[A] Preliminaries

A.1 Some Vocabulary

An *object language* is a language under discussion or being studied (the "object" of our study).

A *metalanguage* is the language used when discussing an object language.

In logic courses we often use English as a metalanguage when discussing an object language consisting of logical symbols along with variables, constants and/or propositional parameters.

As logical studies become more advanced and sophisticated, it becomes more and more important to keep the object language and metalanguage clearly separated.

A *logical system*, or *a "logic"* for short, typically consists of three things (but may consist of only the first two, or the first and third):

1. A *syntax*, or set of rules specifying what expressions are part of the language of the system, and how they may be combined to form more complex expressions/statements (often called "formulæ)".
2. A *semantics*, or set of rules governing the meanings or possible meanings of expressions, and how the meaning, interpretation, evaluation and truth value of complex expressions depend on the meaning or interpretation of the parts.
3. A *deductive system*, or set of rules governing what makes for an acceptable or endorsed pattern of reasoning within in the system.

For example, in your first course on propositional or sentential logic, you learned that the language consisted of propositional parameters or variables, p_0, p_1, etc., symbols such as "$\vee$", "$\neg$", and that they be combined to form complex formulæ such as "$\neg(p_1 \vee \neg p_0)$". This was syntax. You also learned that these formulæ may be true or false and that truth-functions, such as those reflected in truth tables, were used in determing the truth or falsity of complex formulæ. That was *semantics*. Finally, you learned a set of rules for constructing proofs or derivations, such as *modus ponens*: that was a deductive system.

Logical metatheory is the use study of a logical system or logical object language using the logical resources of a metalanguage.

An example of logical metatheory would be giving a proof, *in English*, that a certain logical system is sound and complete, i.e., that every inference its deductive system allows is valid according to its semantics, and vice versa. Since the proof is about a logical system, the proof is not given *within* that logical system, but within the metalanguage.

Though it is not our main emphasis, we will be doing a fair amount of logical metatheory in this course. Our metalanguage will be English, and to avoid confusion, we will use English words like "if", "and" and "not" rather than their corresponding object language equivalents when working in the metalanguage. We will, however, be employing a smattering of symbols in the metalanguage, including generic mathematical symbols, symbols from set theory, and some specialized symbols such as "$\vDash$" and "$\vdash$".

A.2 Basic Set Theory

We shall use these signs *metalanguage* only. (In another logic course, you might find such signs used in the object language.)

A *set* is a collection of entities for which it is determined, for every entity of a given type, that the entity either is or is not included in the set.

An *urelement* is a thing that is not a set.

An entity A is a *member* of set Γ iff it is included in that set.

We write this as: "$A \in \Gamma$". We write "$A \notin \Gamma$" to mean that A is not a member of Γ.

Sets are determined entirely by their members: for sets Γ and Δ, $\Gamma = \Delta$ iff for all A, $A \in \Gamma$ iff $A \in \Delta$.

A *singleton* or *unit set* is a set containing exactly one member.

"$\{A\}$" means the set containing A alone. Generally, "$\{A_1, \ldots, A_n\}$" means the set containing all of $A_1, \ldots, A_n$, but nothing else.

The members of sets are not ordered, so from $\{A, B\} = \{C, D\}$ one cannot infer that $A = C$, only that either $A = C$ or $A = D$.

If Δ and Γ are sets, Δ is said to be a *subset* of Γ, written "$\Delta \subseteq \Gamma$", iff all members of Δ are members of Γ; and Δ is said to be a *proper subset* of Γ, written "$\Delta \subset \Gamma$", iff all members of Δ are members of Γ, but not all members of Γ are members of Δ.

If Δ and Γ are sets, the *union* of Δ and Γ, written "$\Delta \cup \Gamma$", is the set that contains everything that is a member of either Δ or Γ.

The *intersection* of Δ and Γ, written "$\Delta \cap \Gamma$", is the set that contains everything that is a member of both Δ and Γ.

The *relative complement* of Δ and Γ, written "$\Delta - \Gamma$", is the set containing all members of Δ that are not members of Γ.

The *empty set* or *null set*, written "$\varnothing$", "Λ" or "{ }", is the set with no members.

If Γ and Δ are sets, then they are *disjoint* iff they have no members in common, i.e., iff $\Gamma \cap \Delta = \varnothing$.

