

Russell, His Paradoxes, and Cantor's Theorem: Part II

Kevin C. Klement

August 21, 2009
(forthcoming in *Philosophy Compass*)

Contents

1	Introduction	2
2	Russell's Rejected Solutions	2
2.1	Theories of Limitation of Size	3
2.2	The Zigzag Theory	6
2.3	Metaphysical Types	7
3	The Retreat from Pythagoras	10
3.1	The Predication Paradox	10
3.2	Class-Intensions and Descriptive Senses	13
3.3	Russell's No Classes Theory	14
3.4	Higher-Order Quantification	17
4	Conclusion	22

1 Introduction

This article is a sequel to “Russell, His Paradoxes, and Cantor's Theorem: Part I,” in which various Cantorian diagonal paradoxes either discovered or considered by Bertrand Russell were outlined. These include Russell's famous class paradox involving the class of all classes not members of themselves, his predication paradox involving the property of non-self-instantiation, as well as similar paradoxes involving propositions, descriptive senses, class-intensions and equivalence classes of coextensive properties. Four different lines of solutions considered by Russell were also discussed: (i) the theory of limitation of size, (ii) the zigzag theory, (iii) logical types of things, and (iv) the “no classes (etc.)” theory. (In what follows, it is assumed that the reader has read Part I.) In this sequel, we examine in further detail the impact of these paradoxes on Russell's own philosophy, his consideration of possible solutions of all these kinds, and his reasons for, in the end, moving towards a rejection of robust metaphysical realism about many kinds of abstract objects in favor of viewing them as “logical fictions,” or mere *façons de parler*, so that such abstract things need not be taken as included among the ultimate furniture of reality.

2 Russell's Rejected Solutions

Russell spent the bulk of his intellectual energy between 1902–1908 trying to find responses to paradoxes such as these that both seemed philosophically sound, and allowed his project in the foundations of mathematics to escape inconsistency and paralysis. He changed tack often, and his manuscripts from the period are filled with half-

starts, rejected proposals and admissions of uncertainty.¹ It is worth discussing the major approaches he considered but rejected at least briefly.

2.1 Theories of Limitation of Size

Surprisingly, perhaps, given their prevalence today, Russell gave the least consideration to limitation of size approaches. The general suggestion for the class paradox in particular is that when a certain property or condition holds of too many things, those things do not form a class or extension as a single object, even though those things instantiating a less popular property may form such a unit. Russell had many reasons for not finding this approach very attractive. For one, it does not tell us how big is too big. Certain proposals have been made, but none without drawbacks. Some might draw the line at an infinite collection, but this cripples the use of classes in mathematics. Others might draw the line at the size of the entire universe, per Boolos's (New V), but without knowing the size of the universe, we are still left in the dark about what classes we can assume to exist. Russell insisted upon an independently *philosophically well-motivated explanation* of this.

Moreover, the conceptions of classes or sets that lend themselves to this sort of view are out of sorts with Russell's own interests. "Sets" in ZF and similar theories, understood as structures built up iteratively by union and powerset operations beginning with the empty set, seem almost unrelated to those understood in the Boolean/Whiteheadian logical tradition as involved in all categorical judgments: the objects denoted by such phrases as "all humans," etc. (*PoM* 67). Russell understood by a "class" the extension of a

¹Those from 1902 through mid-1905 have been published in his *Collected Papers* v3-4, but those from 1906 through 1909 are currently only available at the Russell Archives at McMaster University.

concept, or the collection we're talking about when we make a claim about those things, all of which share a common property, but in which the truth of the content asserted depends only on the makeup of that collection. From this perspective, the supposition that classes only sometimes exist, depending on their would-be size, is tantamount to suggesting that there's a radical difference in kind between talk of "all humans" (which are finite in number, and so not too many to form a class) and talk of "all numbers" (which are too many), which seems implausible. Russell had assumed that, on this understanding, if there are classes at all, there should be such "large" classes as the universal class or class of classes, which would be involved in any discourse about "all classes" or "all things." Recall (from Part I) that it is such collections that initially led Russell to be skeptical about Cantor's theorem.

Another reason Russell was likely unattracted overall to this approach is that its prospects are dim for solving the *other* Cantorian paradoxes he considered. The analogous suggestion for the predication paradox would be that a property constitutes an object or logical subject, and hence can be predicated truly or falsely of itself, only when it doesn't apply to too many (other) things. Yet it seems very arbitrary to think that there should be such an object as *Humanity*, but not such an object as *Number* (Numberhood?), and it's hard to imagine a philosophical theory that could support such a position. Indeed, even those who are attracted to a "limitation of size" solution to Russell's class paradox usually opt for a solution of a different stripe for the predication paradox, or fail to address it at all. Similar comments apply with regard to the others. Consider the propositions paradox. The "limitation of size" approach suggests that there is a proposition of the form *all propositions with property F are true* when, but only when, the property *F* does not apply to too many things.² Yet it would seem bizarre to hold that, e.g., the

²Something along these lines also seems like an unintended consequence

proposition *all propositions expressed explicitly in this paper are true* exists but not *all propositions entailed by things expressed in this paper are true*, simply because I explicitly express only a finite number of propositions, but they entail countless more. This paradox is rarely discussed, but it seems difficult to imagine that even those endorsing the limitation of size approach for the class paradox would think a similar solution would apply here. Yet, Russell himself was convinced that, as he put it, “the close analogy” between the paradoxes “strongly suggests that the two must have the same solution, or at least very similar solutions” (*PoM* 527).³

of the view prevalent in contemporary philosophy of language discussions on which structured propositions (sometimes misleadingly called “Russellian propositions”) are thought of as set-theoretic constructs. If embedded within a set-theory like NBG, this leads to the result that there are no structured propositions about proper classes (since proper classes cannot be members of any other classes)—a result that, in my opinion, constitutes a *reductio* of their view.

