

THE ORIGINS OF THE PROPOSITIONAL FUNCTIONS VERSION OF RUSSELL'S PARADOX

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Russell discovered the classes version of Russell's Paradox in spring 1901, and the predicates version near the same time. There is a problem, however, in dating the discovery of the propositional functions version. In 1906, Russell claimed he discovered it after May 1903, but this conflicts with the widespread belief that the functions version appears in *The Principles of Mathematics*, finished in late 1902. I argue that Russell's dating was accurate, and that the functions version does not appear in the *Principles*. I distinguish the functions and predicates versions, give a novel reading of the *Principles*, section 85, as a paradox dealing with what Russell calls *assertions*, and show that Russell's logical notation in 1902 had no way of even formulating the functions version. The propositional functions version had its origins in the summer of 1903, soon after Russell's notation had changed in such a way as to make a formulation possible.

INTRODUCTION

Russell discovered the classes version of Russell's Paradox in the spring of 1901 (see *Papers* 3: xxxii). In their correspondence in 1906, Russell described to Philip Jourdain the chronological development of his views with regard to its solution:

My book [*The Principles of Mathematics*] gives you all my ideas down to the end of 1902: the doctrine of types (which in *practice* is almost exactly like my

present view) was the latest of them. Then in 1903 I started on Frege's theory that two non-equivalent functions may determine the same class.... But I soon came to the conclusion that this wouldn't do. Then, in May 1903, I thought I had solved the whole thing by denying classes altogether; I still kept propositional functions, and made ϕ do duty for $z^2(\phi z)$. I treated ϕ as an entity. All went well till I came to consider the function W , where

$$W(\phi) \equiv \phi \cdot \sim \phi(\phi)$$

This brought back the contradiction, and showed that I had gained nothing by rejecting classes.¹

This reminiscence is puzzling. Wasn't Russell already aware of the propositional functions version of the paradox while writing the *Principles*? Wouldn't it have occurred to him that in making propositional functions "do duty" for classes, a version of the paradox for propositional functions was a possibility? Yet, at least for a short period starting in May 1903, he seems to have thought that the paradox had been completely solved. He wrote in his journal on 23 May, "four days ago I solved the Contradiction—the relief of this is unspeakable" (*Papers* 12: 24). Whitehead gave him his "heartiest congratulations" (*Papers* 4: xx). It is difficult to pin down exactly how long the ill-gained euphoria lasted, but if Russell was already aware of the propositional functions version of the paradox prior to May 1903, it is difficult to understand how he could have forgotten about it for what must have been days, weeks or even months.

There are, I think, three *primâ facie* plausible arguments that Russell was aware of the propositional functions version of the paradox while writing the *Principles*:²

¹ Quoted in Ivor Grattan-Guinness, *Dear Russell—Dear Jourdain* (New York: Columbia U. P., 1977), p. 78.

² These attitudes are, I think, widespread. One should compare, at least, the following: Gregory Landini, "Russell to Frege, 24 May 1903: 'I Believe I Have Discovered That Classes Are Entirely Superfluous'", *Russell*, n.s. 12 (1993): 160–85, and *Russell's Hidden Substitutional Theory* (New York: Oxford U. P., 1998), Chap. 3; Grattan-Guinness, *The Search for Mathematical Roots 1870–1940* (Princeton: Princeton U. P., 2000), Chap. 6; Michael Kremer, "The Argument of 'On Denoting'", *Philosophical Review*, 103 (1994): 249–97; Peter Hylton, *Russell, Idealism and the Emergence of Analytic Philosophy* (Oxford: Clarendon P., 1990), Chaps. 4–5; Nino Cocchiarella, *Logical Studies in Early*

- (1) In the *Principles*, Russell explicitly formulates a version of the paradox involving predicates not predicable of themselves (along with a few other versions); this version is often equated with the propositional functions version of the paradox.
- (2) At the end of Chapter VII of the *Principles*, Russell gives an argument that “the ϕ in ϕx is not a separate and distinguishable entity” by invoking a similar paradox. This definitely looks like the propositional functions version of the paradox.
- (3) In both Chapters X (§102) and XLIII (§348), Russell gives a Cantorian argument for thinking there must be more propositional functions than terms or objects. Given that similar Cantorian argumentation to the effect that there must be more classes than individuals lead him to the classes version of the paradox, surely he would have been led in similar way to the propositional functions version of the paradox.

My thesis is that these arguments are not convincing and, indeed, the propositional functions version of the paradox was a *new* discovery in May 1903. In the *Principles*, he had formulated at least two closely related versions of the paradox, but it was only after his views on the nature of functions and the role they were to play in his logic had changed that he explicitly formulated the paradox with propositional functions in mind. I begin by considering the three arguments above in turn.

PREDICATES VERSUS PROPOSITIONAL FUNCTIONS

In the *Principles*, §§78, 96, 100–1, Russell speaks of a contradiction that arises considering predicates. Some predicates, e.g., *human*, *wise*, etc., are not predicable of themselves. Humanity is not human and wisdom is not wise. Other predicates, such as *non-human*, are predicable of themselves. Non-humanity is non-human. He then goes on to say that while it is natural to suppose that there is some predicate that holds of all and only predicates not-predicable of themselves, this supposition is impossible, because we could then ask whether or not this predicate is pre-

dicable of itself, and it would turn out that it is if and only if it is not. Some writers on Russell seem to mark no distinction between this paradox and the one concerning propositional functions. But they are not the same, and this becomes clearer once it is realized that early Russell did not equate predicates with propositional functions.

First, what is a “predicate” for Russell? Contrary to contemporary usage, he does not mean anything linguistic. Russell’s 1903 ontology centred around the notion of *propositions*, understood as mind-independent complex entities. Constituents of a proposition occur either “as term”, i.e., as logical subject, or “as concept”, i.e., predicatively. Which entities occur in which way can roughly be determined by considering the grammar of the sentence used to express the proposition (§46). Russell uses the word “predicate” for those entities that correspond to grammatical adjectives or adjectival phrases. So the word “wise” in the sentence “Socrates is wise” indicates a predicate. The predicate occurs *as concept* and not as logical subject in the corresponding proposition. A Russellian predicate is understood (roughly) as a Platonic universal, and indeed, he later uses that terminology for them (e.g., *PP*₂, p. 93). However, he is quite clear that predicates can occur otherwise than as concept. In *Wisdom is a virtue*,³ the same predicate found in *Socrates is wise* occurs as term.

Unlike Frege and many others,⁴ Russell also thinks that the copula “is” in “Socrates is wise” indicates a relation, albeit of a unique sort (§53). What makes it unique is that wisdom still occurs as concept as a relatum of this relation, whereas in other cases of relational propositions, e.g., *Callisto orbits Jupiter*, the relata of the relation occur as term. Wisdom cannot be replaced by something other than a predicate in *Socrates is wise*; something such as “Socrates is Plato” does not represent a proposition if “is” still indicates the copula (as opposed to identity). For this reason, Russell distinguishes the proposition expressed by “Socrates is wise”, in which wisdom occurs as concept, from that expressed by “Socrates has wisdom”, wherein wisdom occurs as term (§57). The “has” in

³ When a complex phrase is put in italics, it should be taken as a name for the entity indicated or expressed by that phrase. In this case, for example, *Wisdom is a virtue* is the proposition expressed by the sentence “Wisdom is a virtue”.

⁴ See G. Frege, “On Concept and Object”, in *Collected Papers on Mathematics, Logic and Philosophy*, ed. Brian McGuinness (Oxford: Basil Blackwell, 1984), p. 183.

the latter example represents the relation of *instantiation* or *exemplification* that holds only between an individual and a universal. The propositions *Socrates is wise* and *Socrates has wisdom*, while equivalent, have distinct forms. Wisdom can be replaced by Plato in *Socrates has wisdom* to obtain *Socrates has Plato*, but this proposition is simply false, since Plato is not a universal that Socrates instantiates.

