

# Russell, His Paradoxes, and Cantor's Theorem: Part I

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## 1 Introduction

In 1961, W. V. Quine described a philosophical method he dubbed “The Ways of Paradox.” It begins with a seemingly well-reasoned argument leading to an apparently absurd conclusion. It continues with careful scrutiny of the reasoning involved. If we are ultimately unwilling to accept the conclusion as justified, the process may end with the conclusion that, “some tacit and trusted pattern of reasoning must be made explicit and be henceforward avoided or revised” (“Ways” 11). This method perfectly describes Bertrand Russell's philosophical work, especially from 1901–1910, while composing *Principia Mathematica*.

Russell is most closely associated with the class-theoretic anti-nomy bearing his name: the class of all those classes that are not members of themselves would appear to be a member of itself if and only if it is not. This is one of a large collection of paradoxes Russell discovered or considered that shaped his subsequent philosophy. Many, if not most, stem from violations of Cantor's powerclass theorem, the result that every class must have more subclasses than members. Together, they led Russell to be increasingly wary, not only of implicit reasoning involving class existence, but also of the very practice of taking apparent reference to mathematical, abstract, or logically complex “things” at face-value.

In this, the first article in a series of two, we discuss Cantor's powerclass theorem, and how it can be used to generate paradoxes. We then summarize a number of paradoxes thereby generated, either explicitly or implicitly considered by Russell himself. We conclude with a brief summary of the various kinds of solutions they might be given. In the sequel article, the impact of these paradoxes on Russell's own philosophy, and his views about their proper solution, are explored in more detail.

## 2 Cantor's Powerclass Theorem, Russell's Paradox and Frege's Lesson

Cantor's powerclass theorem, also known as the powerset theorem or just "Cantor's theorem," is the widely-accepted result that every class or collection of things can be divided into more subgroups or subclasses than it has members. The "powerclass" of a class is the class of all its subclasses, so the theorem asserts that the powerclass of a class is always larger in size (cardinality) than the class itself. Georg Cantor established this result in 1891 with the following argument. Every class  $c$  has at least as many subclasses as members, since for each member  $a$ , the class of  $a$  alone is a subclass of  $c$ . The core of Cantor's argument involves showing that there cannot be *equally* many subclasses as members. Suppose, for *reductio ad absurdum*, that there were. In that case, the members and subclasses could be paired off so there would be a one-one function,  $f$ , mapping each subclass  $s$  of  $c$  to a distinct member of  $c$ , which we can call  $f(s)$ . Some members might be in the subclass they are mapped from, others not. If  $s = \{a\}$ , for a given subclass  $s$  and member  $a$ , and  $a$  happens to be the object  $f(s)$  that  $s$  is mapped onto, then  $a$  is a member of its corresponding class, but not so if  $a \neq f(s)$ . Consider then the class,  $w$ , consisting precisely of the members of  $c$  that are *not* members of the subclasses that map to them. Since  $w$  is itself a subset of  $c$ , it must be included in the mapping. Hence, there's some member  $r$  of  $c$  such that  $f(w) = r$ . Consider now whether or not  $r \in w$ . We defined  $w$  as the class of all members  $a$  of  $c$  that are not in the class  $s$  such that  $a = f(s)$ , so in the case of  $r$ , which is  $f(w)$ ,  $r \in w$  just in case  $r \notin w$ , which is a contradiction. Cantor concluded that there can be no such one-one function from subclasses of  $c$  to members, and therefore, that there must be more subclasses.

If the class in question is finite or denumerable, Cantor's *re-*

*ductio* reasoning can be represented in tabular form.<sup>1</sup> Arrange the members,  $a_0, a_1, a_2, a_3 \dots$  of  $c$  horizontally, and arrange the subclasses,  $s_0, s_1, s_2, s_3 \dots$ , vertically utilizing the ordering of their corresponding members, so that  $f(s_0) = a_0, f(s_1) = a_1$ , etc. Place a checkmark where the chart intersects for the row of a subclass and the column for any member of it. For example:

	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	...
$s_0$	✓	✓			✓		...
$s_1$				✓		✓	...
$s_2$		✓	✓	✓	✓		...
$s_3$	✓	✓		✓		✓	...
$s_4$				✓		✓	...
$s_5$	✓		✓		✓	✓	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

The problematic subclass,  $w$ , is generated by moving along the chart, diagonally, from the upper left, downwards and to the right, and including precisely those members of  $c$  that do *not* have checkmarks where the diagonal passes through their columns. In the example, this would include  $a_1$  and  $a_4$ , in whose column no checkmark is found along the diagonal, but not the others. Notice, however, that since  $w$  itself should be represented by a row, where the diagonal passes through it, there ought to be a checkmark just in case there is no checkmark, which is clearly impossible. Cantor's method of proof here is therefore called a *diagonal argument* or *diagonalization*.