³Even when “limitation of size” is only used as the criterion for the conditions under which classes exist (and not as a criterion for when properties exist, as objects, or when propositions, senses, etc., exist), doubts may arise as to its formal adequacy. Without a worked-out, robust, theory about the nature of properties, their identity conditions, and so on, it is difficult to estimate, for a given property, how many other properties it is likely to be coextensive with, but it does seem clear that the answer would not be much different for different properties. It seems, for example, that the number of properties coextensive with the property of self-identity is probably the same as the number of properties coextensive with the property of non-self-identity—consider, e.g., that the negation of each property in the former group would be in the latter, and vice-versa. Assuming then, that such collections are not too large to constitute classes (not, I admit, a small assumption), one then has, for each property—even properties too large to have their own extension make up a class—an associated entity (the equivalence class of coextensive properties) which can go proxy for it in class-theoretic reasoning. As Russell was himself aware (*Papers v4* 274), it then becomes possible to give a revised definition of “class membership” where this really means instantiating one of the properties within a given class of properties, and the result is that the size limitation is effectively undone, thereby reintroducing naïve class theory. The version of Russell’s paradox that would then result would not be the original class

2.2 The Zigzag Theory

As an approach to the classes paradox, the zigzag theory, rather than claiming that properties that are true of too many things don’t make-up a class, instead claims that certain conditions don’t define a corresponding class when they have certain intrinsic features. This is a tack Russell considered, on and off, throughout 1902–1905.⁴ His attraction to the view no doubt stemmed from the possibility that it might vindicate his initial doubts about Cantor’s theorem and allow for such un-Cantorian (large) collections as the universal class or class of all classes.

His early ontology included “propositional functions” (ontological correlates of open sentences), themselves capable of occurring as logical subjects in a proposition, and his logic included quantifiers ranging over such propositional functions. Russell’s zigzag approach could then make use *object language* qualifiers for sorting out those propositional functions that do determine a class (“simple” or “predicative” functions) from those that do not (“impredicative” or “quadratic” functions). Russell might write:⁵

$$\sim \text{Quad}(\phi) \supset (\psi)[\hat{x}(\phi x) = \hat{x}(\psi x) \equiv (x)(\phi x \equiv \psi x)]$$

In this regard, Russell’s approach can perhaps be better likened not to Quine’s broadly “zigzag” NF, but rather to the neo-logicist set theories such as those discussed by Shapiro (mentioned in Part I).⁶

form of Russell’s paradox, but rather, what was called the property equivalence class paradox in Part I, but it would be a contradiction all the same.

⁴In his published works, the zigzag theory is only explicitly mentioned in “On Some Difficulties,” though something like it is hinted at with the discussion of “quadratic forms” in *PoM* (104, 487); however it is discussed in many surviving manuscripts from the period: see *Papers v4* parts I–III.

⁵The example is only slightly notationally altered from one of Russell’s manuscripts—see *Papers v4* 9.

⁶From the standpoint of historical accuracy, then, it is somewhat unfortunate

Indeed, if Russell's talk of "impredicative" or "quadratic" propositional functions is taken as synonymous with their notion of "badness," Russell's principle above is equivalent to the neo-logicist genericized form of Boolos's (New V).

Rather than attempting to state a single criterion for badness or impredicativity, however, Russell's approach was more piecemeal. He proposed axioms to the effect that certain simple propositional functions were predicative (or "good") along with certain principles to the effect that certain transformations or combinations of predicative functions must yield new predicative functions. However, it appears that Russell was unable in this way to characterize a notion of predicativity that both excluded all the cases generating paradoxes and also preserved the mathematical reasoning he needed for his logicist project. As he tried out various approaches, the axioms he found it necessary to assume grew, in his own words, "horribly complicated and unobvious" (*DRDJ* 79), and removed from any philosophical insight into the relationship between properties and classes. Sensing a dead end, he abandoned the approach.

2.3 Metaphysical Types

Russell, of course, did eventually endorse a kind of type theory. Although he is often read as doing so, and the matter is still highly controversial, my own interpretation is that Russell's solution to the logical paradoxes in *PM* did not involve postulating different metaphysical *kinds of entities*.⁷ He did, however, flirt with approaches of this sort along the way.

that the most in-depth attempts to give reconstructions of Russell's work in this period, found in Cocchiarella's work, build upon Quine's work instead, although Cocchiarella's reconstructions are fascinating on their own terms.

⁷This is borne out in the recent secondary literature; see especially Landini, *Russell's Hidden Substitutional Theory*; Klement, "Form Before Function"; and Stevens, *Russellian Origins*.