Russell also thinks that predicates are, metaphysically, to be identified with what he calls “class-concepts”, which are the entities expressed by count-noun phrases. The class-concept *man*, Russell tells us, differs only “verbally” from the predicate *human* (§§57–8). Class-concepts are so-called because typically they are invoked when a class is defined as the extension corresponding to a certain intension. From the class-concept *man*, one can obtain the “concept of a class”, *all men* (or simply *men*), which *denotes* the collection or class of all men. Prior to late 1900, Russell had assumed that the use of class-concepts was the primary and indeed perhaps the only way of getting at classes in symbolic logic.⁵ Consequently, he thought that the study of logic involved discovering axioms governing the existence of complex class-concepts (see, e.g., *Papers 2*: 185–95).

However, the paradox involving predicates non-predicable of themselves, which Russell seems to regard as equivalent to the paradox involving class-concepts that are not members of their own extensions,⁶ leads him to be wary of postulating class-concepts or predicates too readily. Indeed, he states the conclusion of his considerations on this topic as that, despite appearances, not every seemingly well-formed grammatical adjective phrase corresponds to a predicate, and not every seemingly well-formed noun phrase corresponds to a class-concept. He writes, “the conclusion ... seems obvious, ‘not predicable of oneself’ is not a predicate,” and similarly, “we must conclude, against appearances, that ‘class-

⁵ Prior to his acquaintance with the work of Peano, which dates from the International Congress of Philosophy in Paris of August 1900, Russell's paradigm for symbolic logic was the Boolean treatment of categorical logic within an algebra of classes, primarily as expounded in Whitehead's *Universal Algebra*. See *Papers 2*: 190–5, *Papers 3*: 44–7.

⁶ In §100, Russell begins by claiming he will examine the version involving predicates not predicable of themselves, then immediately begins speaking of class-concepts, and then, in §101, claims to attempt to state the contradiction itself, and returns to speaking of predicates. This is further evidence that he does not distinguish class-concepts and predicates as entities.

concept which is not a term of its own extension' is not a class-concept" (*PoM*, §101).

This paradox, again, is often taken to be the same as a paradox involving propositional functions that yield a falsehood when taken as argument to themselves. This goes hand in hand with a reading taking propositional functions to be identified in Russell's mind with predicates or class-concepts. However, the text of the *Principles* on propositional functions and the lessons he takes away from the paradox show this reading to be mistaken.

Let me first comment on what I think has led to the misreading. One distorting influence derives from the terminology and work of others of the period, such as Frege. Frege regards every meaningful expression as standing for either a function or an object, and thinks that a grammatical predicate stands for a kind of function, which he calls a "concept" (*Begriff*). Frege also thinks that every sentence containing one or more occurrences of a proper name can be split into the name and the remainder, and the remainder will stand for a concept. Frege regards classes as extensions of concepts. Fregean higher-order quantifiers utilize variables for functions. Frege and Russell shared many views, as Russell himself was the first to point out (*PoM*, §475). They are often lumped together as the two leading proponents of logicism and the driving forces behind the emergence of modern predicate logic. Because Frege thought of concepts and the references of predicates as functions, it is often thought that Russell did as well, even early on. However, Russell's positions on these matters developed almost wholly independently from Frege. Russell had not even read Frege carefully until he had almost completed the *Principles* (see, e.g., p. xviii). When he did, he wrote in the margin of his copy of Frege's *Funktion und Begriff*, next to Frege's claim that an "object is anything that is not a function", that "[t]his is not correct, for predicates etc. seem to be neither."⁷

Another influence on this reading of Russell is the very notation of

⁷ See Bernard Linsky, "Russell's Marginalia in His Copies of Frege's Works", *Russell* n.s. 24 (2004): 34. Here Russell seems to interpret Frege's notion of "objects" (*Gegenstände*) as closest to his notion of "things" (*cf. PoM*, §480), which are those terms that are incapable of occurring as concept in a proposition (*PoM*, §48). It is then obvious why Russell would not regard predicates as objects; the interesting thing here is why they are not functions.

contemporary predicate logic, which writes “ Fa ” for a subject–predicate statement, with the capital letter “ F ” prefixing its “argument” making it appear as a name of a function, and a name of a different *type* of entity than that represented by the letter “ a ”. In contemporary second-order predicate logic, one finds variables which are typically called “predicate variables”, and principles such as comprehension schemata:

$\vdash \ulcorner (\exists \phi)(x)(\phi x \equiv A) \urcorner$, where A is any wff containing
“ x ” but not “ ϕ ” free.

Contemporary second-order predicate logic is related historically to the higher-order logic found in Whitehead and Russell’s *Principia Mathematica*, which involves quantification over propositional functions, utilizing a similar notation. Here again it appears that predicates and propositional functions should be identified.

However, it must be remembered Russell was not aware of contemporary “predicate logic”, and never used the terminology of “predicate variables”. While early Russell did sometimes use the notation “ $\phi(a)$ ”, where the “ ϕ ” is used either as a variable or schematically, Russell did not use a notation such as “ Fa ” for variable-free subject–predicate statements anywhere in his work prior to the 1920s. Moreover, in 1903 he surely would have objected to the notation, since it leaves out the copula. More to the point for our present discussion, Russell does not assimilate variables or constants for propositional functions with variables or constants for predicates or class-concepts. In the *Principles*, he uses the letters “ x ” (§101), “ u ” (§§58, 71, 100) or “ a ” (§§60–1, 73) when he wishes to speak of variable or arbitrary predicates or class-concepts, and never uses “ ϕ ”, “ ψ ”, “ F ”, or “ G ” or anything suggesting a function.

A final distorting influence is the changes that occurred in Russell’s views later on, such as his claims in 1918 and later that a sign for a universal, when used and not mentioned, can only occur predicatively, and that understanding the word “red” requires understanding the form “ x is red”, which he connects with his theory of types (*PLA*, in *LK*, pp. 205–6, 334).⁸ But this view must not be read back into the *Principles*, where

⁸ For more on the changes to Russell’s views in later years, and how these doctrines of 1918 and later were new to Russell after Wittgenstein’s influence, see my “Putting Form before Function: Logical Grammar in Frege, Russell and Wittgenstein,” *Philosophers’*

Russell is explicit that predicates have a twofold nature and can occur both as subject and predicatively (§49) and, indeed, is explicit that predicates are themselves individuals and not in a distinct logical type (§499).

Russell seems to have derived the notion of a propositional function indirectly from Peano, whose influence greatly changed Russell's approach to symbolic logic in late 1900. In 1897, Peano explained certain of his notations as follows:

Let p and q be propositions containing variable letters x, \dots, z . The formula $p \supset_{x, \dots, z} q$ means 'whatever x, \dots, z , are, as long as they satisfy the condition p , they will satisfy the condition q '.

... if p_x is a proposition containing the variable letter x , by $\bar{x} \bar{\epsilon} p_x$ we mean the class of x s which satisfy the condition p_x The whole sign $\bar{x} \bar{\epsilon}$ may be read 'the x such that'.⁹

For instance, the expression " $\bar{x} \bar{\epsilon} (x \in \mathbb{N}. x^2 < 60)$ " would stand for the class of numbers whose squares are less than sixty, and " $x \in \mathbb{N} \supset_x x + 1 \neq x$ " means, that whatever x is, provided it is a (natural) number, then it is non-identical with its own successor. The symbolic logic endorsed by Russell in the *Principles* is explicitly based on that of Peano. Russell, however, criticizes Peano for not distinguishing between expressions for propositions, which must not contain any real (i.e., free) variables, from expressions that do contain such variables. Russell's first mention of propositional functions comes at the *Principles*, §13, where he writes:

I shall speak of propositions exclusively where there is no real variable: where there are one or more real variables, and for all values of the variables the expression involved is a proposition, I shall call the expression a *propositional function*.

There is some use/mention sloppiness here. A propositional function is not an expression or anything linguistic any more than a proposition is. Russell means that propositional functions are the ontological correlates of open formulae, just as propositions are the ontological correlates of closed sentences. Because the use of variables primarily occurs with the

Imprint, 4 (2004): 1–47 [<http://www.philosophersimprint.org/004002/>].