An *ordered n-tuple*, written "$\langle A_1, \ldots, A_n \rangle$", is something somewhat like a set, except that the elements are given a fixed order, so that $\langle A_1, \ldots, A_n \rangle = \langle B_1, \ldots, B_n \rangle$ iff $A_i = B_i$ for all i such that $1 \leq i \leq n$.

An ordered 2-tuple, e.g., $\langle A, B \rangle$ is also called an *ordered pair*. An entity is identified with its 1-tuple.

If Γ and Δ are sets, then the *Cartesian product* of Γ and Δ, written "$\Gamma \times \Delta$", is the set of all ordered pairs $\langle A, B \rangle$ such that $A \in \Gamma$ and $B \in \Delta$. Generally, "Γ^n" is used to represent all ordered n-tupes consisting entirely of members of Γ. Notice that $\Gamma^2 = \Gamma \times \Gamma$.

An *n-place relation (in extension)* on set Γ is any subset of Γ^n. A 2-place relation is also called a *binary relation*. Binary relations are taken to be of sets of ordered pairs. A 1-place relation is also called (the extension of) a *property*.

If R is a binary relation, then the *domain* of R is the set of all A for which there is an B such that $\langle A, B \rangle \in R$.

If R is a binary relation, the *range* of R is the set of all B for which there is an A such that $\langle A, B \rangle \in R$.

The *field* of R is the union of the domain and range of R.

If R is a binary relation, R is *reflexive* iff $\langle A, A \rangle \in R$ for all A in the field of R.

If R is a binary relation, R is *symmetric* iff for all A and B in the field of R, $\langle A, B \rangle \in R$ only if $\langle B, A \rangle \in R$.

If R is a binary relation, R is *transitive* iff for all A, B and C in the field of R, if $\langle A, B \rangle \in R$ and $\langle B, C \rangle \in R$ then $\langle A, C \rangle \in R$.

A binary relation R is an *equivalence relation* iff R is symmetric, transitive and reflexive.

If R is an equivalence relation then, the *R-equivalence class on* A, written "$[A]^R$", is the set of all B such that $\langle A, B \rangle \in R$.

A *function (in extension)* is a binary relation which, for all A, B and C, if it includes $\langle A, B \rangle$ then it does not also contain $\langle A, C \rangle$ unless $B = C$.

So if F is a function and A is in its domain, then there is a unique B such that $\langle A, B \rangle \in F$; this unique B is denoted by "$F(A)$".

An *n-place function* is a function whose domain consists of n-tuples. For such a function, we write "$F(A_1, \ldots, A_n)$" to abbreviate "$F(\langle A_1, \ldots, A_n \rangle)$".

An *n-place operation on* Γ is a function whose domain is Γ^n and whose range is a subset of Γ.

If F is a function, then F is *one-one* iff for all A and B in the domain of F, $F(A) = F(B)$ only if $A = B$.

If Γ and Δ are sets, then they are *equinumerous*, written "$\Gamma \cong \Delta$", iff there is a one-one function whose domain is Γ and whose range is Δ.

Sets Γ and Δ have the *same cardinality* or *cardinal number* if and only if they are equinumerous.

If Γ and Δ are sets, then the cardinal number of Γ is said to be *smaller* than the cardinal number of Δ iff there is a set Z such that $Z \subseteq \Delta$ and $\Gamma \cong Z$ but there is no set W such that $W \subseteq \Gamma$ and $W \cong \Delta$.

If Γ is a set, then A is *denumerable* iff Γ is equinumerous with the set of natural numbers $\{0, 1, 2, 3, 4, \ldots,$ (and so on ad inf.)$\}$.

Aleph null, also known as *aleph naught*, written "$\aleph_0$", is the cardinal number of any denumerable set.

If Γ is a set, then Γ is *finite* iff either $\Gamma = \varnothing$ or there is some positive integer n such that Γ is equinumerous with the set $\{1, \ldots, n\}$.

A set is *infinite* iff it is not finite.

A set is *countable* iff it is either finite or denumerable.

Homework

Assuming that Γ, Δ and Z are sets, R is a relation, and A and B are any entities, informally verify the following:

1. $A \in \{B\}$ iff $A = B$
2. if $\Gamma \subseteq \Delta$ and $\Delta \subseteq Z$ then $\Gamma \subseteq Z$
3. if $\Gamma \subseteq \Delta$ and $\Delta \subseteq \Gamma$ then $\Gamma = \Delta$
4. $(\Gamma \cap \Delta) \cup (\Gamma - \Delta) = \Gamma$

5. If R is an equivalence relation, then ($[A]^R = [B]^R$ iff $\langle A, B\rangle \in R$) and (if $[A]^R \neq [B]^R$ then $[A]^R$ and $[B]^R$ are disjoint).