Russell's deepest exploration of an approach to the classes paradox along these lines was perhaps the theory of types proffered, rather tentatively, in 1902, in Appendix B of *PoM*. The theory there differentiated between individuals, "classes as many" of individuals, "classes as many" of classes as many, and so on. Even here, describing his theory as dividing entities into distinct ontological types is rather misleading given that Russell did not regard a "class as many" as a single entity, but rather, as the name implies, a plurality of distinct entities (*DRDJ* 78). Nevertheless, the theory held that different grammatical types of expressions corresponded to different logical categories of semantic values, and that it was meaningless to place one type of expression where the other ought to go. The approach was short-lived; indeed, Russell seems to have abandoned it by the time *PoM* even appeared in print.

In general, Russell's philosophical scruples committed him throughout this period to embracing a single logical type, that of "individuals" or "logical subjects," encompassing all entities. His argument was that it is always inconsistent to hold that any sort of entity cannot be a logical subject of a proposition, since to do so is tantamount to endorsing a proposition of the form *A is not a logical subject*, whose very form ensures its own falsity (*PoM* 45–48).⁸ Indeed, even when Russell was willing to consider metaphysical types, he held that there must be combined types, including a type subsuming all objects; a concession making this early type-theory much more complicated than usual textbook formulations (e.g., that of Hatcher), and one that might even be utilized to reintroduce the paradoxes.⁹ As Russell himself admitted while offering his 1903

⁸He similarly dismissed Frege's distinction between objects and concepts as guilty of this kind of error (*PoM* 507–10, and in their correspondence: see Frege, *Correspondence* 134–38), no doubt anticipating Geach's observations much later ("Saying and Showing") to the effect that it does not seem possible to state Frege's theory of levels in a way that doesn't violate the theory itself.

⁹For explanation of this latter point, see, e.g., Bostock, "Russell on 'the' in

theory of types, “[t]he fact that a word can be framed with a wider meaning than *term* [individual, logical subject] raises grave logical problems” (*PoM* 55n).

Moreover, Russell did not at the time find the general line of approach suitable as a response to the other paradoxes, explicitly listing the propositions paradox as one it cannot solve. He noted in passing that it might be possible to meet the propositions paradox with the response that “[i]t is possible, of course, to hold that propositions themselves are of various types . . . But this suggestion seems harsh and highly artificial” (*PoM* 528). Nevertheless, in 1908’s “Mathematical Logic as Based on the Theory of Types,” after Russell had already taken up a different sort of line of response for the class paradox, he did consider a view according to which propositions needed to be sorted into various ramified orders. Properties of propositions would similarly need to be relativized to orders based on to what order of propositions they would apply to. A first-order proposition cannot involve quantification over propositions or properties at all, a second-order proposition would involve quantification over (at most) first-order propositions (or properties), and so on. While it might then be possible to define a new proposition for each property of propositions, the order of the proposition so defined would be too high to ask whether or not it has the property it involves, which blocks the diagonalization leading to the Cantorian propositions paradox.¹⁰ Here too, again, however, Russell’s attraction to the approach was short-lived. Almost precisely around the time he wrote “Mathematical Logic,” Russell began to explore

the Plural,” 117–18.

¹⁰When discussing the need for dividing propositions into ramified orders, Russell usually discusses contingent semantic paradoxes such as the Epimenides paradox. Manuscripts of the period, however, suggest that he had Cantorian paradoxes such as the one discussed previously in mind as well; see especially “The Paradox of the Liar.”

the possibility that there may be no propositions at all, apparently finding that approach more promising than the suggestion that there could be a whole hierarchy of different orders of them.¹¹

3 The Retreat from Pythagoras

In *My Philosophical Development*, written in the late 1950s, Russell described the evolution of his philosophy as a “retreat from Pythagoras.” Whereas he had once believed in a wide assortment of logical, mathematical, intensional and other abstract objects—including properties, classes, propositions, numbers, truth-functions, propositional functions, denoting concepts, etc.—as his views developed, he gradually moved towards a position stressing a “robust sense of reality” (*IMP* 135), in which only the simplest raw material of the empirical world was considered ultimately real, and which “swept away many apparent entities . . . [to] result [in] an outlook, which is less Platonic, or least realist in the mediæval sense of the word” (*PoM* 2nd ed., xiv): the abstracta he had formerly believed in reduced to “logical fictions” or mere *façons de parler*. This retreat was by no means overnight, and had its origins even earlier in Russell’s work, but was pushed on to a large extent by the desire to find uniform resolutions to the Cantorian paradoxes.

3.1 The Predication Paradox

The core of the position is evident already with Russell’s first proposed solution to one form of the properties or predication paradox.¹² In 1903, Russell used the word “predicate” not for anything

¹¹See especially, “On the Nature of Truth” (1907) and “Logic in which Propositions are not Entities.”

¹²Cantorian paradoxes of “properties” can be found in more than one form in Russell’s philosophy given the distinction between concepts, predicates or

linguistic, but for the ontological correlate of an adjective phrase, understood as a Platonic universal. He first stated, and proposed to solve, a version of the paradox involving predicates, thusly:

If x be a predicate, x may or may not be predicable of itself. Let us assume that “not-predicable of itself” is a predicate. Then, to suppose either that this predicate is, or that it is not, predicable of itself, is self-contradictory. The conclusion, in this case, seems obvious: “not-predicable of oneself” is not a predicate. (*PoM* 102)

Russell’s solution is neither to rescind the claim that predicates are individuals or logical subjects in propositions,¹³ nor to hold that in general that there are no propositions, even true propositions, to the effect that a certain definite predicate is not predicable of itself, e.g.:

universals on one hand, and propositional functions on the other. A fuller treatment of this topic would require differentiating these versions of the paradox, and discussing their different treatment by Russell at different points in his career. For further discussion, see Klement, “Origins.”