⁹ See Giuseppe Peano, "Studies in Mathematical Logic", in *Selected Works of Giuseppe Peano*, ed. H. Kennedy (London: Allen and Unwin, 1973), pp. 193, 197.

notation for *formal implication*, represented by Peano's " \supset_x ", or for class abstracts using Peano's " $\overline{x \in} \dots$ ", Russell's discussion of propositional functions in the *Principles* is usually linked with his discussion of formal implication and the notion of *such that*.¹⁰

The propositional function expressed by "x is a man" arises in the following way:

In any proposition, however complicated, which contains no real variables, we may imagine one of the terms, not a verb or an adjective, to be replaced by other terms: instead of "Socrates is a man," we may put "Plato is a man," "the number two is a man," and so on. Thus we get successive propositions all agreeing except as to the one variable term. Putting x for the variable term, "x is a man" expresses the type of all such propositions. (PoM, §22)

Here we begin to see ways in which Russell's understanding of propositional functions deviated from his understanding of predicates. The propositional function captures what a certain set of propositions have in common. In this case, what they have in common is not simply the class-concept humanity. They also all contain "is-a", which Russell at the time regarded as representing a relation between an individual and a class-concept, "nearly, if not quite identical with" the relation expressed by "has" in "Socrates has humanity" (see PoM, §79 and p. 55n.). But more than this, they exhibit a certain *constancy of form*: they consist of one entity occurring as subject related by the *is-a* relation to humanity. While *Humanity is a concept* contains both humanity and the *is-a* relation, it is not a value of the propositional function *x is a man* because it is not of the appropriate form. Hence it becomes easy to see that Russell did not equate propositional functions with predicates or class-concepts.

¹⁰ Russell lists *such that* as one the primitive notions of logic in the very first paragraph of the *Principles*. He clearly has in mind the notion as borrowed from Peano's logic of classes. Peano's notation for "such that" changed through the 1880s and 1890s. At first he used the notation " $[x \in] \dots x \dots$ " for the class of all x such that... Later he used the notation " $\overline{x \in} \dots x \dots$ ", and finally the notation " $x \ni \dots$ ", which is adopted by Russell in his early logical writings. Peano described *such that* as the "inverse" of the membership relation \in , because while " $x \in$ " written before the name of a class creates a name for a proposition, " $x \ni$ " written before the name of a proposition creates a name of a class. Using brackets, overlining, and writing signs upside down in general represented *inversion* in Peano's notation during various periods. However, this notion of inversion is obscure at best. See Grattan-Guinness, *Search for Mathematical Roots*, Chap. 5.

Propositional functions, like propositions but unlike their simple constituents, presuppose forms or structures. Russell also has different terminology for the relationship between a term and a propositional function that yields a true proposition when taking the term as argument than for the relationship between a term and a predicate/concept it exemplifies or instantiates. For the former relation, Russell uses the word “satisfy” and never “has” or “belongs to” (e.g., §§24, 80).

THE SOLUTION TO THE PREDICATES VERSION OF THE PARADOX

Russell’s reaction to the predicates or class-concepts version of the paradox is only intelligible if predicates are distinguished from propositional functions. As we have seen, he rejected the assumption that there is any such predicate as “non-predicable of self”. However, he did not conclude that there is no such class as that consisting of predicates which cannot be predicated of themselves, or even that this class cannot be denoted. Instead, he concludes that defining classes using Peano’s *such that* notation prefacing a sign for a propositional function is not equivalent to defining a class as the extension of some predicate:

It must be held, I think, that every propositional function which is not null defines a class, which is denoted by “ x ’s such that ϕx .” ... But it may be doubted—indeed the contradiction with which I ended the preceding chapter gives reason for doubting—whether there is always a defining predicate of such classes. Apart from the contradiction in question ... “being an x such that ϕx ,” it might be said, may always be taken to be a predicate. But in view of our contradiction, all remarks on this subject must be taken with caution.

(*PoM*, §84)

The exact point established by the above contradiction [the paradox regarding predicates] may be stated as follows: A proposition apparently containing only one variable may not be equivalent to any proposition asserting that the variable in question has a certain predicate.

(*PoM*, §96)

We shall maintain, on account of the contradiction there is not always a class-concept for a given propositional function ϕx , i.e. that there is not always, for every ϕ , some class-concept a such that $x \in a$ is equivalent to ϕx for all values of x

(*PoM*, §488; see also §§77, 80)

None of these remarks would make sense if Russell equated propositional functions with class-concepts.

Russell's conclusion from the paradox was that there are some classes that can be legitimately defined as all entities satisfying a given propositional function, but whose members do not share a common predicate or class-concept. For this to have any connection with the predicates version of the paradox, the class of all predicates not predicable of themselves must be such a class. Hence he admits the propositional function $\sim(x \text{ is } x)$, and holds it to define a class, i.e., *the x 's such that $\sim(x \text{ is } x)$* ; he simply denies that a common predicate held of the members of this class. Allowing the function to be an entity does not generate a contradiction, even if it is taken as its own argument. For values of x which are not themselves predicates, the function $\sim(x \text{ is } x)$ does not even yield a proposition (*PoM*, p. 20n. and §83), because only predicates can occur as concept as second relatum to the special relation indicated by the copula. So the propositional function $\sim(x \text{ is } x)$ could only be satisfied by predicates. Since the function is not a predicate, it does not and cannot satisfy itself, and its not satisfying itself does not lead back to the result that it does. One might instead attempt to formulate the paradox with the function $\sim(x \text{ has } x)$. In that case, the function would satisfy itself, because it does not bear the exemplification relation to itself. However, its satisfying itself does not lead to the conclusion that it does not, because that result would only follow if the satisfaction relation were the same as the exemplification relation indicated by "has", which it is not.

I think this dispatches the first argument to the effect that the propositional functions version of the paradox is to be found in the *Principles*. The predicates version is a distinct paradox, and while Russell is rather explicit that it led him to be cautious about positing a predicate for every adjective phrase or as a defining feature for every class, this caution did not carry over to a similar caution about propositional functions (nor, as far as I can tell, should *this* version of the paradox have given him any reason for such caution).

ASSERTIONS VERSUS PROPOSITIONAL FUNCTIONS

The next argument is relatively more difficult to counter. In the single paragraph of *Principles*, §85, we find mention not of a propositional

function of the form $\sim(x \text{ is } x)$ or $\sim(x \text{ has } x)$ but one of the form $\text{not-}\phi(\phi)$. This looks far more like a propositional function taking itself as argument. However, to understand §85, we must probe a bit further into exactly what propositional functions were understood to be, and how the variable “ ϕ ” is used in the *Principles*. We saw earlier that Russell’s understanding of propositional functions involves a class of propositions sharing the sort of constancy of form that can be found when each member of the class can be obtained from the other by replacing one of its constituents with something else. But what is the propositional function itself? Is it the class itself, or something that denotes the class, or something that denotes the members of the class severally, or some other entity connected with the class? We shall return to this question. What is most important for understanding §85, however, is Russell’s *rejection* of an account on which a propositional function is identified with what remains of a proposition when a certain constituent is simply removed, a view which he finds in Frege (§482).

Frege suggests at many places in his writings that functions are to be understood as “incomplete” or “unsaturated” entities, gotten at by “pulling out” some object from a unified whole.¹¹ This goes along with Frege’s claim that in mathematical notation such as “ $2x^3 + x$ ”, the variable letter “ x ” only indicates the argument, not a part of the function, and so the function would be better written “ $2.()^3 + ()$ ”.¹² Prior to Frege, it was commonplace to equate a function with what Euler called an “analytic expression”,¹³ an expression containing a variable that might form one half of an equation such as “ $y = 2x^3 + x$ ”, representing the relationship between a dependent and independent variable. Frege

¹¹ Frege makes such claims many times throughout his career. However, especially in context of his mature philosophy, the view is problematic. The functions Frege typically has in mind are located at the level of reference, and the reference of a complex expression containing a name is not a complex whole containing the reference of the name. E.g., the reference of “ $2 + 3$ ” is five, and five does not contain “2” as a part. Therefore it is unclear exactly what it is one is supposed to pull 2 out of in order to get the function $() + 3$. The *sense* of “ $2 + 3$ ” may be complex, but this does not help if the function is located at the level of reference. For further discussion see my *Frege and the Logic of Sense and Reference* (New York: Routledge, 2002), pp. 67–8.

¹² See Frege, “Function and Concept”, *Collected Papers on Mathematics, Logic and Philosophy*, p. 140.