A.3 Mathematical Induction

The *principle of mathematical induction* states:
If (ϕ is true of 0), then if (for all natural numbers n, if ϕ is true of n, then ϕ is true of $n + 1$), then ϕ is true of all natural numbers.

To use the principle mathematical induction to arrive at the conclusion that something is true of all natural numbers, one needs to prove the two antecedents, i.e.:

Base step. ϕ is true of 0

Induction step. For all natural numbers n, if ϕ is true of n, then ϕ is true of $n + 1$

Typically, the induction step is proven by means of a conditional proof in which it is assumed that ϕ is true of n, and from this assumption it is shown that ϕ must be true of $n + 1$. In the context of this conditional proof, the assumption that ϕ is true of n is called the *inductive hypothesis*.

From the principle of mathematical induction, one can derive a related principle:

The *principle of complete* (or *strong*) *induction* states:
If (for all natural numbers n, whenever ϕ is true of all numbers less than n, ϕ is also true of n) then ϕ is true of all natural numbers.

In logical metatheory, we most commonly use certain corollaries of the above.

The *principle of wff induction* states that:
For a given logical language, if ϕ holds of the simplest well-formed formulas (wffs) of that language, and ϕ holds of any complex wff provided that ϕ holds of those simpler wffs out of which it is constructed, then ϕ holds of all wffs.

This is a corollary of mathematical induction. Actually, it is a version of it. Let ϕ' be the property a number has if and only if all wffs of the logical language having that number of logical operators have ϕ. If ϕ is true of the simplest well-formed formulas, i.e., those that contain zero operators, then 0 has ϕ'. Similarly, if ϕ holds of any wffs that are constructed out of simpler wffs provided that those simpler wffs have ϕ, then whenever a given natural number n has ϕ' then $n+1$ also has ϕ'. Hence, by mathematical induction, all natural numbers have ϕ', i.e., no matter how many operators a wff contains, it has ϕ. In this way wff induction simply reduces to mathematical induction.

Similarly, this principle is usually utilized by proving the antecedents, i.e.:

Base step. ϕ is true of the simplest well-formed formulas (wffs) of that language; and

Induction step. ϕ holds of any wffs that are constructed out of simpler wffs provided that those simpler wffs have ϕ.

Again, the assumption made when establishing the induction step that ϕ holds of the simpler wffs is called the inductive hypothesis.

We may also use:

The *principle of proof induction*:
In a logical system that contains a deductive system, if ϕ is true of a given step of the proof whenever ϕ is true of all previous steps of the proof, then ϕ is true of all steps of the proof.

The principle of proof induction is an obvious corollary of the principle of complete induction. The steps in a proof can be numbered; we're just applying complete induction to those numbers.

Homework

Answer any of these we don't get to in class:

1. Let ϕ be the property a number x has just in case the sum of all numbers leading up to and including x is $\frac{x(x+1)}{2}$. Use the principle of mathematical induction to show that ϕ is true of all natural numbers.
2. Let ϕ be the property a number x has just in case it is either 0 or 1 or it is evenly divisible by a prime number greater than 1. Use the principle of complete induction to show that ϕ is true of all natural numbers.
3. Let ϕ be the property a wff A of classical propositional logic has if and only if has a even number of parentheses. Use the principle of wff induction to show that ϕ holds of all wffs of propositional logic. (If needed, consult the next page for a definition of a wff in classical propositional logic.)
4. Consider a logical system for propositional logic that has only one inference rule: *modus ponens*. Use the principle of proof induction to show that every line of a proof in this system is true if the premises are true.

[B] Classical Propositional Logic

B.1 Syntax

A *propositional parameter* or *propositional variable* is any member of the list: $p_0, p_1, p_2, \ldots$.