¹³Russell did eventually, under the influence of Wittgenstein, come to hold such a position—see *PLA* 67—but this was not a requirement of his theory of types, and he held the opposite view even after the publication of *Principia Mathematica*: see, e.g., “Analytic Realism,” 135. Even when Russell did hold such a view, however, he did not connect it directly with the paradoxes, and continued to maintain a distinction between “predicates,” entities involved in simple predications, and “properties” or “propositional functions,” where only the former were thought of as entities at all (e.g., *MPD* 166), a distinction which by itself solves the paradox for predicates, regardless of whether or not these entities are in a distinct type. Russell also expresses misgivings about the Wittgenstein-inspired view putting universals in a distinct type, writing in an (unpublished) 1921 letter to Moore, “there are difficulties in this view, beginning with the fact that it cannot be stated without apparent self-contradiction,” echoing his 1903 criticisms of Frege. For further discussion, see Klement, “Form Before Function.”

(1) Redness is not red.

Instead, he adopts a kind of ontological conservatism: not everything that appears as a well-formed, independently meaningful adjectival phrase actually corresponds to a single entity. We might rephrase (1) thusly:

(2) Redness is not predicable of itself.

Or even:

(3) Redness has the property of not being predicable of itself.

Russell concluded that the predicate phrases in these last two formulations, unlike in the first, do not represent any single entity.

Russell is not denying that the *parts* of the complex adjectival phrase “not predicable of itself,” like “not” or “itself,” have entities that they represent. There just isn’t any *one* object meant by the *entire phrase as a whole*. It may be a unit grammatically and syntactically, but according to Russell, this is no guarantee that it corresponds to a single, identifiable constituent of the corresponding proposition or fact. The make-up of the latter, one might say, is intelligible given only the “atoms” of meaning: the truth of (2) and (3) only need the entities involved in the truth of (1)—Redness (twice over), negation, and whatever corresponds to the copula—and not some additional complex property of non-self-instantiation. Without taking that property as a single, unified “thing” of which some properties hold and others don’t in its own right, the paradox cannot get off the ground. Although this solution is not quite as radical as some of the others discussed below, since Russell is not offering a wholesale “no properties” view, but only a “no complexly-defined properties” view, in many ways it is still a good indication of the trajectory of his thought, and how the paradoxes would shape his metaphysical views as a whole.

3.2 Class-Intensions and Descriptive Senses

The entities involved in what, in part I, were called “the class-intension paradox” and “the descriptive sense paradox” were understood early on by Russell as two different kinds of what he called “denoting concepts.” In 1905, Russell abandoned this earlier theory in favor of his celebrated theory of descriptions of “On Denoting.”¹⁴ This theory helps to block these paradoxes in a fashion that has many similarities to the earlier response to the predicates paradox. Whereas Russell had previously regarded the “all humans” part of “all humans are mortal” as an independently meaningful part, with a unified entity, the denoting concept *all humans*, occupying a discrete part of the proposition, Russell now puts forth a view according to which “all humans are mortal” is interpreted to mean:

$$(x)(x \text{ is human} \supset x \text{ is mortal})$$

Here, there is no unified part of the analysans corresponding to the phrase “all humans,” and hence, for Russell, the question doesn’t arise as to whether *the* thing that this phrase means, the class-intension, is, or is not, human. Yet, one can still make sense out of *most* discourse about “all humans”; doing so still requires thinking

¹⁴While Russell did often connect his work with solving the paradoxes in mathematical logic with his work on descriptions (see, e.g., *DRDJ* 79, *Auto* 150, *IMP* 136, etc.), the precise intellectual motivations for the theory of descriptions remain controversial among Russell scholars, and there isn’t overt evidence that paradoxes exactly like the class-intension or descriptive sense paradoxes were a direct factor. It is striking, however, that the difficult argument Russell explicitly gave in “On Denoting” against denoting complexes involved the problem of understanding the nature of propositions actually about denoting complexes themselves rather than what they denote, and notice that it is precisely the ability to speak of denoting complexes themselves as opposed to their denotations, i.e., to predicate properties of the complexes themselves, that is needed to make these paradoxes formulable.

of the word “human” as representing something in the proposition or state-of-affairs, just not the entire phrase “all humans.”¹⁵

The same considerations apply to definite descriptions, and so Russell’s theory of descriptions similarly spares him from the descriptive sense paradox. According to this theory, the proposition expressed by “the author of *Frankenstein* is a woman” is analyzed:

$$(\exists x)((y)(y \text{ authored } \textit{Frankenstein} \equiv y = x) \ \& \ x \text{ is a woman})$$

This respects the intuition that this proposition differs in meaning from “Mary Shelley is a woman,” but does so without postulating a distinct singular entity, a descriptive sense or denoting concept, that has or lacks properties on its own. Without thinking of descriptive senses as independent entities, the descriptive sense paradox poses no threat.