This reasoning is validated within most forms of set theory, and is difficult to counter. However, it is not completely incon-

<sup>1</sup>Even when the class isn't denumerable, it is often worthwhile to imagine imperfectly the resulting chart abstractly anyway, just as a heuristic. The fact that such a chart isn't technically possible doesn't invalidate the core argumentative strategy.

trovertible. In particular, the supposition that  $w$  corresponds to a well-defined subclass of  $c$  might be open to doubt, since it is defined in terms of a function whose domain is  $c$ 's powerclass, and perhaps there is a vicious circle in this if  $w$  is to be included in that very range. More on this below.

Russell's initial reaction to Cantor's theorem was to regard it as guilty of error.<sup>2</sup> Cantor himself concluded from the theorem that there was no greatest cardinal number, since for any number of things, the number of their subclasses would be greater. Russell and others have regarded this as paradoxical (and indeed the problem here has sometimes been called "Cantor's paradox" or "the paradox of the greatest cardinal"). Certain classes—such as the universal class containing everything, or the class of all classes—it would seem, cannot have more subclasses than members, because all their subclasses *are* members. Indeed, at first blush, it scarcely seems possible that any collection could be larger in size (cardinality) than such huge classes as the universal class or class of all classes. Russell attempted to contravene the alleged impossibility of mapping each subclass of the class of all classes to a member by mapping each subclass to *itself*, i.e., letting  $f(s)$  be  $s$  itself. Cantor's diagonal class  $w$  is then the class of all classes of classes not included in *themselves*. This class too is mapped to itself, and a contradiction results by asking if it is a member of itself. Drop the assumption that  $w$  need only contain those classes *of classes* that are not members of themselves, and this becomes Russell's paradox in its famous form. Russell was explicit in many places that Cantor's theorem was his inspiration.<sup>3</sup> Russell soon communicated it to Giuseppe Peano and Gottlob Frege, whose logical systems it rendered inconsistent.

<sup>2</sup>See Russell, *Papers* v3, xxxii, and "Mathematics and the Metaphysicians"; for discussion of the historical details, see Coffa, "Humble Origins," and Griffin, "Prehistory."

<sup>3</sup>See e.g. *PoM* 101, *IMP* 136, *MPD* 158, *Auto* 150.

Cantor's diagonalization method generalizes beyond mappings involving classes or sets. Given certain assumptions about the nature of properties (or predicates, attributes, universals, etc.), it establishes that the number of properties applicable (or not) to a certain logical kind of thing must always exceed the number of things of that kind. If we reconceive  $a_0, a_1, a_2, a_3 \dots$  in the chart as the things of the kind in question, and  $s_0, s_1, s_2, s_3 \dots$  as properties applicable to them, and view the checkmarks as indicating which things instantiate which properties, we are prompted to ask whether or not there is such a property as *not instantiating the corresponding property in the mapping*. If so, it should be included in the mapping, but then the object that corresponds to it instantiates it if and only if it does not. Here, however, the supposition that there must be such a property, merely because we seem able to describe its instantiation conditions, is even more open to doubt.

Gottlob Frege's reaction to the inconsistency in his logical system, published in an appendix to volume II of his *Grundgesetze*, usefully illustrates the matter. Rather than "properties," Frege spoke of what he called "concepts," understood as a kind of function from objects to truth-values. Thinking of these functions extensionally, Frege equated concepts satisfied by all and only the same objects.<sup>4</sup> Frege traced the presence of Russell's paradox in his system to his Basic Law V, which could be written:<sup>5</sup>

$$(\dot{\alpha} F(\alpha) = \dot{\alpha} G(\alpha)) \equiv (x)(F(x) \equiv G(x))$$

This states that the extension of the concept  $F( )$ , or class of all  $F$ s, is identical to the extension of the concept  $G( )$ , or the class

<sup>4</sup>Frege would invoke his sense/reference distinction to explain away apparent problems with equating coextensive concepts; see his "Comments on Sense and Reference."