¹³ See Leonhard Euler, *Introductio in Analysin Infinitorum* (Lausanne, 1748), I: 4.

complained that this confused the function and the value of the function for an arbitrary argument. This complaint was later taken up by such notables as Alonzo Church, whose lambda abstracts were introduced in part to disambiguate between a function itself and an arbitrary value.¹⁴ For those of us influenced by later trends, it can be difficult to return to our pre-Fregean naïveté. As we shall see, however, Russell's way of thinking about functions, in 1902 at least, is much closer to the older Eulerian tradition, and shared at least some of the pitfalls of that tradition. For Russell, the variable was not only part of the notation for the function, the presence of a variable is *what makes* such an expression functional. Certainly, Russell's views of functions shares little with the Fregean conception of something incomplete gotten at by *removing* a constituent from a whole.

Indeed, Russell believed that it was *in general* impossible to simply remove a constituent from a unity and have the remainder constitute a single separable entity. This is arguably the major lesson of Chapter VII of the *Principles*. Russell does claim that it is *sometimes* possible. In particular, he claims that it is possible when the proposition in question is a simple relational proposition of the form bRa . In such a case, the entity b can be removed, and the remainder, which Russell writes "... Ra ", is called an "assertion". Russell claims that the same assertion can be asserted of different subjects, and from this process we get propositions such as cRa , dRa , etc. This assertion is in effect, "the constant part" of the propositional function xRa , i.e., the part that all the values have in common.

The notion of *assertion* is an important and often neglected one in Russell's early philosophy.¹⁵ There are reasons for thinking that assertion is an earlier notion of which the notion of a propositional function was a successor. In his 1902 notes detailing his "plan" for the book, we see that Chapter VII was originally to be called "Assertions" instead of

¹⁴ See his *Introduction to Mathematical Logic* (Princeton: Princeton U. P., 1956), pp. 15–23.

¹⁵ The notion receives brief discussion in Kremer, pp. 261–5; Hylton, pp. 177, 214–15; and Landini, *Russell's Hidden Substitutional Theory*, p. 65, but little discussion in other writings on Russell's early philosophy. We must be careful to distinguish this notion of "assertion" from that involved in the distinction between asserted and unasserted propositions from the *Principles*, §38, which receives a separate listing in the index.

“Propositional Functions” (see *Papers* 3: 211). We noted earlier that Russell seems to have derived the notion of a propositional function from reflecting on the “propositions containing variables” Peano had used in his notations for class abstracts formed with “such that”, and for implications for all values of a variable. In Chapter VII, Russell begins by posing the question as to what extent the notion of assertions can be used in explaining the notion of *such that* (§§80–1). Similarly, earlier in the *Principles* (§§42–4), although expressing doubts, Russell gave some initial credence to the view that the formal implication expressed by “ x is a man \supset_x x is a mortal” can be seen as derived from a relation between the assertions expressed by “... is a man” and “... is a mortal”. However, he comes to the conclusion that a view involving assertions alone is “too simple to meet all cases” (§42), and for this reason he concludes that “propositional functions” (as entities *distinct* from assertions) “must be accepted as ultimate data” (§84).

Russell introduces the cases in which there is no assertion as separable entity by asking us to consider the proposition *Socrates is a man implies Socrates is mortal*. He continues:

... when we omit Socrates, we obtain “... is a man implies ... is a mortal.” In this formula, it is essential that, in restoring the proposition, the *same* term should be substituted in the two places where dots indicate the necessity of a term. It does not matter what term we choose, but it must be identical in both places. Of this requisite, however, no trace whatever appears in the would-be assertion, and no trace can appear, since all mention of the term to be inserted is necessarily omitted. (*PoM*, §82)

If the assertion is simply what remains when a constituent is removed, there is a problem for cases in which the removed term appeared twice in the original proposition. With only gaps remaining, there is nothing to force the resulting empty spots to be filled by the same entity. This interferes with his initial plan to “explain propositional functions by means of assertions” (§82), because the difference between the function expressed by “ x is a man implies x is a mortal” and that expressed by “ x is a man implies y is a mortal” is lost. Accordingly, Russell criticizes Frege for being unable to distinguish $2x^3 + x$ and $2x^3 + y$ when the former is written “ $2.()^3 + ()$ ” (§482). Variable letters must be used in the notation for propositional functions to indicate in what positions the same term must be restored. If one simply erases the variables, the result-

ing expression does not indicate a genuine entity. He concludes this line of reasoning as follows:

It would seem to follow that propositions may have a certain constancy of form, expressed in the fact that they are instances of a given propositional function, without its being possible to analyze the propositions into a constant and a variable factor. (PoM, §82)

Subject-predicate propositions, and such as express a fixed relation to a fixed term, could be analyzed, we found, into a subject and assertion; but this analysis becomes impossible when a given term enters into a proposition in a more complicated manner than as referent of a relation. Hence it became necessary to take propositional function as a primitive notion. A propositional function of one variable is any proposition of a set defined by the variation of a single term, while the other terms remain constant. But in general it is impossible to define or isolate the constant element in a propositional function, since what remains, when a certain term, wherever it occurs, is left out of a proposition, is in general no discoverable kind of entity. Thus the term in question must be not simply omitted, but replaced by a variable. (PoM, §106)

Russell criticizes the Fregean view that a function is simply what remains when an entity is removed from a proposition, calling this “in general a non-entity” (§482), and chides Frege for thinking that “if a term *a* occurs in a proposition, the proposition can always be analyzed into *a* and an assertion about *a*” (§475).

Russell's terminology in §106 and elsewhere (especially §§22, 82, 93, 254) suggests that he understands a propositional function as a proposition-like unity containing a variable in place of a definite term; rather than simply removing the term to be varied, it is replaced by a *variable*. In general, neither propositions nor terms are linguistic entities in Russell's parlance; if this suggestion is to be taken seriously, Russell must also take variables as non-linguistic entities. Indeed, in Chapter VIII on “The Variable”, Russell endorses a view of the variable as an extra-linguistic object. We shall return to consider the nature of variables below; what is important in this context is that a propositional function can be thought of as having a variable as a constituent. The lesson of Chapter VII is that the remainder of a propositional function *excluding* the variable is usually not itself to be understood as a single and separable entity.

§85 OF “THE PRINCIPLES OF MATHEMATICS”

At the very end of Chapter VII we find the disputed §85, allegedly containing the propositional function version of the paradox. It begins:

It is to be observed that, according to the theory of propositional functions here advocated, the ϕ in ϕx is not a separate and distinguishable entity: it lives in the propositions of the form ϕx , and cannot survive analysis... [This] has the merit of enabling us to avoid a contradiction arising from the opposite view. If ϕ were a distinguishable entity, there would be a proposition asserting ϕ of itself, which we may denote by $\phi(\phi)$; there would also be a proposition not- $\phi(\phi)$, denying $\phi(\phi)$. In this proposition we may regard ϕ as variable; we thus obtain a propositional function. (*PoM*, §85)

This section must be read carefully and placed in its proper context at the end of Chapter VII. It then emerges that the paradox discussed here does not involve the negation of a propositional function taking itself as argument, but an assertion denied of itself. Confusion about this stems from misunderstandings regarding Russell's use of the Greek letter “ ϕ ”, albeit very natural misunderstandings given Russell's later work. Most readers take for granted that Russell always used “ ϕ ” as a variable for propositional functions. At least here, however, I think he does not. This requires explanation.

It will be recalled that Russell thinks that propositional functions are the ontological correlates of open sentences: sentences containing letters used for variables. Thus “ x is human” represents a propositional function. As we have seen, the variable letter “ x ” is an ineliminable part of the symbolism. In this case, the assertion ... *is human* is a separable constituent, but this is often not the case. Other propositional functions include those corresponding to “ x is a man implies x is a mortal” or “ x loves Socrates or Socrates loves x ”, etc. When Russell wishes to speak of about an arbitrary propositional function, as opposed to a specific example, he usually uses the notation “ ϕx ”, “ ψx ” or “ $f x$ ”¹⁶; it is worth noting that he *does not* use “ ϕ ” or “ f ” by itself.¹⁷ In the case of a proposi-

¹⁶ See §§22, 24, 33, 83, 84, 86, 88, 93, 104, 482.