3.3 Russell’s No Classes Theory

Russell’s solution to the class paradox is still too often described as involving the kind of theory of types that separates reality into logical divisions, where individuals, classes of individuals, classes of classes of individuals, are all taken to be different kinds of beings about which the same things cannot meaningfully be said. In fact, however, this way of describing things is no more true for Russell’s mature theory of types than it was for his 1903 response to the predication paradox. Russell summarized his view around the time of 1910’s *Principia Mathematica* thusly:

¹⁵And it should be noted that nothing in the approach even requires that if the phrase is more complicated, such as with “all yellow horses” or “all predicates not predicable of themselves,” that we treat the antecedent of the analysans as involving a *single* property or constituent of the proposition; perhaps Yellowness and Horseness need to be constituents of the proposition expressed by $(x)(x \text{ is a horse} \ \& \ x \text{ is yellow} \supset x \text{ is tame})$, but Yellow-Horseness needn’t.

I have . . . discovered that it is possible to give an interpretation to all propositions which verbally employ classes, without assuming that there really are such things as classes at all. . . . That it is meaningless . . . to regard a class as being or not being a member of itself, must be assumed for the avoidance of a more mathematical contradiction; but I cannot see that this could be meaningless if there were such things as classes. (“Some Explanations” 357)

Instead, his response is akin to the other solutions we’ve considered. Expressions that seem to represent names of classes should not be taken at syntactic face-value. For Russell, class-terms, like definite description phrases, are dubbed, “incomplete symbols,” which means that they can be interpreted as making a contribution to the make-up of the meaning of sentences in which they appear, but without any one single “thing” constituting their meaning or semantic value in the proposition expressed or state of affairs represented. In *Principia Mathematica*, we find the following contextual definition for formulæ involving class abstracts, $\hat{z}(\dots z \dots)$:

$$A\{\hat{z}(\psi z)\} =_{Df.} (\exists \phi)[(z)(\phi!z \equiv \psi z) \& A\{\phi!\hat{x}\}]$$

A statement seemingly about a class can be rewritten with higher-order quantification along with whatever is involved in specifying the membership conditions of the class. The statement, “Socrates $\in \hat{z}(z \text{ is human})$ ” (or *Socrates is a member of the class of all humans*) becomes:

$$(\exists \phi)[(x)(\phi!x \equiv x \text{ is human}) \& \phi!(\text{Socrates})]$$

I.e., there is a predicative propositional function satisfied by all and only humans, and Socrates satisfies it. Discourse about “classes” is interpreted but without the assumption that the class term “ $\hat{z}(z \text{ is}$

human)” has its own independent meaning. (The rigamarole involving second-order quantification is a fancy way of ensuring that, unless embedded within a further intensional context, any sentence involving a class term “ $\hat{z}(\psi z)$ ” will remain true regardless of which of numerous coextensive properties is used.)

There is, however, an important disanalogy here between this and the solutions to other paradoxes sketched above. The theory of descriptions provides not a theory about what we, in our pre-paradox naïveté, would have taken to be a proposition *about* a descriptive sense. It rather provides us with a reinterpretation of propositions which we might have hitherto interpreted as utilizing such senses to speak about their referents. Indeed, the theory provides no reinterpretation of propositions about the senses themselves, although reflection on the new theory may convince us that no such reinterpretation is necessary. Because of this, it is not even in *danger* of providing us with a way of reconstructing the reasoning involved with the descriptive sense paradox in a way that might lead back to contradiction. However, the treatment of class-terms as incomplete symbols *does* provide a reinterpretation of what we would have previously, naïvely, have thought to be about classes. Hence, the question still arises as to whether, once we reinterpret the talk of “classes” involved, it is possible to translate the class paradox into a form where, despite not really involving classes, contradiction still results.

It is here and only here that the theory of “types” enters in to Russell’s solution. His contextual definition requires that the class abstract appear in a position that syntactically allows a higher-order or “propositional function” variable of the type that takes as argument values of the variable used within the class abstract (the “ z ” in “ $\hat{z}(\psi z)$ ”). In Russell’s system, propositional function variables are never of the same type as the variables for their possible arguments, and hence, a statement to the effect that a class is, or is

not, a member of itself is uninterpretable.¹⁶ This is the genesis of the apparent metaphysical division between classes of different types; really, however, “classes” are just a way of speaking, for Russell.

3.4 Higher-Order Quantification

The question still remains, however, whether or not the system of different types of propositional function variables and higher-order quantification commits Russell to metaphysical divisions of types of entities.

Russell’s precise understanding of propositional functions, and higher-order quantification generally, is an incredibly complicated topic, and controversial among Russell scholars. We shall not be able to do more than scratch the surface here.¹⁷ Russell’s own views changed. There was a period, from roughly 1902 through late 1905, during which Russell believed in a special category of entities corresponding to open sentences, entities which might be “named” by such expressions as “ \hat{x} is human \supset \hat{x} is mortal.” By 1906, however, Russell had come to the conclusion that “to assume a separable ϕ in ϕx is just the same, essentially, as to assume a class defined by $\phi x \dots$ ” and that by having “treated ϕ as an entity” he “brought back the contradiction” even after he had “thought [he] had solved the whole thing by denying classes altogether” (*DRDJ* 78).

At the time, he had hoped that he could treat propositional functions also as a mere *façons de parler* by replacing talk of propositional functions with talk of substitutions within propositions.

¹⁶A statement to the effect that a class whose “members” are coextensive propositional functions satisfies one or more of those propositional functions is similarly uninterpretable, blocking the equivalence class version of the paradox.