<sup>5</sup>Here we allow ourselves to Russellize Frege's notation somewhat, and restrict our focus to concepts as opposed to other functions.

of all  $G$ s, if and only if, all and only  $F$ s are  $G$ s. Frege understood the notation “ $\dot{\alpha}(\dots\alpha\dots)$ ” as representing a “second-level function,” a function that takes a concept as argument and returns an object as value. Originally, it was to stand for the function that takes a concept as argument, and returns as value its corresponding class, or extension. In the appendix, Frege argues that the left-to-right half of this biconditional must come out false, *regardless* of what second-level function “ $\dot{\alpha}(\dots\alpha\dots)$ ” is taken to represent. While not explicitly presented as such, the reasoning is straightforwardly Cantorian. Rewrite “ $s_0, s_1, s_2, \dots$ ” in the chart above as “ $F_0(\ ), F_1(\ ), F_2(\ ), \dots$ ” for different concepts, and rewrite “ $a_0, a_1, a_2, \dots$ ” as “ $\dot{\alpha}F_0(\alpha), \dot{\alpha}F_1(\alpha), \dot{\alpha}F_2(\alpha), \dots$ ,” and the connection becomes clear. If it were possible to map concepts to objects to yield distinct objects for distinct (i.e. non-coextensive) concepts, then, by what amounts to diagonalization, we could always produce a contradiction: just consider the concept of *being an object in this mapping that does not fall under the concept from which it’s mapped*, i.e., the concept of being an  $x$  such that:<sup>6</sup>

$$(\exists F)(x = \dot{\alpha}F(\alpha) \ \& \ \sim F(x))$$

The object that results by applying the function  $\dot{\alpha}(\dots\alpha\dots)$  to *this* concept would be such as to fall under that concept if and only if it does not. This is equally true whether “ $\dot{\alpha}(\dots\alpha\dots)$ ” is interpreted to yield the “extension” of the concept to which it is applied, or whether it yields some *other* kind of object that would be different for different or non-coextensive concepts.

Let us call the result that it is impossible to generate a mapping from concepts, properties or (propositional) functions to objects that results in distinct objects for different or non-coextensive properties, “Frege’s lesson.” We might be tempted simply to take

<sup>6</sup>Notice that if  $x$  is a class, and “ $\dot{\alpha}F(\alpha)$ ” is interpreted to mean “the class of  $F$ s” then this precisely gives the condition for  $x$ ’s not being a member of itself.

this lesson in stride, except that there are many cases in which it *seems* possible to generate a distinct object for distinct properties. In such cases, one must either react by explaining why the initial impression that a distinct object can be generated for each property was mistaken, or explain how this might be possible without diagonalization leading to contradiction.

### 3 A Plethora of Paradoxes

Russell felt the impact of Frege’s lesson early.<sup>7</sup> In a Sept. 1902 letter to Frege, Russell despaired that “from Cantor’s proposition that any class contains more subclasses than objects we can elicit constantly new contradictions.”<sup>8</sup> It is worth listing several examples.

The first we have already discussed:

**Intuition 1.** Obviously for any two properties that are not coextensive it is possible to generate distinct objects: their extensions or corresponding classes!

**Diagonalization Result: Russell’s Class Paradox.** Consider the property an extension has just in case it does not have its defining property (or, equivalently, is not a member of itself). This property has its own distinct extension, but that extension has that property just in case it does not.

<sup>7</sup>Indeed, he was aware of the main gist prior to reading Frege’s appendix—see *PoM* 103, and his letter to Frege of 24 July 1902, in Frege, *Philosophical and Mathematical Correspondence* 139.

<sup>8</sup>See Frege, *Correspondence* 147. Unfortunately, Frege does not seem to have fully appreciated the importance of the paradox Russell went on to describe, and that it threatened his philosophy as much as it did Russell’s; see Klement, “Russell’s Paradox in Appendix B,” and *Frege and the Logic of Sense and Reference*, chap. 6.

Many philosophers believe that properties or concepts can be considered objects or logical subjects in their own right. Those that do must ponder the following:<sup>9</sup>

**Intuition 2.** Obviously for any two properties (whether coextensive or not) it is possible to generate distinct objects, viz., the properties themselves.

**Diagonalization Result: Russell’s Predication Paradox.** Consider the property a property has just in case it does not instantiate itself. Does it, as an object, instantiate itself? It does just in case it does not.

Russell’s early ontology included “propositions” understood as mind-independent complex entities, the bearers of truth or falsity. Many other philosophers believe in similar intensional entities, though with widely varying details and vocabularies (e.g., thoughts, states-of-affairs, possible facts, belief-contents, etc.) Consider:

**Intuition 3.** There are as many propositions as there are properties thereof. For each property of propositions, one can generate a distinct proposition, such as the proposition that every proposition has that property, or the proposition that all propositions with that property are true.