Following Priest, we use the letters p, q, s, t, etc., *without subscripts* as placeholders for *any arbitrary* propositional parameter. Note that strictly speaking, these are part of the metalanguage, whereas those with subscripts are part of the object language. The schema $\ulcorner p \supset (q \supset p)\urcorner$ represents *any* of "$p_0 \supset (p_1 \supset p_0)$" or "$p_1 \supset (p_2 \supset p_1)$" or "$p_2 \supset (p_3 \supset p_2)$", etc. Strictly speaking, these latter examples are object language expressions, but $\ulcorner p \supset (q \supset p)\urcorner$ is not in the object language at all. Worrying about this difference is pure fussiness we shall usually ignore.

A *propositional connective* or *propositional operator* is any of the signs: $\neg$, $\supset$, $\equiv$, $\wedge$, $\vee$. The sign $\neg$ is a monadic connective; the others are binary connectives.

You may be used to other symbols:

Priest's sign	Alternatives	Meaning
$\neg$	$\sim, -$	Negation
$\supset$	$\rightarrow, \Rightarrow$	Material conditional
$\equiv$	$\leftrightarrow, \Leftrightarrow$	Material equivalence
$\wedge$	$\&, \bullet$	Conjunction
$\vee$	(none)	Inclusive disjunction

A *well-formed formula* (or *wff* for short) is defined recursively as follows:
(i) propositional parameters are wffs;
(ii) if A is a wff, then so is $\neg A$;
(iii) if A and B are wffs, then so is $(A \supset B)$;
(iv) if A and B are wffs, then so is $(A \equiv B)$;
(v) if A and B are wffs, then so is $(A \wedge B)$;
(vi) if A and B are wffs, then so is $(A \vee B)$; and
(vii) nothing that cannot be constructed from repeated applications of (i)–(vi) is a wff.

As a convention, matching outermost parentheses of a wff may be omitted. Thus, "$((p_0 \vee p_1) \supset \neg(p_2 \wedge p_1))$" will be written simply "$(p_0 \vee p_1) \supset \neg(p_2 \wedge p_1)$".

The letters A, B, C, etc., are used (in the metalanguage) schematically to represent any arbitrary formulæ of the object-language. Hence, $\ulcorner A \supset \neg B\urcorner$ represents not only the form of "$p_0 \supset \neg p_1$" and "$p_2 \supset \neg p_0$" but also that of "$(p_1 \wedge \neg p_1) \supset \neg(p_0 \equiv p_1)$" and "$(p_0 \equiv \neg(p_0 \wedge p_3)) \supset \neg\neg\neg p_2$", etc. (Some other books may reverse the conventions for upper vs. lowercase for these purposes.)

B.2 Semantics

An *interpretation* or *truth-value assignment* of the language is a function ν which assigns either of the truth values, true (1) or falsity (0) to each of the propositional parameters of the language. (I.e., for each p, either $\nu(p) = 1$ or $\nu(p) = 0$, but not both.)

Indirectly, each interpretation determines a truth value for every complex formula as well, according to the truth functions represented by the connectives, applied recursively:

$$\nu(\neg A) = \begin{cases} 1, & \text{if } \nu(A) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu(A \supset B) = \begin{cases} 1, & \text{if } \nu(A) = 0 \text{ or } \nu(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu(A \equiv B) = \begin{cases} 1, & \text{if } \nu(A) = \nu(B); \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu(A \wedge B) = \begin{cases} 1, & \text{if } \nu(A) = 1 \text{ and } \nu(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu(A \vee B) = \begin{cases} 1, & \text{if } \nu(A) = 1 \text{ or } \nu(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

If Δ is a set of wffs, and A is a wff, then A is a *semantic consequence* of Δ (and the argument $\Delta \therefore A$ is said to be *logically valid*) just in case there is no interpretation ν such that for every member B of Δ, $\nu(B) = 1$ but $\nu(A) = 0$ (i.e., it is impossible to make all the premises true and the conclusion false).

This may be written *in the metalanguage*:

$$\Delta \vDash A$$

If Δ has finitely many members $B_1, B_2, \ldots, B_n$, we may write simply:

$$B_1, B_2, \ldots, B_n \vDash A$$

A wff A is said to be a *tautology* or *logical truth* or *logically valid* on its own just in case $\varnothing \vDash A$, or, equivalently, *every* interpretation ν is such that $\nu(A) = 1$.

This may be written simply:

$$\vDash A$$

Truth tables (which we shall not review fully) represent decision procedures for determining logical validity; each row represents a different possible interpretation.