¹⁷For fuller discussion of the development of Russell’s views, see especially Klement, “Form Before Function” and “Russell’s 1903–05 Anticipation of the Lambda Calculus.”

The suggestion centers around a four-place relation, written

$$p/a; b!q$$

which means that q (typically, a proposition) results from the substitution of the entity b for a wherever a occurs as logical subject in (proposition) p . Rather than considering a function \hat{x} is human, one could utilize a “matrix” consisting of the proposition *Socrates is human* and Socrates. While maintaining only one kind of variable, the theory yielded results very similar to a simple type-theory, and Russell’s paradox is excluded because there is no way to represent a matrix taking “itself” as argument, because something like “ $p/a; p/a!q$ ” is ungrammatical. Russell’s main attraction to the theory was that it offered an explanation for what goes wrong with the paradoxes without positing different ontological types of entities.¹⁸

The approach didn’t last. It banished thinking of propositional functions as genuine objects but at the cost of necessitating thinking of propositions that way. Hence it made the propositions paradox—along with certain variants of it particular to the substitutional theory¹⁹—unsolvable along similar lines. Indeed, after abandoning it, Russell’s move was predictable: he concluded that propositions too should be understood as “logical fictions” or incomplete symbols. While “that Socrates is human” in the sentence, “Plato believes that Socrates is human,” may appear to constitute a syntactic unit and thereby suggest a unified thing it “names,” Russell now holds that only the individual words making up the clause are representative. Belief must be understood not as a dyadic relation between a believer and proposition believed, but rather as a “multiple relation” between a believer and the various components that would make

¹⁸See especially his “The Substitutional Theory of Classes and Relations,” and “On ‘Insolubilia.’”

¹⁹For further discussion, see Landini, *Russell’s Hidden Substitutional Theory*, chap. 3; Stevens, *Russellian Origins*, chap. 3.

up the corresponding fact were it true (*PM* 43–44, “Nature of Truth and Falsehood”). This “multiple relation” theory of judgment was also not long-lived in Russell’s philosophy, but it is telling that Russell never returned to a realism about “propositions” understood as mind- and language-independent intensional entities.

However, by abandoning propositions, Russell was forced to abandon the substitutional theory along with it, which left him without a clearcut way of making sense of higher-order quantification, which he took as necessary for mathematical logic. He returned to a vocabulary of “propositional functions”, but seemed wary of thinking of open sentences as representing a distinct kind of thing about which the same things cannot be said as of individuals. In an early draft of a section on “Types” of *PM*, Russell wrote:

A function must be an incomplete symbol. This seems to follow from the fact that $\phi!(\phi!\hat{z})$ is nonsense. The whole difficulty lies in reconciling this with the fact that a function can be an apparent [i.e., bound] variable.

Readers of *PM* provide different answers as to whether, or how, Russell reconciled this tension. According to one popular reading, Russell returned to a realist view of “propositional functions” as mind-independent complex entities of various metaphysical types.²⁰ On more sophisticated versions of this account, propositional functions are still not taken as ontologically on par with more basic entities such as particulars and universals (see esp. Linsky, *Russell’s Metaphysical Logic*, chap. 2). They are instead taken as “derived entities,” metaphysically constructed out of more basic stuff. For the construction to be possible, a propositional function cannot have as arguments anything presupposing the function itself, and hence cannot take itself as argument (cf. *PM* 48f).

²⁰See, e.g., Hylton, *Propositions, Functions and Analysis*, Linsky, *Russell’s Metaphysical Logic*, and Potter, *Reason’s Nearest Kin*.

According to a newer but increasingly popular account, which I favor, Russell’s mature account of “propositional functions” is that they are nothing more than open sentences, and that higher-order quantification in *Principia Mathematica* is to be understood in terms of linguistic substitutions in sentences, so that a quantified formula gets its truth-conditions in terms of the truth or falsity of the results of well-formed replacements one can make for the variable.²¹ Different “types” of variable correspond to substitutions of different kinds of expressions of various complexities (and “order” restrictions involving what kinds of further quantifiers are allowed in the legitimate substitutions for a variable of a certain kind, to ensure that the truth conditions so generated are non-circular).²² Ultimately, the truth or falsity of higher-order quantified statements is resolved recursively in virtue of the truth or falsity of lower-order statements, eventually terminating in basic forms such as elementary propositions, which only involve such “entities” as simple universals and particulars. This reading has in common with the view of propositional functions as derivative entities, then, the view that in some sense, truths apparently or actually about such things depend metaphysically on the more basic stuff, and the nature of this dependence rules out such apparent statements involving a propositional function taking “itself” as argument as having well-defined truth-conditions at all.

I cannot pretend to have fully argued for, or even fully explained, this general line of interpretation for Russell’s understanding of propositional functions in *PM*, but I do think it is in keeping with the general approach to paradox-dissolution Russell had applied

²¹I am, in effect, here suggesting that Russell’s held a “substitutional” theory of higher-order quantification; for interpretations along these lines, see Sainsbury, *Russell*, Landini, *Russell’s Hidden Substitutional Theory*, Stevens, *Russellian Origins*, and Klement, “Form Before Function.”