**Diagonalization Result: The Propositions Paradox.** Fasten on any one of these mappings: take the latter. Consider the

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<sup>9</sup>Depending on what we interpret “properties” here to mean, Russell discusses different interpretations of this paradox in different places. Interpreted to mean what early Russell called “predicates,” by which he meant something like Platonic universals, it occurs in *PoM* (80, 102). Interpreted to mean what he called “propositional functions” it occurred only later. See Klement, “Origins” for discussion of the difference.

property,  $\phi$ , a proposition in this mapping has when it doesn’t have the property of propositions of which it asserts all instances are true. For example, the proposition *all atomic propositions are true* is not itself an atomic proposition, so it has  $\phi$ ; whereas the proposition *all true propositions are true* is itself true, so it does not have  $\phi$ . Consider then the proposition *all propositions with  $\phi$  are true*: does it have  $\phi$ ? It does just in case it does not.<sup>10</sup>

Frege, famously, but also many other philosophers, including Russell prior to “On Denoting,” believe in special abstract “semantic” objects: senses, meanings, individual concepts, denoting complexes, and so on. At least *some* (and perhaps all) of these entities are understood as picking out their “referents” or “denotations” in virtue of their unique possession of some property or other.

**Intuition 4.** There are as many descriptive senses as there are properties. For each property, we can generate a descriptive sense that picks out as denotation whatever object (if any) uniquely holds that property. While the object picked out may be the same for descriptive senses generated from distinct properties, the descriptive sense itself would be different for different properties.

**Diagonalization Result: The Russellian Descriptive Sense Paradox.** Consider the property,  $H$ , which a descriptive sense

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<sup>10</sup>A version of this paradox, dealing simply with classes of propositions rather than properties of propositions was formulated by Russell in *PoM* (527–28). Russell also formulated it in terms of “propositional functions” instead of classes of propositions in a letter to Frege (see Frege, *Correspondence* 159–60). Antinomies of this form were independently rediscovered by John Myhill in the context of evaluating certain later forms of intensional logic; see, e.g., Myhill, “Problems,” Anderson, “Semantic Antinomies,” and Klement, “The Number of Senses” and “Does Frege Have Too Many Thoughts?” It is sometimes called “the Russell-Myhill Antinomy.”

has when it lacks the property in virtue of which it presents a denotation, if any. The sense *the author of Waverly* did not write *Waverly*; hence, it has *H*. On the other hand, the sense *the self-identical thing* is a self-identical thing, so it lacks *H*. Now consider the sense *the H*; does it have *H*? It does just in case it does not.<sup>11</sup>

By a slight variation, we could consider an old fashioned “intension” understood as a semantic entity that represents the entire class of things having a certain property, rather than just *the* thing having it, as above. Early Russell called these “concepts of a class,”<sup>12</sup> but I shall call them “class-intensions” instead.

**Intuition 5.** There are as many class-intensions as there are properties. For each property, there is a class-intension that represents the class of things having that property. While the corresponding extension or class may be the same for different class-intensions, the class-intensions themselves are different for different properties.

**Diagonalization Result: The Russellian Class-Intension Paradox.** Consider the property, *K*, a class-intension has when it lacks the property it uses to collect together its class, if any. The class-intension *(all) teaspoons* is not a teaspoon; hence, it has *K*. The other hand, the class-intension *(all) class-intensions* is a class-intension, so it lacks *K*. Now consider the class intension *(all) class-intensions having K*; does it have *K*? It does just in case it does not.

<sup>11</sup>A paradox of this form is discussed at greater length in Klement, “Cantorian Argument.”

<sup>12</sup>In *PoM* (67), Russell distinguishes the “concept of a class,” *all humans*, from the “class-concept” *human*. The difference is subtle, and we could generate a Cantorian paradox from either one, though I think that the “class-concept” is really just the property itself, and so the resulting paradox is just the predication paradox.

(If class-intensions simply *are* properties, then this paradox collapses into the predication paradox above.)

We needn’t necessarily generate a distinct intension for each property defining a class; it’s enough to generate *one* corresponding to the equivalence class of coextensive properties, since it still would hold that we’d get different ones for non-coextensive properties. Indeed, a version of the paradox could be formulated dealing with that equivalence class itself:

**Intuition 6.** We can map properties to equivalence classes of properties where the associated equivalence relation is coextensionality. For any two non-coextensive properties, the equivalence class to which they would be mapped would be different. The property of *having a heart* would be mapped to the same equivalence class as the property of *having a kidney*, but not to the same equivalence class as the property of *being a featherless biped*; though the latter would be mapped to the same equivalence class as *being human*.

**Diagonalization Result: The Russellian Property Equivalence Class Paradox.** Consider the property  $\psi$  that an equivalence class of coextensive properties has just in case it doesn’t have any (—or, if you prefer, *all*, since they’re coextensive—) of the properties it contains. Now, consider the equivalence class of all properties coextensive with  $\psi$ : does it have  $\psi$ ? If it does have  $\psi$ , then it doesn’t have  $\psi$ , since it’s one of the properties in the equivalence class. If it does not have  $\psi$ , then it must have at least one property coextensive with  $\psi$ , in which case, it must have  $\psi$  as well—so we get a contradiction either way.