B.3 Deductive Validity

We might here present a traditional deductive system for classical propositional logic. However, I assume you already familiar with at least one such system, whether it is a natural deduction system or axiom system. All such standard systems are equivalent and yield the same results.

We write:

$$\Delta \vdash A$$

to mean that A can be derived in a deductive system using only members of Δ as (non-logical) premises. If A can be derived from the rules alone without any additional premises, we may write simply:

$$\vdash A$$

Notice the difference, but similarity, between the double turnstile $\vDash$ for capturing semantic validity and $\vdash$ for capturing the deductive or *proof-theoretic* conception of validity. It is sometimes also called the *syntactic* conception of validity, because deductive rules of inference are usually stated in syntactic terms.

If our deductive system is set up properly, we should have the following metatheoretic results:

Soundness: If $\Delta \vdash A$ then $\Delta \vDash A$, for all Δ and A.

Completeness: If $\Delta \vDash A$ then $\Delta \vdash A$, for all Δ and A.

Together these results entail the equivalence of semantic and deductive validity.

Proofs of these results with Priest's tableaux method of constructing proofs or derivations are given in the book. We now present this method as an alternative to more customary styles of deductive systems.

B.4 Tableaux

A semantic tableau (plural, *tableaux*), also called a *truth tree*, represents a kind of demonstration that a certain formula or collection of formulæ is either logically possible or impossible. Indirectly, the technique can be used to establish that a formula is a logical truth by showing that its negation is impossible, or that an argument is valid by showing that the combination of the premises and negation of the conclusion is impossible.

One way of understanding the technique is akin to the method of abbreviated truth tables. For example, a full truth table can be used to establish the tautologyhood of $(p \supset q) \vee (s \supset p)$:

$$\begin{array}{ccccccc} (p & \supset & q) & \vee & (s & \supset & p) \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}$$

A less cumbersome method however would be attempt to construct a row on which it is false and show this cannot be done. We'd start by filling in a 0 under the main operator:

$$\begin{array}{ccccccc} (p & \supset & q) & \vee & (s & \supset & p) \\ \hline & & & 0 & & & \end{array}$$

Because disjunctions are only false when both disjuncts are false, we'd get:

$$\begin{array}{ccccccc} (p & \supset & q) & \vee & (s & \supset & p) \\ \hline & 0 & & 0 & & 0 & \end{array}$$

And then by the rule governing $\supset$:

$$\begin{array}{ccccccc} (p & \supset & q) & \vee & (s & \supset & p) \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \uparrow & & & & & & \uparrow \end{array}$$

Now we see that we have assigned p both 1 and 0 in the same row, which is impossible. The obvious conclusion is that no truth value assignment (interpretation) can make the formula false, so it must be logically valid.

The same process can be used to show that a formula isn't logically valid if the process continues until the entire row is filled out, but nothing is given incompatible assignments:

$$\begin{array}{ccccccc} (p & \wedge & q) & \supset & (p & \supset & s) \\ \hline 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}$$

This technique could be used to show that an argument is valid or invalid by starting with assigning 1 to each premise and 0 to the conclusion, and attempting to fill in the rest of the row until it is completed successfully (providing a counterexample, or "counter interpretation") or one is forced into an inconsistent assignment (in which case the argument is valid).

This process gets messy however, if there are multiple ways for a certain formula or subformula to have a certain truth value. Consider:

$$\begin{array}{ccccccc} (p & \vee & q) & \equiv & (q & \supset & p) \\ \hline & & & 0 & & & \end{array}$$

Here there are two ways for the biconditional to be false: for the left to be true and the right to be false, or the left to be false while the right is true:

$$\begin{array}{ccc} \multicolumn{3}{c}{(p \lor q) \equiv (q \supset p)} \\ \hline 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}$$

Continuing the process, we'll see that we'd need to "split" the rows again, since there are multiple ways for the disjunction to be true in the first row, and there are multiple ways for the conditional to be true in the second row. Some of these possibilities will lead to inconsistent assignments; some will not. The result would be a rather messy table.

Tableux can be seen as doing the same thing, but without the mess.

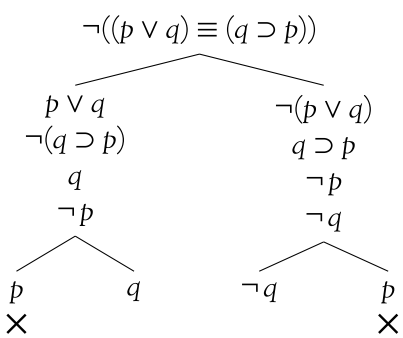
Rather than using 1s and 0s or Ts and Fs, we begin by simply writing a formula or its negation (or a collection thereof).