²²This is in essence the hierarchy of truth and falsity of *PM* 42ff.

elsewhere. Fuller discussion must be left for another occasion. I mention here only that Russell claimed later that “In the language of the second order, variables denote symbols, not what is symbolized” (*IMT* 192) and that “Whitehead and I thought of a propositional function as an expression” and indeed as “nothing but an expression” (*MPD* 62, 69). In *The Philosophy of Logical Atomism* (96), he wrote that, “a propositional function is nothing, but, like most things one wants to talk about in logic, it does not lose its importance through that fact.”

If this reading is right, then, for Russell, considered extra-linguistically, propositional functions are nothing. Classes are nothing. (And since Russell defined relations-in-extension, numbers and other mathematical structures as classes, they too, are nothing.) Descriptive senses and/or denoting concepts are nothing. In short, every category of entity one is tempted to imagine populated in sufficient numbers so as to transgress the Cantorian lessons he (and Frege) had learned so well is a category of nothing. Ockham has triumphed, and Pythagoras has retreated. But notice that Russell’s advocacy in favor of ontological parsimony is not a simple-minded attitude of “fewer is better” or “simplicity is preferable to complexity.” Russell was convinced that the Cantorian paradoxes were unsolvable in a uniform and non-*ad hoc* way if the reality of these would-be entities were taken at face-value. Notice, moreover, that Russell’s eschewal of these abstract entities did not take a form of a straightforward reduction in which it is admitted that these are things, just not “fundamental things.” According to Russell, thinking of propositions, classes, denoting concepts, etc. as things at all is an illusion created by the surface grammar of ordinary language. Instead, he offered a replacement locution in which terms for such “abstract things” could be wholly eliminated in terms of logical forms and/or some stock of primitive expressions for the concrete, non-complex, beings of the empirical world and their properties

and relations: the “logical atoms” of logical atomism.

4 Conclusion

Bertrand Russell engaged in a decade-long struggle with various forms of Cantorian diagonal paradoxes. Growing from it was his logical atomism,²³ whereupon “none of the raw material of the world has smooth logical properties, but that whatever appears to have such properties is constructed artificially in order to have them” (*PoM* 2nd ed., xi). This seems to me to be a remarkable moment in the history of philosophy, and one which, even a few generations later, we have yet to appreciate fully.

In 2009, no one would take a formulation of set theory seriously if it did not offer any kind of response to Russell’s paradox.²⁴ Similarly, no one would take a theory of truth to be complete until it had something to say about the liar paradox and its generation from Tarski’s T-schema. It strikes me as highly odd that so many philosophers feel no compunction at all about positing the reality of abstract or intensional entities such as (Fregean or quasi-Fregean) senses, concepts, intensions and propositions, without even addressing the possibility of Cantorian contradictions.²⁵ Nor should the investigation of such matters be viewed as an annoyance or a hindrance to discovery in the areas of intellectual thought where senses, concepts, intensions, and the like may be of theoretical use. It is sometimes said that the “foundational crisis” in set theory, sparked

²³Russell says (*PLA* 35) that logical atomism is a position that “forced itself upon” him “while thinking about the philosophy of mathematics” but demurs from the hardline position that one position entails the other.

²⁴This is true even if that response amounted to something like embracing the contradiction while containing its ill-effects, as with certain paraconsistent logics.

²⁵A similar point is made recently by Harry Deutsch in his review of J. C. King’s recent book.

by the recognition of the paradoxes, fueled the growth and development of set theory as a branch of mathematics throughout the 20th century. I fear we have not given “the ways of paradox” with regard to other areas of abstract philosophy enough of a chance to do the same. We may end up with a better sense of what these entities could or might be, or we might end up (as Russell would have us) concluding that thinking of them as “things” at all is misguided, but that realization may (as I think Russell hoped) not come at the price of the insights engendered by the theories that originally prompted us to postulate the existence of such entities.

Even when narrowly focused on the philosophy of mathematics and set/class-theory, where awareness of the possibility of Cantorian paradoxes is still kept at the forefront of researchers’ minds, there are lessons to be drawn from Russell’s engaged assault on them. Russell despaired to Phillip Jourdain in 1907, no doubt thinking of the writings of Zermelo and von Neumann, that “I have given up expecting much of solutions [of the set-theoretical paradoxes]” (*DRDJ* 54). The “solutions” offered within their approaches, still dominant if not hegemonic today, were rejected by Russell, and for compelling reasons that have scarcely been discussed since. Russell’s own proposed solution to the class-theoretic paradoxes has, I think, not been well understood or thoroughly evaluated as an alternative. Even where there is, today, movement towards the creation of a new kind of philosophically minded set-theory, as within the neo-logicist movement, Russell’s name usually takes a backseat to Frege’s, or others’.²⁶ (The Russellian notion of a logical construction or incomplete symbol provides a compelling alternative to the metaphysics of abstract entities found within this literature.) The familiarity of those works by Russell cited within the “canon” of

²⁶Although on the positive side, at least there seems to be some engagement with the Russellian notion of indefinite extensibility; see, e.g., Shapiro, “Prolegomenon”; Shapiro and Wright “All Things”.

analytic philosophy, unfortunately, I think, gives the impression that his work has already been thoroughly pillaged of its sources of philosophical inspiration. This impression is, I think, however, thoroughly mistaken.

Acknowledgements

Thanks to an anonymous referee for helpful comments on a previous draft; indeed, space constraints have prevented me from doing them full justice.