¹⁷ One difficulty with making sense of these passages is that Russell does not distinguish as he should between speaking of *an arbitrary* expression containing the variable “ x ”, which would best be represented using a metalinguistic schematic letter, e.g., with

tional function such as “ x is human”, Russell says, “a fixed assertion is made of a variable term” (§77). “ ϕx ” in general represents an arbitrary or unspecified assertion made of a variable term. The “ ϕ ” represents the assertion, and “ x ” the variable subject for the assertion. Russell is explicit about this in a manuscript from June/July 1902 entitled “General Theory of Functions”, where he writes, “we may regard the assertion as the ϕ in the notation ϕx ” (*Papers* 3: 687).

Bearing this in mind when reading §85, it becomes clear that Russell’s argument there that “the ϕ in ϕx is not a separate and distinguishable entity” is nothing more than the contention that the remainder of the function above and beyond the variable is not always a separate entity, which is just the general lesson of Chapter VII. In the table of contents (p. xxiii), Russell summarizes §85 by writing that “a propositional function is in general not analyzable into a constant and a variable element”. Russell thought he had already established this in §82, but notes in §85 that this allows him to avoid a problem that would arise from taking “the opposite view”. If every propositional function were analyzable into a constant element and a variable, or, what amounts to the same, if every proposition were analyzable into a logical subject and assertion, then we could consider any assertion, e.g.

... is human

and we could then consider it asserted of itself:

(... is human) is human

The result would be a proposition. There would also be the negation of this proposition:

\sim ((... is human) is human)

$\lceil A(x) \rceil$ representing any expression containing “ x ” free, versus speaking of an expression such as “ ϕx ”, with “ ϕ ” used as some sort of object-language variable. This has long been a source of frustration in making sense of Russell’s logical writings. See, e.g., Landini, *Russell’s Hidden Substitutional Theory*, pp. 258–67, and my “Russell’s 1903–1905 Anticipation of the Lambda Calculus”, *History and Philosophy of Logic*, 24 (2003): 32.

Since we've assumed that the assertion is an individual entity, we can replace it by a variable, whence we get the propositional function:

$$\sim \phi(\phi)$$

A propositional function is what we get when an entity in a proposition is replaced by a variable. So " $\sim \phi(\phi)$ " represents a propositional function, and its constituent variable is indicated by the letter " ϕ ". However, this does not tell us what the *values* of the variable are for which the propositional function yields a proposition. We saw earlier that the function expressed by " $\sim (x \text{ is } x)$ " only yields propositions when the variable takes values that are predicates. The function expressed by " $\sim \phi(\phi)$ " only yields a proposition when the variable takes values that are *assertions*. So, again, the contradiction is not one that arises when this propositional function is itself taken as the value of the variable, as the result there would not be a proposition. The problem arises in a more indirect fashion. Notice that §85 continues:

The question arises: Can the *assertion in* this propositional function be asserted of itself? The assertion is non-assertibility of self, hence if it can be asserted of itself, it cannot, and if cannot, it can. This contradiction is avoided by the recognition that the *functional part* of a propositional function is not an independent entity. (Emphasis added.)

Per the assumption made in the *reductio ad absurdum*, every propositional function can be analyzed into constant and variable parts. The remainder of the function above and beyond the variable is always a separable entity: an assertion. The assertion contained in the propositional function indicated by " $\sim \phi(\phi)$ " might be written as " $\sim \dots (\dots)$ ". The paradox arises, Russell tells, us when we ask whether the assertion in $\sim \phi(\phi)$, viz., $\sim \dots (\dots)$, can or cannot be asserted of itself. As we have seen, the function $\sim \phi(\phi)$ yields a proposition when the variable takes values that are assertions. When it takes $\sim \dots (\dots)$ as argument, we get a paradox. However, Russell tells us, this contradiction is avoided by the "recognition that the functional part of a propositional function is not an independent entity". The "functional part" of a propositional function, when it is a separable entity, is an assertion. So the problem is solved when we reject the assumption that an assertion can always be extracted from a propositional function. This, Russell, thinks is indepen-

dent confirmation of what he had already established in §82.

So it would be very misleading to think of the paradox discussed in §85 as being a version of the propositional functions version of the paradox. Indeed, the very spirit of Chapter VII is an argument to the effect that propositional functions must be taken as fundamental and that the work they do in logic cannot be relegated to assertions. The paradox in §85 is another argument against having assertions doing the work of propositional functions; it is not an argument *against* taking propositional functions as entities, as it is often read.¹⁸

PROPOSITIONAL FUNCTIONS AS LOGICAL SUBJECTS?

So far we've established that neither Russell's discussion of the predicates version, nor the *Principles*, §85, can plausibly be read as an explicit mention of a propositional functions version of the paradox. There are, moreover, no other passages of the *Principles* that can plausibly be read as *explicitly* discussing the propositional functions version. However, it might still be maintained that even without an explicit mention in the *Principles*, Russell could not have failed to consider such a version of the paradox, especially given the very similar versions that do receive explicit mention. Fuel for this sentiment comes from §102 and §348, where Russell outlines Cantorian reasoning for the conclusion that there must be more propositional functions than terms. Russell discovered the classes version of Russell's paradox by considering Cantor's theorem that every class has more subclasses than members. When applied to the class of all classes, this seems impossible, since all its subclasses would appear to *be* members. One might then seek a potential counterexample to Cantor's theorem by considering a mapping from the class of all subclasses of the universal class into the universal class that correlates each subclass with itself. Applying Cantor's diagonal argument, this mapping must leave out the class of all classes not members of themselves. This is allegedly how Russell discovered his problematic class.¹⁹ Similar reason-

¹⁸ Cf. Cocchiarella, pp. 27, 199; Landini, *Russell's Hidden Substitutional Theory*, pp. 67, 69–70; and Kremer, pp. 262–3. Indeed Kremer *misquotes* §85 in a way that hinders his ability to interpret it correctly. Alasdair Urquhart, editor of *Papers* 4, also seems to read §85 of the *Principles* this way. See *Papers* 4: 49.

¹⁹ See his letter to Frege dated 24 June 1902, in Frege, *Philosophical and Mathematical*

ing shows that there can be no mapping from propositional functions into the class of all terms. It would seem, then, that Russell need only consider the potential mapping from propositional functions into terms that correlates each propositional function with itself; the diagonal argument would then invite us to consider the propositional function that yields a truth if and only if its argument is a propositional function that does not yield a truth for itself as argument. In §102 and §348, Russell discusses the impossibility of correlating propositional functions with terms. Given his reasoning in the classes case, one might suggest, it would have only been natural for him to have gone through the same reasoning for the functions case.

I can only respond, however, that I can find no evidence that Russell ever (prior to May 1903) took the last step by considering the mapping just mentioned or the paradox it engenders. I think this is explained by two facts: (a) Russell does not seem to have had a fully worked out view about what a propositional function *is* as an entity, and (b) using the logical notation he employed at the time, there would have been no way even to represent a function taking itself as argument. Let us consider (b) first. Earlier we explained a propositional function as the ontological correlate of an open sentence. So “*x* is human” represents a propositional function. How, then would we represent the value of this function for itself as argument? It is tempting to represent it as “(*x* is human) is human”. What we meant to have was a *proposition* in which a propositional function occurs as logical subject. However, on a purely syntactic level, “(*x* is human) is human”, still contains the variable “*x*” free. Interpreted at face value, it represents not a proposition, but a propositional function. Specifically, it would seem to represent a propositional function whose values are propositions such as (*Socrates is human*) is human and (*Humanity is human*) is human, which involve predicating humanity of different propositions. Without further comment, this notation does not seem to give us what we want. This is not to say that Russell

Correspondence, ed. G. Gabriel *et al.* and B. McGuinness (Chicago: U. of Chicago P., 1980), pp. 133–4. It is perhaps worth noting that in that letter, Russell discusses a paradox involving the function $\sim \phi(\phi)$, and claims it leads him to doubt “whether the ϕ in ϕx can be regarded as anything at all.” Obviously, I read this in the same way I read §85 of the *Principles*, though here there arises the complicating factor that Russell interprets Frege’s functions as most similar to assertions (see *PoM*, §§480, 482).

couldn't have, or shouldn't have, developed notation for representing a proposition containing a propositional function occurring as term. Those familiar with Russell's later works would no doubt suggest the notation " $(\hat{x}$ is human) is human". However, Russell was not yet using the circumflex notation for indicating a function itself as opposed to an arbitrary value—that notation did not appear until an April 1904 letter from Whitehead to Russell (see *Papers* 4: xxiv). Prior to having a notation for function abstraction, i.e., for naming functions themselves as opposed to arbitrary values, Russell had no way of even formulating the propositional functions version in his logic. In short, his notation was too Eulerian for the task.