Other examples could be given, but the above suffice to establish the general pattern of how Cantor’s theorem, or, more specifically, “Frege’s lesson,” generates Russell-style paradoxes almost *ad*

*nauseam*.<sup>13</sup> One needs only mention a category of entity—most likely, an abstract entity—that can be correlated or related to properties (or classes) in a systematic way<sup>14</sup> and where the identity conditions are fine-grained enough that the entities correlated with non-coextensive properties can be distinguished.

## 4 Kinds of Solutions

In a 1905 paper entitled “On Some Difficulties in the Theory of Transfinite Numbers and Order Types,” largely dedicated to Russell’s paradox, Russell identified three broad approaches for finding a solution. It is fair to say that most contemporary approaches can still be seen as falling under one of these categories, though we shall discuss some exceptions below. Responses to the other Cantorian paradoxes can be sorted under roughly the same headings. The three categories, as Russell dubbed them, are (1) the theory of limitation of size, (2) the “zigzag theory” and (3) the no classes theory. We discuss these in turn.

### 4.1 Theories of Limitation of Size

Consider, again, those classes that led Russell to suspect an error in Cantor’s proof: the universal class, and the class of all classes.

<sup>13</sup>There are, to be sure, other important paradoxes in the same family that don’t fit quite as well into the rubric provided by our previous discussion, such as the paradox of relations Russell discusses in “Mathematical Logic” (222–23), the paradox Kaplan discusses in “A Problem in Possible World Semantics,” or what is called the “class/sense paradox” in Klement, “The Number of Senses.” The puzzle called “Cantor’s paradox” (see sec. 2) concerning whether or not there is a greatest cardinal number is of course another paradox related to Cantor’s theorem that doesn’t neatly fit this rubric.

<sup>14</sup>Above, we often say “generated from” but this metaphor should not be taken too seriously.

Cantor himself called such things “inconsistent multiplicities” (in a letter to Dedekind) meaning that their size is too large for them to be considered “one thing.” Axiomatic set theories now prevalent among mathematicians, such as Zermelo-Frænkel (ZF) set theory, also disavow the existence of a universal set, or set of all sets. This is keeping with an “iterative” conception of a set, whereupon sets are thought to be built up out of successive applications of powerset and union operations.<sup>15</sup> More complicated theories, such as von Neumann-Bernays-Gödel (NBG) set theory, distinguish sets from “proper classes,” where the latter are considered too large to be members of any set or class. Here, while there may be a class of all sets, it is a proper class, and hence not a member of itself, nor are those subclasses which are also proper classes members. Insofar as it has a “powerclass” at all, it would only contain *sets* that are subclasses of it, not all subclasses whatever. Even here, then, there is no class of *all* classes, both improper and proper.

These theories escape contradiction by denying that a distinct object can be generated for *every* property or characteristic of sets (or at least classes, in the case of NBG). Those properties that are true of “too many things” have no class (or no distinctive class) associated with them. There are only as many classes as there are properties that aren’t too widespread. This general line of avoiding inconsistency is perhaps clearer in the case of the “Limitation of Size”-based set theory developed more recently by George Boolos (“Saving Frege” and elsewhere), formulated in a second-order logic, where Frege’s Basic Law V is replaced with (New V):

$$(\dot{\alpha} F(\alpha) = \dot{\alpha} G(\alpha)) \equiv ((Big(F) \& Big(G)) \vee (x)(F(x) \equiv G(x)))$$

This commits us to as many classes (Boolos calls them “subtensions”) as there are non-big properties. Boolos defines a big prop-

<sup>15</sup>This way of describing things derives from Boolos, “The Iterative Conception.”

erty as one that is instantiated by as many things as there are things—though not necessarily by all things. Whether one adopts Boolos’s proposal or a related one, the condition of not being a member of oneself, “ $x \in x$ ,” is thought to represent a property that holds of too many things to have a distinctive class associated with it, and hence, according to this approach, there is no such class as that which would be involved in Russell’s class paradox. It also denies the existence of a universal class or class of all classes, thereby escaping the worries Russell initially entertained about Cantor’s theorem.

The limitation-of-size approach has not been pursued much, or as directly, with regard to the other paradoxes listed in the previous section. Indeed, it is not entirely clear how to extend this approach to them in a plausible way. We shall return to this issue in the sequel article.