If the assumption of that formula's truth or falsity requires parts of it to be definitely true or false we write those results underneath.

If the assumption of its truth or falsity can be realized in more than one way, we create separate "branches" for each possibility.

We apply the same method to the results on each branch until we get both a formula and its negation on the same branch, which is the same as getting an incompatible assignment in an abbreviated truth table, or we successfully identify an interpretation consistent with our original hypothesis. Branches leading to inconsistencies are closed with the absurdity symbol ×.

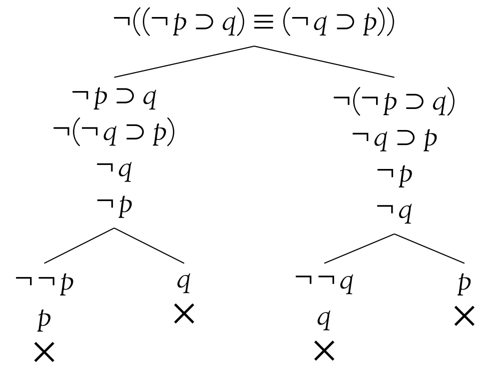
Here is the tableau for our previous example:



If a completed tableau contains "open" branches (those not closed by ×), it represents a truth-value assignment consistent with the original assumption(s).

Here the two open branches in the middle each represents a truth-value assignment showing that "$(p \lor q) \equiv (q \supset p)$" is not a tautology. The middle left branch dhows that it is false when q is true and p is false; the middle right shows that it is false when p and q are both false.

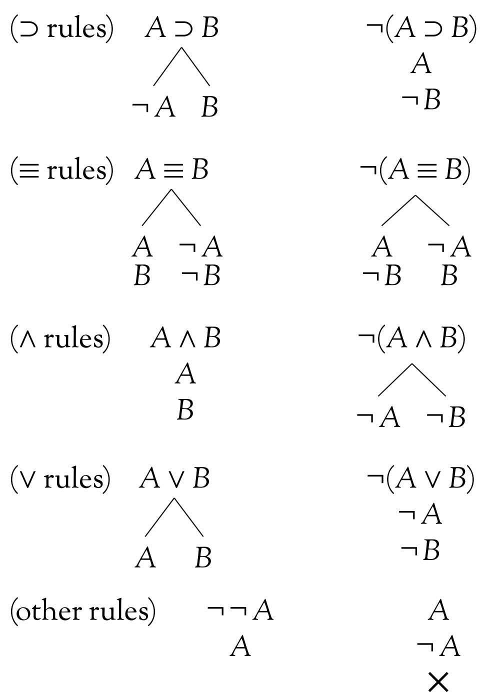
On the other hand, if we begin a tableau with the negation of a tautology, all branches will close:



To complete a tableau, we must apply the appropriate rule to each formula on the tree. The rule is applied to all open branches (if any) under the formula.

Once a branch closes, we may cease working on it.

For each connective, we have two tableau rules: one for formulæ of that form, and one for negations thereof. We also have a rule for double negation and for closing branches. (There are no special rules for atomic propositional parameters or their negations; these are used simply to determine the truth-value assignment should any branches remain open.)



Hopefully it is obvious that these match the semantic rules for the language. Perhaps it is also obvious how these

rules and a tableau style proof generally could be transformed into a more traditional proof (using, e.g., *reductio* or *indirect proof* argumentation with branches representing proofs by cases).

To avoid dealing with too many branches it is generally easiest to apply non-branching rules *before* branching rules. It is also helpful to place checkmarks next to each formula as you apply the rule to it to ensure that none is forgotten.

To establish the validity of the argument: $p \supset \neg\neg q, \neg p \vee \neg q \therefore \neg p$, we show that the tableau assuming its premises true and conclusion false has no open branches.

$$\begin{array}{ccccc}
 & \checkmark\, p \supset \neg\neg q & & & \\
 & \checkmark\, \neg p \vee \neg q & & & \\
 & \checkmark\, \neg\neg p & & & \\
 & p & & & \\
\neg p & & & \checkmark\, \neg\neg q & \\
\times & & & q & \\
 & & \neg p & & \neg q \\
 & & \times & & \times
\end{array}$$

A branch is said to be *closed* iff it terminates in $\times$.