However, I do not wish to suggest that Russell's failure to consider the functions version of the paradox during the writing of the *Principles* was entirely due to lack of imagination with regard to notation. I think Russell's *philosophical understanding* of the nature of a function itself as a separate entity from its values was not sufficiently developed or clear for him to have a proper awareness of what it would be for a function itself to occur as logical subject in a proposition. This brings us to (a). Russell himself admits that the subject of the nature of propositional functions is "full of difficulties" and puts forth his own views with the qualification that they "are put forward with a very limited confidence in their truth" (§80). However, he seems to give different accounts at different occasions as to what a propositional function is in and of itself. In the appendix on Frege, when contrasting his views to Frege's, Russell gives a list of the following allied notions:

We have, then in regard to any unity [e.g., a proposition], to consider the following objects:

(1) What remains of the said unity when one of its terms is simply removed, or, if the term occurs several times, when it is removed from one or more of the places in which it occurs.... This is what Frege calls a function.

(2) The class of unities differing from the said unity, if at all, only by the fact that one of its terms has been replaced, in one or more of the places where it occurs, by some other terms....

(3) Any member of the class (2).

(4) The assertion that every member of the class (2) is true.²⁰

²⁰ I take it that in this context, by "assertion" Russell does not mean "assertions" in the sense of §82, but in the sense of §38, i.e., an asserted proposition. See my note 15.

(5) The assertion that some member of the class (2) is true.

(6) The relation of a member of the class (2) to the value which the variable has in that member. (§482; cf. *Papers* 3: 687–8)

As we have seen, as part of his criticism of Frege, Russell rules out (1) as an account of propositional functions. As he puts it, “[i]nstead of the rump of a proposition considered in (1), I substitute (2) or (3) or (4) according to circumstances” (p. 509). What he means is that different ways of using expressions containing variables will be explained, ontologically, in different ways. When a variable is bound by a quantifier, one gets either (4) or (5). However, of course, a propositional function is not to be identified with a quantified proposition. Entities of types (2), (3) and (6) represent better candidates for what propositional functions themselves might be. Unfortunately, however, Russell seems to prefer different answers at different places in the *Principles*.

Later in the same section (§482), Russell considers defining them as either (2) or (6) but claims this cannot be made a formal definition because “propositional functions are presupposed in defining the class of referents and relata of a relation.” This remark corresponds to remarks made earlier in the *Principles* that the sort of class mentioned at (2), and the sort of relation mentioned at (6), are different and in some ways more fundamental than other classes and relations. With regard to (2), recall that the general way of defining a class in the quasi-Peanist logic endorsed by Russell is by means of *such that* and a propositional function, which can be used even in those cases in which the members of the class have no common predicate. Hence, a class in general could be defined as something that can be denoted by means of *such that* along with the sort of class of propositions that is connected with a propositional function, i.e., the sort described at (2). Russell writes:

The notion of a class of propositions of constant form is more fundamental than the general notion of *class*, for the latter can be defined in terms of the former, but not the former in terms of the latter. (§86)

Hence, even if Russell were to identify propositional functions with entities of type (2), as he sometimes seems to (§91), it would not follow that notation for propositional functions could be gotten through the means of the notation he typically uses for classes. Russell also sometimes intimates that (6) is really how a propositional function should be

understood, writing even that “the function, as a single entity, is (6) above” (§482), but as with (2), claims that the normal apparatus of relations cannot be used for getting at such relations (*cf.* §80).

Of course, propositional functions cannot be both (2) *and* (6); these seem to be rival ways of making sense of the nature of propositional functions without identifying them with the entities considered and rejected with (1).²¹ In the body of the *Principles*, however, most of the time when Russell speaks of propositional functions, he seems to identify them with (3); the function indicated by “*x* is human” is not itself the class of all propositions differing from *Socrates is human* by at most having some other term occurring for Socrates, but instead it is “any member” of said class (see, e.g., §106). This requires further scrutiny. By saying that a propositional function is *any* member of a certain collection defined by constancy of form, he is not equating a propositional function with some one *particular* member. To see this we must say more about variables and Russell’s general conception of *any*.

We saw earlier that Russell’s writing sometimes suggests that he thinks of a propositional function as a proposition-like unity containing a variable instead of a definite term at one or more places. The details of his understanding of a variable as an extra-linguistic object are obscure, to say the least, and he himself, while professing the variable to be “the most distinctively mathematical of all notions” (§§86–7),²² admits from the outset that “in the present work a satisfactory theory as to its nature ... will hardly be found” (§6). One thing that is clear, however, is that his understanding of variables is connected with his views on *denoting* discussed in Chapter v. Russell defines a denoting concept this way: “A concept *denotes*, when, if it occurs in a proposition, the proposition is not *about* the concept, but about a term connected in a certain way with the concept” (*PoM*, §56). The concept-of-a-class, *all men*, as we have seen, denotes the class of humans, and when this denoting concept

²¹ Russell also suggests that the arguments in Chapter VII that assertions are not always separable entities may make using “ ϕ ” as a variable seem problematic. However, he suggests this can be remedied by understanding the range of the variable as being (2) or (6) rather than entities of type (1). See §§103, 482.

²² Contrast this with Frege’s claim that “variables are not a part of the proper subject-matter of arithmetic.” See his *Posthumous Writings*, ed. H. Hermes, F. Kambartel and F. Kaulbach (Chicago: U. of Chicago P., 1979), p. 238.

occurs in a proposition, the proposition is about the class. Besides denoting concepts of this stripe, there are such denoting concepts as *a man*, *some man*, *any man*, *every man* and *the man*. Different kinds of denoting concepts denote different kinds of objects. Those of the *all*-stripe denote classes, and classes are understood as “numerical conjunctions” (§59) of objects; the class of all men consists of *many* different people. Using grammar as his guide, he concludes, however, that the denoting concept *any number* must denote only one number, since the denoting phrase “any number” is grammatically singular, but he notes that this leads to the problem that while it denotes only one number, there is no number in particular that it denotes:

Any number is neither 1 nor 2 nor any other particular number, whence it is easy to conclude that *any number* is not any one number, a proposition at first sight contradictory, but really resulting from an ambiguity in *any*, and more correctly expressed by “*any number* is not *some* one number.” (§75)

He concludes that *any*-variety denoting concepts denote “variable conjunctions”, which he contrasts with classes in that while a class is a combination of individuals as many, forming a plurality, a variable conjunction is a non-plural combination of individuals. The variable conjunction of numbers is *one* number rather than many, although it is not any one in particular (§59). Russell admits that this sort of indefinite object is “very paradoxical” (§62) and admits that it leads to difficulties he does not know how to solve (§75). I think Russell himself was not fully happy with these views, yet, as unpalatable as they are, they do seem to be the official doctrines of the *Principles*.

In Chapter VIII, Russell links his understanding of variables with the entities denoted by denoting concepts of the *any*-variety. Indeed, he says that “*x*, the variable, is what is denoted by *any term*, and ϕx , the propositional function, is what is denoted by *the* proposition of the form ϕ in which *x* occurs” (§86). He admits, however, that when this understanding is applied to propositional functions with more than one variable, some complication is in order, since the difference between “*x* loves *x*” and “*x* loves *y*” is eliminated if both are rendered as “any term loves any term”. Hence he concludes that a “variable is not *any term* simply, but any term as entering into a propositional function. We may say, if ϕx be a propositional function, that *x* is *the* term in *any* proposi-

tion of the class of propositions whose type is ϕx ". This suggests the following understanding of a propositional function. The denoting concept involved with "x" in "x is human" denotes one individual only, but no individual in particular. The phrase "x is human" as a whole can be understood as representing a "dependent variable", of which "x" represents the "independent variable"²³; just as "x" represents one individual, but no individual in particular, "x is human" indicates one proposition, but no proposition in particular. In sum, "x is human" represents a non-plural combination of propositions just as "any number" represents a non-plural combination of numbers.