## 4.2 The Zigzag Theory

The previous approach requires that, contrary to our “intuitions,” it’s untrue, after all, that we can generate a new object for *every* property (or every subclass) of our original group of things; for those that apply to too many things, there is no distinct associated object of the type suggested by the “intuition.” The zigzag approach works differently. It grants the “intuition” that for every property—or at least, for every *unexceptional* property—we can generate a distinct object. However, it denies that for every grammatically well-formed condition, we have the kind of unexceptional property to which the intuition correctly applies. In particular, the conditions used to specify the would-be diagonal subclasses or properties are thought to be exceptional or problematic in some sense, and that this undermines the diagonal reasoning behind Cantor’s theorem.

Recall that Cantor’s argument begins by assuming that a one-one mapping exists between subclasses of  $c$  and members of  $c$ , and then uses that very mapping to define *another* subset  $w$  of  $c$  which, it is alleged, cannot be included in the mapping. This is because nothing in the mapping *could* be *it*, given how  $w$  is described. The argument concludes the mapping does not exhaust the subsets of  $c$ . One might counter by questioning whether or not just any description of a subclass of  $c$  necessarily corresponds to a genuine subclass of  $c$ . In effect, one could exploit the impossibility of  $w$ ’s occurring in the map in question in the other direction, arguing that there can be no such subclass as  $w$ . The so-called subclass that would be generated from reversing the arrangement of checkmarks along the diagonal is no actual subclass at all, but merely an empty description to which nothing need answer. To provide a complete solution, one would need to specify conditions under which a description of a subclass (i.e., specification of conditions for inclusion in that subclass) of a given class can or cannot be guaranteed to define a subclass.

Apart from Russell’s own experiments with this approach, which we leave for the sequel, and reconstructions thereof, the most thorough examination of an approach along these lines is perhaps Quine’s system NF, which takes a form similar to naïve set theory, except that the class abstraction schema:

$$(\exists y)(x)(x \in y \equiv \dots x \dots)$$

rather than holding for all open sentences “ $\dots x \dots$ ” not containing “ $y$ ” free, is only allowed for instances in which the open sentence “ $\dots x \dots$ ” has certain syntactic properties. In particular, it needs to be *stratified*, i.e., a function must exist assigning natural numbers to terms flanking the membership sign wherever it occurs in “ $\dots x \dots$ ” so that the number assigned to the term left of “ $\in$ ” is one lower than that assigned to the term on the right. In NF, Cantor’s

theorem is unprovable (and indeed, demonstrably false for many instances), since the diagonally generated class  $w$  in Cantor's proof would be defined by an illegitimate formula. Instead, NF embraces classes that have all their subclasses as members, including a universal class and a class of all classes. Russell's class paradox is also blocked, since it too would be defined by an illegitimate formula, which is not surprising given that it can be thought of as generated by diagonalization.

Simply taken as a solution to Russell's class paradox, the overall strategy is neutral between an interpretation according to which the problematic diagonal condition does correspond to a "property," albeit an "exceptional one" with no corresponding subclass, and an interpretation according to which the condition, although it can be stated in a syntactically well-formed way, does not "comprehend" a genuine property at all. For the general approach, however, to solve some of the other paradoxes mentioned in sec. 3, particularly the predication paradox, something more like the latter interpretation seems more promising. (Note that for the other paradoxes, admitting the property but denying a well-defined subclass won't be enough, since some other entity, or even the property itself, is involved instead.) This interpretation could then allow that for every property, there is a distinct corresponding object (itself, or some proposition about it, or some descriptive sense involving it, etc.), but deny that there are such "diagonal" properties as non-self-instantiation,  $H$  from the descriptive sense paradox, or  $\phi$  from the propositions paradox. Again, to be fully plausible, the theory would need to explain under what conditions the specification of the exemplification conditions for a would-be property does or does not suffice to guarantee the existence of a property so delineated.

On the other interpretation, a property is admitted, but is regarded as exceptional in some sense, and therefore does not have a

unique corresponding object. It is difficult to assess which of these interpretations is right for Quine, whose nominalistic tendencies steer him away from speaking in terms of "properties" rather than linguistic formulas. Quine states the limitation on what "conditions" define classes in the metalanguage, and in terms of syntactic features of the open sentence used to describe a class. For those who, unlike Quine, embrace second-order logic, the requirement could instead take the form of *object language* qualifiers for sorting out those properties that determine a corresponding object from those that do not. For a theory involving which properties define classes, this tack is compatible with certain neo-logicist set theories that adopt a genericized version of Boolos's (New V), in which talk of properties too "big" to generate classes is replaced by more neutral talk about properties that are "bad" or non-distinct-class-generating (Cf. Shapiro 65):

$$(\dot{\alpha} F(\alpha) = \dot{\alpha} G(\alpha)) \equiv ((Bad(F) \& Bad(G)) \vee (x)(F(x) \equiv G(x)))$$