References

- Boolos, George. “Saving Frege from Contradiction.” *Logic, Logic and Logic*. Cambridge, Mass.: Harvard University Press, 1998. 171–82.
- Bostock, David. “Russell on ‘the’ in the Plural.” *Russell vs. Meinong: The Legacy of “On Denoting”*. eds. Dale Jacquette and Nicholas Griffin. New York: Routledge, 2008. 113–43.
- Cocchiarella, Nino. *Logical Studies in Early Analytic Philosophy*. Columbus: Ohio State University Press, 1987.
- Deutsch, Harry. “Review of J. C. King, *The Nature and Structure of Content*.” *Notre Dame Philosophical Reviews*. <http://ndpr.nd.edu/review.cfm?id=13165> (2008).
- Geach, Peter. “Saying and Showing in Frege and Wittgenstein.” *Acta Philosophica Fennica* 28 (1976): 54–70.
- Grattan-Guinness, Ivor, ed. *Dear Russell–Dear Jourdain (DRDJ)*. New York: Columbia University Press, 1977.

- Hatcher, William. *The Logical Foundations of Mathematics*. Oxford: Pergamon Press, 1981.
- Hylton, Peter. *Propositions, Functions and Analysis*. Oxford: Clarendon Press, 2005.
- Klement, Kevin. "Russell's 1903–05 Anticipation of the Lambda Calculus." *History and Philosophy of Logic* 24 (2003): 15–37.
- . "Putting Form Before Function: Logical Grammar in Frege, Russell and Wittgenstein." *Philosopher's Imprint* 4 (2004): 1–47.
- . "The Origins of the Propositional Functions Version of Russell's Paradox." *Russell* 24 (2005): 101–32.
- . "A Cantorian Argument Against Frege's and Early Russell's Theories of Descriptions." *Russell vs. Meinong: The Legacy of "On Denoting"*. eds. Dale Jacquette and Nicholas Griffin. New York: Routledge, 2008. 65–77.
- Landini, Gregory. *Russell's Hidden Substitutional Theory*. Oxford: Oxford University Press, 1998.
- Linsky, Bernard. *Russell's Metaphysical Logic*. Stanford: CSLI Publications, 1999.
- Potter, Michael. *Reason's Nearest Kin: Philosophies of Arithmetic from Kant to Carnap*. Oxford: Oxford University Press, 2000.
- Russell, Bertrand. *The Principles of Mathematics (PoM)*. Cambridge: Cambridge University Press, 1903 (second edition 1931).
- . "On Denoting." *Mind* 14 (1905a): 479–93.
- . "On the Nature of Truth." *Proceedings of the Aristotelian Society* 7 (1907): 28–49.

- . "Mathematical Logic as Based on the Theory of Types." *American Journal of Mathematics* 30 (1908): 222–262.
- . "On the Nature of Truth and Falsehood." *Philosophical Essays*. New York: Longmans & Green, 1910a.
- . "Some Explanations in Reply to Mr. Bradley." *Mind* 19 (1910b): 373–378.
- . *Introduction to Mathematical Philosophy (IMP)*. London: Allen & Unwin, 1919.
- . *An Inquiry into Meaning and Truth (IMT)*. London: Allen & Unwin, 1940.
- . *My Philosophical Development (MPD)*. London: Unwin, 1958.
- . "On Some Difficulties in the Theory of Transfinite Numbers and Order Types." *Essays in Analysis*. New York: George Braziller, 1973 (first published 1905)b. 135–64.
- . "On 'Insolubilia' and their Solution by Symbolic Logic." *Essays in Analysis*. New York: George Braziller, 1973 (first published 1906)a. 190–214.
- . "The Substitutional Theory of Classes and Relations." *Essays in Analysis*. New York: George Braziller, 1973 (written 1906)b. 165–89.
- . *The Philosophy of Logical Atomism (PLA)*. Chicago: Open Court, 1985.
- . "Analytic Realism." *Collected Papers of Bertrand Russell v6*. London: Routledge, 1992 (first published 1911). 132–47.

- . *The Collected Papers of Bertrand Russell v3*. London: Routledge, 1993.
- . *The Collected Papers of Bertrand Russell v4*. London: Routledge, 1994.
- . “Logic in which Propositions are not Entities.” *Unpublished manuscript*. Bertrand Russell Archives, McMaster University (forthcoming-a).
- . “The Paradox of the Liar.” *Unpublished manuscript*. Bertrand Russell Archives, McMaster University (forthcoming-b).
- . “Types [draft].” *Unpublished manuscript*. Bertrand Russell Archives, McMaster University (forthcoming-c).
- Shapiro, Stewart. “Prolegomenon to Any Future Neo-Logicist Set Theory: Abstraction and Indefinite Extensibility.” *British Journal of the Philosophy of Science* 54 (2003): 59–91.
- Shapiro, Stewart and Wright, Crispin. “All Things Indefinitely Extensible.” *Absolute Generality*. eds. Agustín Rayo and Gabriel Uzquiano. Oxford: Oxford University Press, 2006. 255–304.
- Stevens, Graham. *The Russellian Origins of Analytical Philosophy*. London: Routledge, 2005.
- Whitehead, A. N. and Russell, Bertrand. *Principia Mathematica (3 vols)*. Cambridge: Cambridge University Press, 1910–1913 (second edition 1925–1927).