This sort of view is very difficult to comprehend, and leads to many complications. One complication that derives from his early theory of denoting in general is the need to disambiguate discourse about the denoting concept itself as opposed to the object it denotes. In the case of those denoting concepts such as *any number* or *any term*, however, the problem is in a way even more acute. It would seem that for a statement in which "any number" is used in its typical fashion, the assertion made would have to be true of the individual numbers 0, 1, 2, 3, 4 separately, and would not be true *either* of the concept, or of the variable conjunction the concept denotes. It would appear that the things of which it is true to say "is even or odd" are 0, 1, 2, 3, etc., not the variable conjunction, even if it is only one number and not many. As Russell himself admits, if "I met a man" is true, the thing I met was "an unambiguous, perfectly definite man" (§62), something with "with a tailor and a bank-account or public-house and a drunken wife" (§56). However, noting that "I met a man" does not say the same thing as, e.g., "I met Peter", Russell concludes even if it was Peter that I met, Peter cannot be what the denoting concept *a man* denotes, and hence cannot be what the proposition is about. He concludes that what is denoted is a special kind of non-plural combination, in this case a *variable disjunction* of men.

The oddity, however, is that, if *a man* denotes this variable disjunction in *I met a man* then it seems that what I met was the variable disjunction. But can this really be maintained? Russell claims that a prop-

²³ Russell uses this sort of terminology in the *Principles* at §254, but the usage is more common in manuscripts from the subsequent period, such as the 1903 manuscripts "Dependent Variables and Denotation", "Points about Denoting" and "On Meaning and Denotation". See *Papers* 4: 297–343.

osition such as that expressed by “Peter is a man”, on one understanding (taken as *Peter is a-man*, not *Peter is-a man*), represents an identity between Peter and an ambiguous man (p. 54n.). But if an ambiguous man is simply the variable disjunction of men, then Peter must be identical to the variable disjunction. This solves the problem of my having met such an odd thing as a variable disjunction; if they’re identical, to meet one is to meet the other. But this leads right into the problem that since *Paul is a man* is also true, by the symmetry and transitivity of identity, we get that Peter is Paul. If Peter is to be *distinguished* from the variable disjunction denoted by *a man*, and 1, 2, 3, etc., are each to be distinguished from the variable conjunction denoted by *any number*, then Russell needs not only to be able to distinguish truths about the concept *any number* from truths about the numbers themselves, he needs also to be able to distinguish those truths that hold only of the special sort of combinations denoted by such denoting concepts. However, Russell doesn’t quite seem to see this, and, when discussing it, he seems to misrepresent his own views:

When a class-concept, preceded by one of the six words *all*, *every*, *any*, *a*, *some*, *the*, occurs in a proposition, the proposition, is, as a rule, not *about* the concept formed of the two words together, but about an object quite different from this, in general not a concept at all, but a term or complex of terms. This may be seen by the fact that propositions in which such concepts occur are in general false concerning the concepts themselves. At the same time, it is possible to consider and make propositions about the concepts themselves. “Any number is odd or even” is a perfectly natural proposition, whereas “*Any number* is a variable conjunction” is a proposition only to be made in a logical discussion. (§65)

His example here suggests that the denoting concept *any number* is a variable conjunction. In §62, however, Russell strongly suggests that the variable conjunction is not the concept, but what it denotes (although *cf. Papers* 3: 196–8), and so the example here does not seem accurate. At other points, Russell hints that, contrary to his general contention that everything must be capable of occurring as logical subject in a proposition, “complexes of terms denoted by *any* and cognate words” may not be terms, i.e., may not be possible logical subjects (§§49, 483). If so, then variable conjunctions would be unique entities indeed. Truths about them could not take the paradigmatic form in which they themselves

occur as logical subject in a proposition. Moreover, those propositions containing denoting concepts that denote them represent truths about the individuals they variably conjoin, rather than truths about the variable conjunctions themselves. At no time in 1902 does Russell explore in any detail the nature of propositions about the unusual objects that *any*-style denoting concepts denote, as opposed to either the concepts or the definite individuals they combine.

Russell's failure to give this issue the attention it deserves, however, *explains* his inability to fully appreciate or bring to light the propositional functions version of the paradox. If his understanding of propositional functions is bound up with his understanding of variables, which is in turn bound up with his understanding of *any*-style denoting concepts and the objects they denote, even formulating the propositional functions version of the paradox would have required getting clearer about the nature of propositions about denoting concepts and variable conjunctions themselves as opposed to "unambiguous and definite" objects. We've seen reason for thinking that Russell thinks of a propositional function as an *ambiguous* proposition, as *any* member of some class of propositions of like form. Without a philosophical understanding of the nature of propositions about such odd entities, Russell is not even in a position to develop a notation for talking about propositional functions themselves as opposed to the specific propositions that are their values. Without such notation, the functions version of the paradox cannot be formulated.

THE PROPOSITIONAL FUNCTIONS PARADOX AT LAST

As noted earlier, Russell only made a close study of Frege's works after having finished the body of the *Principles*; the appendix devoted to Frege's work was written afterwards, and completed in November 1902. While the appendix contains criticism of Frege, over the next year, Russell's philosophical interests and positions take a decidedly Fregean turn. After the second volume of Frege's *Grundgesetze* appeared in early 1903, Russell added a last-minute footnote to the *Principles* (p. 522), endorsing Frege's ill-fated solution to the classes version of Russell's paradox, which involves supposing that two non-coextensional propositional functions can determine the same class. As Russell notes to Jourdain in the letter quoted at the start of this paper, Russell was himself attracted to the idea

(see, e.g., *Papers* 4: 3–37), but soon thought better of it. However, throughout 1903 one can see Frege’s influence in Russell’s work in other ways. Russell’s manuscripts from the year bear titles reminiscent of Frege’s own, such as “Functions and Objects” (*Papers* 4: 50–2) and “On Meaning and Denotation” (*Papers* 4: 314–58). Although his views during this period were not in all ways Fregean, the direction of his thought was as much guided by Frege’s influence as it was by Peano two years prior.

This brings us to May 1903, when Russell had thought he had “solved the whole thing”. The best record of Russell’s exuberance can be found in a letter to Frege dated 24 May 1903, where he writes, “I believe I have discovered that classes are entirely superfluous. Your designation $\dot{\epsilon} \phi(\epsilon)$ can be used for ϕ itself, and $x \cap \dot{\epsilon} \phi(\epsilon)$ for $\phi(x)$.”²⁴ Russell goes on to describe methodology for making functions do the work of classes in their shared logicist enterprise. For example, rather than defining numbers in terms of a cardinal similarity relation that holds between classes when they stand in 1–1 correspondence, Russell suggests defining them in terms of a similarity relation that holds between functions. Russell now uses “ ϕ ” and “ f ” for the propositional functions themselves rather than for assertions.²⁵ He alludes to the smooth-breathing notation

²⁴ See Frege, *Philosophical and Mathematical Correspondence*, p. 158. Much of their correspondence up to this point had concerned itself with formulations of various related paradoxes, and possible solutions. For further discussion, see my “Russell’s Paradox in Appendix B of *The Principles of Mathematics*: Was Frege’s Reply Adequate?”, *History and Philosophy of Logic*, 22 (2001): 13–28, and my *Frege and the Logic of Sense and Reference*, Chap. 6 (cited at n. 11).