To count as a "zigzag theory", "badness" would need to be spelled out in terms of the internal properties of a property rather than, e.g., the range of its applicability. Again, there has been very little by way of exploration of approaches along these lines as applied to other paradoxes.<sup>16</sup>

### 4.3 The No Classes Theory

The third, and most radical, kind of solution to these paradoxes involves eschewing the kind of would-be entity that appears to violate Cantor's theorem altogether. Thinking of the classes paradox, Russell called this approach the "no classes theory." Here,

<sup>16</sup>Though see Cocchiarella, "Russell's Paradox of the Totality of Propositions," and Cantini, "On a Russellian Paradox," for exceptions.

one would deny that there are any such “things” as classes, and suggest that discourse apparently about classes, to the extent that it is not meaningless or confused, is not to be taken at face-value. Such discourse would be meaningful precisely to the extent that it is possible to reword it in a form in which no explicit mention of a class is made. For example, the claim that *the class of sedans is a subclass of the class of cars* can be reworded simply by saying that *all sedans are cars*. The solution to Russell’s class paradox comes in insisting that *certain* kinds of talk about classes cannot be so reworded. In particular, the claim that a class is a member of itself is to be regarded as meaningless, along with derivative expressions, such as that a class is not a member of itself. Hence the description used to define Russell’s paradoxical class is not meaningful, and therefore does not determine a condition or property that defines a class.

For the other paradoxes, it would be more appropriate to speak of the “no properties theory” or the “no propositions theory,” and so on. Of course, philosophers are likely familiar already with arguments showing that there is no such “thing” as Redness, or (false) propositions such as *Jupiter is in my pocket*. To be fully plausible, however, these approaches must make sense of the apparent discourse about these entities which seems unquestionably true, such as the claim that, “Euclid proved the proposition that there are infinitely many primes.” It also must ensure that the paraphrase given of such discourse is not by itself enough to generate the paradoxes. Again, the suggestion is likely that while some discourse about these apparent entities can be reworded in a form in which they are not mentioned, the discourse giving rise to the paradoxes cannot. Notice that it is not enough simply not to take the entities as *sui generis*. Replacing abstract “propositions” in favor of, e.g., classes of synonymous sentences, does not help solve the paradoxes if enough such classes are posited to violate Cantor’s theorem.

Nevertheless, it is approaches of this stripe that, by and large,

Russell himself gravitated towards, especially from late 1905 and afterwards (after discovering his theory of descriptions). We shall take up his views in the sequel article.

#### 4.4 Logical Types

Another broad kind of approach, not listed by Russell in “Some Difficulties,” though, ironically, often attributed to him, posits logical *types* of things.<sup>17</sup> Strategies of this sort can be seen as attempting to maintain *modified or more sophisticated forms* of the “intuitions” listed for the paradoxes, which, in the end, are found not to be inconsistent with Cantor’s theorem. Maintaining that entities and the properties applicable to them fall into distinct logical types, and in keeping with the “intuition” behind each paradox, one might suggest that for each property applicable to entities in a given category, it is possible to generate a distinct new entity, but insist that this new entity is in a separate logical category from the entities to which the original group of properties were applicable. Hence, any property applicable (or not) to these new entities is not among the original group.

For the classes paradox, for example, it amounts to dividing classes into type 1, or classes of individuals, type 2, or classes of classes of individuals, type 3, or classes of classes of classes of individuals, and so on, where it is not simply false but meaningless to ask whether  $a \in b$  unless  $b$  is of a type one higher than  $a$ . Then, we are free to postulate a distinct class for every property applicable to individuals, but this class is not one of the entities to which that property may or may not apply. Diagonalization never gets off the ground, since the properties involved in the mapping are not such as to apply, or not apply, to the entities to which they’re mapped, and so no system of “checkmarks” (to recall our visualization ear-

<sup>17</sup>As I argue in the sequel article, this attribution is contentious at best.

lier) is appropriate.

The approach has in common with the “no classes” theory the suggestion that expressions of the form “ $x \in x$ ” or “ $x \notin y$ ” are not meaningful. In this, they contrast with the limitation of size and zig-zag theories insofar as the latter regard such constructions as at least grammatically well-formed, even if they do not define classes. Despite this similarity, their explanations for their meaninglessness differ. In the “no classes” theory, no sentence of the form “ $a \in b$ ” is to be taken as about some entity of any type denoted by “ $b$ ”; instead, the entire sentence as a whole must be reworded into a form in which no class is mentioned, which is deemed impossible in this instance. In the kind of theory mentioned here, “ $b$ ” is an independently meaningful expression; it simply differs in what *kind* of thing it means from “ $a$ ”, and it engenders nonsense to attempt to say the same things of  $a$  one would say of  $b$ .

