\phi)(\alpha = [(\forall\beta)\phi\beta] \wedge \neg\phi\alpha)$$

That is, $\psi$ is the property a fact has if it makes true or false a formula of the form $(\forall\beta)\phi\beta$ but it itself does not have the $\phi$ in question. We can then consider the fact $[(\forall\beta)\psi\beta]$ making it true or false (well, false) that all facts have $\psi$. Arguably, this fact itself

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satisfies $\psi$ just in case it doesn't:

$$\psi([(\forall\beta)\psi\beta]) \leftrightarrow \neg\psi([(\forall\beta)\psi\beta])$$

For suppose $\psi([(\forall\beta)\psi\beta])$. Then $[(\forall\beta)\psi\beta]$ is identical to some fact $[(\forall\beta)\phi\beta]$ for some $\phi$ it does not have. But presumably facts are individuated so they correspond to the generalization or existence of a unique property, so $\phi$ can only be $\psi$, and since it does not have $\phi$, it must not have $\psi$, so our supposition cannot be true. Suppose instead $[(\forall\beta)\psi\beta]$ does not have $\psi$. This means it has every property $\phi$ for which it is the same as the fact $[(\forall\beta)\phi\beta]$, including $\psi$ itself, so it does have $\psi$ after all: a contradiction.

Russell could try to resist this by pointing out the impredicativity of $\psi$—we are assuming $\psi$ itself is a possible value to the $(\exists\phi)$ quantifier it contains. Unfortunately, the axiom of reducibility might provide a coextensive function to $\psi$ which is in the range of the $(\exists\phi)$. Russell could argue that facts must be subdivided into logical types depending on what they quantify over, so that $[(\forall\beta)\psi\beta]$ is not of the right type to be an argument to $\psi$, but again, if general facts are considered ontologically real, it is unclear why a fact cannot involve quantification over a range that includes itself.

Another response Russell could give would be to deny that it is possible to name or quantify over facts at all, and indeed, in his logical atomism lectures, he claims that facts cannot be named. This appears to have been a point that he took from early Wittgenstein and he does not argue for it or explain it in the detail it deserves. Sometimes he seems only to mean that facts are best represented with complex expressions rather than simple ones (Russell, 1956c [1918], 189). This does not mean that we cannot quantify over or form complex names of (descriptions for) facts. More explanation would need to be given to determine whether or not it would give him grounds for resisting our notation $\ulcorner[A]\urcorner$, and with it, a way to block this paradox. A strong interpretation of this position, one on which it is wholly impossible to quantify over facts, or talk about them similarly to regular

things, seems to create problems for Russell similar to those that led Frege to ask for his pinch of salt. What are we to make of such claims as that "there are general facts in addition to atomic facts", or "the proposition 'Socrates is human' is made true by a fact" if facts cannot be named or quantified over? Even the intermediate view that quantification over facts is possible but that no fact can involve quantification over a range that includes itself is difficult to express by its own lights. Is that claim itself made true by a fact, and if so, is it included in its own range of applicability? Unfortunately, Russell's later views on the nature of general facts and logical forms are probably too unclear to fully assess. He might be better off not positing a general fact for every quantified judgment. This would involve suggesting not only that there are no false facts but that negations of false general propositions are not always made true by distinct facts. It might be sufficient to postulate just one general fact, i.e., that the list of individuals in the world is comprehensive. In that case, there is no risk of postulating as many general facts as propositional functions satisfiable by them, which would ease any such Cantorian worries.

# 8    Conclusion

Frege and Russell are often heralded as two of the principal founders of analytic philosophy and their views have been remarkably influential. I have attempted to sketch the development of their views on the nature of (propositional) functions and higher-order quantification. I have argued that the views of each developed as they did due in part to the difficulties that resulted from taking functions to be objects, or to be correlated with objects in a way postulating as many objects as there are functions. I have further explained how each held a mature position on which their own metaphysical views on the nature of functions or general facts were difficult to state from within their own position. These issues are still of ongoing concern. Perhaps most significantly is that contemporary researchers need to take seriously the warning

Frege wanted to be "visible from afar": that if we do adopt things such as higher-order/higher-level functions or properties into our logical systems, we must be very careful not to take them to be objects in their own right or even correlate them one-to-one with objects, or else we run the risk of contradiction.

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