²⁵ It is difficult to pin down the precise date for the switch. In his notes entitled “Frege on the Contradiction” (*Papers* 4: 608–19), probably written in February 1903, Russell comments on Frege’s use of functions having function as argument, writing “consider a function M which may have as argument any function of a term. Thus if $g\xi$ be such a function, $M(g)$ is admissible”, but criticizes Frege by writing, “[b]ut the argument is only the assertion, so that ξ forms no part of the argument” (p. 608). Frege used the signs “ ξ ” and “ ζ ” to indicate the distinct argument spots of those functions that take more than one argument. However, Frege thought that when a function took a function as argument, the higher-level function mutually saturates with the argument function, and the argument spot of the argument function is somehow completed. His general notation for this was “ $M_{\beta}(g(\beta))$ ”. See Frege, *Basic Laws of Arithmetic*, ed. M. Furth (Berkeley: U. of California P., 1964), §25. Here, the “ β ” is understood in a way as *part* of the function M . Russell seems to interpret Frege’s “ ξ ” and “ ζ ” as akin to his variables, and since the “ g ” part of “ $g\xi$ ” stands for the remainder of the function above and beyond the variable, interprets M as taking the assertion g as argument, not the function

" $\dot{\epsilon} \phi(\epsilon)$ " used by Frege for what he called the "*Wertverläufe*", or value-ranges, of functions (which, in the case of concepts, Frege identified with their extensions or corresponding classes). Russell, however, suggests that the notation be used *for functions* rather than classes: thereby inventing a kind of function abstraction. He continues, "In this way we can do arithmetic without classes. And this seems to me to avoid the contradiction."²⁶ Russell elaborates on the notation in his manuscript "Functions and Objects":

If ϕ denotes the function, $\phi|x$ will be used to denote the value of the function for argument x ; and conversely, if X denotes an expression containing x , $\dot{x}(X)$ will be used to denote the function involved. Thus $\dot{x}(X)|x$ will be another symbol for X ; and if we denote by Y what X becomes when y is substituted for x , $\dot{x}(X)|y$ will be another symbol for Y . (Papers 4: 50)

For example, then " $\dot{x}(x > 7)$ " would constitute a name of the propositional function whose value for 9 as argument is the proposition that $9 > 7$, and so on.

Given the greater role functions were now to play in his logicism, the abstraction notation was a must for being able to name and talk about specific complex propositional functions.²⁷ However, precisely what

gξ. In those notes, he also describes a quadratic form, i.e., a statement taking the form " $\phi(f(\phi))$ ", as "a variable assertion concerning a term which varies with the assertion" (Papers 4: 614). Elsewhere in those notes, however, he does not seem to respect the assertion/function distinction, possibly because the notes are on Frege, whom he believes to have conflated the two.

When Russell adopted the smooth-breathing abstract notation in May 1903, however, the variable which is part of the function is maintained in a different way. In " $f|g$ ", the " g " can stand for the entire function consisting of both assertion and variable, since the allowable instances of " g " look like, e.g., " $\dot{x}(x > 7)$ ", so that we might write " $f|\dot{x}(x > 7)$ ". The abstraction notation contains the variable letter " x ". Here the argument is more than just the assertion. Later in 1903, Russell seems to return to something more like the old notation and usage of " ϕ " or " f " by itself as standing for the assertion and only " $f\xi$ " for the function (see, e.g., Papers 4: 345–8). By the time of the letter to Jourdain in 1906, " $W(\phi)$ " would seem to be a function with functions (not assertions) as argument. Of course, during that period, Russell was working on the substitutional theory, in which functions are not treated as entities, and so he then had no "official notation" for them.

²⁶ See Frege, *Philosophical and Mathematical Correspondence*, p. 159.

²⁷ At one point in early 1903, Russell had apparently considered a notation " $\psi x \div x$ "

makes this notation so useful in proxying talk of classes also makes it possible to formalize the propositional functions version of the paradox, because the abstracts stand for the functions themselves, as opposed to arbitrary values. Sometime soon afterwards Russell considered the function $\hat{x}(\sim x|x)$, and from it formalized the functions version of the contradiction. This appears explicitly for (what I think is) the first time in a manuscript entitled “No Greatest Cardinal” (*Papers* 4: 62–3), probably written sometime in the summer of 1903.²⁸ This surely brought an end to his “unspeakable” relief. However short-lived, the illusion of having solved the whole thing would not have been possible if he had, prior to this time, ever explicitly considered the functions version of the paradox. If he had, surely, his reaction would have been calmer, and less exuberant. Frege, of course, saw that Russell’s proposed solution fared no better in solving the problem, writing back to Russell over a year later claiming that Russell’s suggestion “would lead to the same difficulties as my value-range notation”, and sketching a version of the contradiction using function abstraction.²⁹ Russell wrote back to Frege in late 1904 admitting that “I have known already for about a year that my attempt to make classes entirely dispensable was a failure, for essentially the same reasons as you give.”³⁰

Russell’s immediate reaction to the functions version of the paradox was to hold that the notation “ $\hat{x}(\dots x \dots)$ ” was inadmissible in certain cases but not others, and this became his “zig-zag theory”, according to which functions cannot be abstracted from complexes when the complexes have certain features. Later, in late 1905, Russell abandoned propositional functions as individual entities altogether as part of his new

for “a function itself, without its argument” (*Papers* 4: 37), but apparently did not make the same sorts of realizations, probably because he was not yet proxying classes in terms of functions, and so “ $\psi x \div x$ ” would have at most a small role to play in his logicist project.

²⁸ The manuscript is undated, but due to notational clues, it appears to have been written not much later than May 1903. Russell seems aware of the problem in the manuscript “On the Meaning and Denotation of Phrases” (see *Papers* 4: 291), which we have good reason to date earlier than 22 August 1903 (see *Papers* 4: 283).

²⁹ Frege, *Philosophical and Mathematical Correspondence*, pp. 161–2. Frege there uses a “rough-breathing” (*spiritus asper*) for function abstraction “ $\hat{\epsilon}(\dots \epsilon \dots)$ ”, rather than the smooth-breathing (*spiritus lenis*) he elsewhere used for class/value-range abstraction.

³⁰ *Ibid.*, p. 166.

“substitutional theory”. I have discussed the development of Russell’s views on propositional functions from May 1903 to the birth of the substitutional theory elsewhere.³¹ However, his views changed rapidly during this period, largely involving the nature of the relationship of functions to denoting complexes and to the variable. The story we have told with regard to the origins of the propositional functions version of the paradox sheds considerable light on exactly why it is that Russell’s views developed as they did. The notation, “ $\hat{x}(x > 7)$ ”, after Whitehead’s letter of April 1904, became “ $(\hat{x} > 7)$ ”. This too is a device for speaking of a function itself as opposed to one of its values. No longer positing such strange combinations of objects as variable conjunctions and variable disjunctions, Russell now portrays a function as itself a denoting complex or meaning containing a variable (rather than the denotation), and portrays its specific values as the entities denoted by that meaning. On this view, to talk about a function itself is to talk about a meaning. He writes:

The circumflex has the same sort of effect as inverted commas have. E.g. we say

Any man is a biped;
“Any man” is a denoting concept.

The difference between $p \supset q \cdot \supset \cdot q$ and $\hat{p} \supset \hat{q} \cdot \supset \cdot \hat{q}$ corresponds to the difference between any man and “any man”. (*Papers* 4: 128–9)

As we have seen, formulating the functions version of the paradox requires a notation for talking about the function itself as opposed to its values. This explains why it is that Russell explored the notion of denoting with renewed interest in 1904–05, when his main philosophical task was the search for a solution to the paradoxes. As he put it in later in the same 1906 letter to Jourdain:

... in April 1904 I began working at the Contradiction again, and continued at it, with few intermissions till January 1905. I was throughout much occupied by the question of Denoting, which I thought was probably relevant, as it proved to be.³²

³¹ See my “Russell’s 1903–1905 Anticipation of the Lambda Calculus” (cited at n. 17).

³² Quoted in Grattan-Guinness, *Dear Russell—Dear Jourdain*, p. 79. Cf. *Auto.* 1: 155, *MPD*, p. 60.

His exploration of denoting, of course, led him to the theory of descriptions in 1905, and, in “On Denoting”, to call into question the very coherence of disambiguating between meaning and denotation within those theories that invoke such a divide.³³ The topic of the “Gray’s Elegy Argument” is almost *precisely* the same issue that the propositional functions version of the paradox had raised for him in 1903.

³³ For discussion, see my “Russell on ‘Disambiguating with the Grain’”, *Russell*, n.s. 21 (2001): 101–27.