Frege’s theory of “levels” of concepts, according to which there are objects or “saturated entities,” first-level concepts (under which objects may or may not fall), second-level concepts (within which first-level concepts may or may not fall), etc. could be used to provide a response of this stripe to Russell’s predication paradox.<sup>18</sup> For each first-level concept (i.e., property), or concept applicable to objects, there is indeed an entity, that concept itself; but that concept is not itself an object, and the question as to whether it applies to itself is meaningless; one can only ask whether or not second-level concepts are applicable to it.

Addressing the other paradoxes with this kind of strategy

<sup>18</sup> Although, actually, I think this description of the situation is somewhat misleading, given that, for Frege, the extension of a concept “has its being” in the concept itself, and various other suggestions in his work to the effect that the extension of a concept simply *is* the concept treated as a logical subject. From this perspective, the class paradox and predication paradox are indistinguishable for Frege, which is why, I think, Frege describes Russell’s description of the predication paradox in his letter to him only as “imprecise” rather than erroneous.

would involve, for example, arguing that although a distinct proposition can be derived for each property, the resulting proposition is not of the right sort either to have or not to have that property. But this is precisely what the definition of  $\phi$  from the propositions paradox assumes, and hence, it is poorly defined. Similarly, while distinct descriptive senses might be generated from all properties, they would not be the kind of thing to which such properties may or may not apply.

## 4.5 Other Approaches

Lastly, there are kinds of solutions that fall into none of the above categories. These include radical approaches as might be taken by a dialethist who simply embraces the contradictions as true, while trying to insulate their harmful effects by means of a non-explosive paraconsistent logic. Such approaches raise other philosophical issues we cannot fully explore here.

Another less radical approach, however, might stem from noting that Cantor’s theorem or Frege’s lesson are not automatically violated by just any function that maps properties of things to things. If the same object may result as value for non-coextensive properties as argument, then the function doesn’t postulate as many objects as subclasses. One might then hope to maintain the spirit of the “intuitions” lying behind the paradoxes, but without the supposition that the entity generated is always “distinct.”

Notice, however, that it is not enough to allow that sometimes different properties may generate the same entity in the mapping; one must allow that sometimes non-coextensive properties may generate the same entity. With regard to classes or extensions, pushing this line of response is more or less tantamount to arguing that non-coextensive properties may have the same extension, a proposal which sounds absurd on its face. Nevertheless, Frege him-

self endorsed such a proposal in his appendix on Russell's paradox, and Russell himself was for a time attracted to it.<sup>19</sup> However, without philosophical support provided by an independent theory,<sup>20</sup> the supposition that non-coextensive properties may determine the same class, or, in the case of the Property Equivalence Class paradox, be included in the same equivalence class of coextensive relations seems bewildering. The situation is even worse with the paradoxes involving intensional entities. Intensions are supposed to be *finer-grained* in their identity conditions than extensions, yet to solve the descriptive sense and class-intension paradoxes we'd have to allow that *the F* and *the G* be identical *senses* in some cases even when *F* and *G* aren't even coextensive, or that two distinct classes may be generated by the same class-intension.<sup>21</sup> There does not seem to be much to be said in favor of such approaches.<sup>22</sup>

<sup>19</sup>This is evinced by the last minute footnote to *PoM* (p. 496), in which Russell calls it "very likely the correct solution," as well as in manuscripts of the period (*Papers* v4, 17–37). Indeed, Russell even seems to have hoped that it might work in other cases too, mentioning it in connection with the propositions paradox in particular to Frege in a letter; see Frege's *Correspondence* 159–60. Frege's own proposed "solution" was later found to lead to more complicated contradictions. See Quine, "Frege's Way Out" and Landini, "Ins and Outs" for further discussion.

<sup>20</sup>Notice that Boolos's (New V) system is technically a theory allowing non-coextensive "big" properties to have the same "class," although the explanation of this makes use of a different sort of theory, and talk of "extensions" replaced by talk of "subtensions."

<sup>21</sup>For further discussion of related issues, see Klement, "A Cantorian Argument" 73.

<sup>22</sup>Merely bearing the possibility of such "solutions" in mind, however, forces us to be rather more careful about the precise formulation of the paradoxes. Nicholas Denyer deflected what amounts to a Fregean version of the propositions paradox formulated by Adam Rieger by pointing out that Rieger's mapping didn't necessarily generate a distinct proposition from each property; however, a fairly insignificant modification to Rieger's proposal is immune to like treatment; see Klement, "Too Many."

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