A branch is said to be *complete* iff it is either closed or the appropriate rule has been applied to every formula on the branch that is not either an atomic propositional parameter or its negation, and any contradictions have resulted in $\times$.

A tableau is said to be *complete* iff every branch is complete.

A branch is said to be *open* iff it is not closed.

A tableau is said to be *open* iff it is complete and has at least one open branch.

Result: If a branch of a tableau is complete and open, there is at least one interpretation or truth-value assignment ν such that, for every formula A on that branch, $\nu(A) = 1$.

There is a full metatheoretic proof in the object language of this result in the book, but it may be obvious informally. Choose an assignment ν such that, for every atomic parameter p, $\nu(p) = 1$ if p occurs alone on the branch, and $\nu(p) = 0$ if $\neg p$ occurs instead. (If neither occurs, $\nu(p)$ may be assigned either 0 or 1, indifferently.) The connection between tableau rules and semantic rules will guarantee that $\nu(p) = 1$ for the more complex formulæ on the branch as well.

Corollary: If the tableau for the formulæ $B_1, B_2, \ldots, B_n, \neg A$ is open, then there is at least one interpretation ν making all of $B_1, B_2, \ldots, B_n$ true and A false; hence the argument $B_1, B_2, \ldots, B_n \therefore A$ is not logically valid. (In this instance, the interpretation ν is called a *counter-model*.)

Tableaux represent a method of finding a counter-model if one exists.

Corollary of corollary: By transposition, if the argument $B_1, B_2, \ldots, B_n \therefore A$ is logically valid, then the tableau for it will close.

This effectively establishes the completeness of this method; a tableau proof can be created for any valid argument. All that remains to be noted is that it is always possible to complete a tableau for a finite number of formulæ in a finite number of steps. This is guaranteed in classical propositional logic by the fact that no formula contains infinitely many binary connectives, and each application of a rule reduces the number of binary connectives in the resulting formulas. Since the number cannot decrease forever, one will eventually "bottom out" in atomic propositional parameters and their negations, and so the tableau will be complete.

Homework

Priest, §1.14, problem 1, any 7 of (a)–(j) (pp. 18–19).

B.5 *Limitations and Criticisms of Classical Logic and the Material Conditional*

Many of the motivations for deviating from classical logic can be appreciated by considering the motivations *for* certain other specific approaches. For example, intuitionist logic has its origins in certain views in the philosophy of mathematics according to which certain mathematical statements might be well-formed without either them or their negations being true, just given the nature of mathematics. Other rationales for accepting truth value gaps might include semantic failures (non-referring names, etc.), category errors, incomplete semantic rules, vagueness, etc.; arguments for truth value gluts might include logical and semantic antinomies, and more. Hopefully we'll have the opportunity to discuss these issues in more depth as we look at systems that aim to address them.

Priest focuses most of his discussion in chapter 1 with issues surrounding what are perceived as defects or inadequacies with the material conditional $\supset$ when understood with its classical semantics. Perhaps that is not the right way to frame the issue: it would better to say that there

appears to be a mismatch between the meaning or interpretation of "$A \supset B$", when interpreted as equivalent to "$\neg A \vee B$", and the meaning or interpretation of the natural language (English) phrase "if A then B".

Here are some (overlapping) concerns, sometimes (misleadingly?) called *the paradoxes of material implication*:

1. $A \supset B$ is true regardless of whether or not there is a connection between A and B, but consider:
   - If $2 + 2 = 4$, then it will rain this week.
   - If dinosaurs are still alive, then I can fly.
2. $A \supset B$ can be true even when there seems to be the wrong sort of connection between A, provided A is false or B true for other reasons; consider:
   - If McCain is president, then a Democrat is president.
   - If Oswald didn't shoot Kennedy, then no one else did.
   - If $2 + 2 = 5$, then $2 + 2 \neq 5$.
3. $A \supset B$ is always true when B is a tautology, or A a contradiction.
   - If pigs fly and pigs don't fly, then Kevin is the ruler of the universe.
   - If Katy Perry sings at UMass, then either Genghis Khan will come back to life or he won't.
4. $A \supset B$ does not distinguish between accidental occurrences and real links between claims:
   - If you drink Coke, you will die.
   - If you visit the Middle East, you will be killed by terrorists. (Spoken to someone who will never visit there.)
5. $A \supset B$ at best represents "indicative" conditionals and not subjunctive conditionals:
   - "If Oswald didn't shoot Kennedy, someone else did" vs. "If Oswald didn't shoot Kennedy, someone else *would* have."
   - "If you were nice to her, she was rude to you." vs. "If you had been nice to her, she would have been rude to you."
6. Some formal results that seem surprising if read as "if":
   $\neg(A \supset B) \vDash A$
   $\vDash (A \supset B) \vee (B \supset C)$
   $A \supset (B \vee C) \vDash (A \supset B) \vee (A \supset C)$
   $(A \wedge B) \supset C \vDash (A \supset C) \vee (B \supset C)$
   $(A \supset B) \wedge (C \supset D) \vDash (A \supset D) \vee (C \supset B)$
   But consider, e.g.:
   - It's not true that if God doesn't exist, then morality is impossible. Therefore, God doesn't exist.
   - Either if I make any typos on this page, Romney will be elected, or if Romney is elected, World War III will ensue.
   - If I wear a blindfold and I drive, I'll get in trouble; hence, either if I wear a blindfold, I'll get in trouble, or if I drive, I'll get in trouble.

One possible response to these worries comes from the notion of *conversational implicature*. Perhaps "if A then B" doesn't *mean* something stronger than $A \supset B$, but it is reasonable to make further inferences from the additional assumption that the speaker of "if A then B" is cooperating conversationally. If the only reason for believing "$A \supset B$" is the independent knowledge of $\neg A$ or independent knowledge of B, then one could reasonably be expected to assert the logically stronger information. Such explanations are often accepted in debates regarding disjunctions.

Priest also relates an informal argument given in favor of the equivalence of "$\supset$" and "if ... then ..." taken from Faris (1968).

1. First it is clear that the truth of "if A then B" entails the truth of $A \supset B$. First, either A or $\neg A$. If the latter, then, by its semantics, $A \supset B$ holds. If A, then from the fact that if A then B, by modus ponens, B holds, and if B holds, so does $A \supset B$. (This half of the argument is *relatively* uncontroversial, and most stronger conditionals we will explore later in the course do entail the material $A \supset B$.)
2. Next, it is argued that if $A \supset B$ holds, that it must also hold that if A then B. One assumption made here is something like this: if B can be inferred or concluded from something else known along with A, then it holds that if A then B. But modus ponens is clearly valid for $A \supset B$, since $A \supset B$ is equivalent with $\neg A \vee B$ and B follows from it and A by disjunctive syllogism, so B may be inferred from it along with A; therefore if A then B. (This half of the argument is relatively more controversial; Priest points out here that it requires the use of disjunctive syllogism, or equivalently, modus ponens for $A \supset B$ when interpreted as $\neg A \vee B$.)

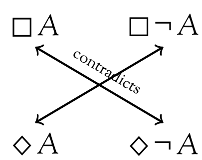
[C] System K and Possible Worlds Semantics

C.1 Introduction to Modal Logic

We now make a brief foray into modal logic, which may be defined narrowly or broadly.

Narrowly defined, *modal logic* is the study of the logic of necessity and possibility.

Broadly defined, *modal logic* is the study of pairs of intensional operators $\Box$ and $\Diamond$ (called *modal operators*) obeying the *modal square of opposition*:



The usual interpretation of $\Box$ is "necessarily …" or "it is necessarily the case that …"; and the usual interpretation of $\Diamond$ is "possibly" or "it is possibly the case that…". One can then understand the modal square of opposition: something is necessary just in case it is not possibly not the case; something is possible when it is not necessarily not the case.

Other interpretations include deontic or moral ones, such as $\Box$ for "it is (morally) obligatory that …" and $\Diamond$ for "it is (morally) permissible that …", temporal ones, (e.g., $\Box$ for "always" and $\Diamond$ for "sometimes"), and so on.

The semantics for logical languages involving modal operators is typically stated using the notion of *possible world*. Exactly what a possible world is is a matter of some debate, which we'll pick up later. The important thing logically is that interpretations for such languages make the truth or falsity of a formula relative to one world or another. One might then interpret $\Box A$ as being true in a given world iff A is true in all (accessible) worlds; and $\Diamond A$ as meaning A is true in at least one such world.

The use of possible worlds for the purposes of formal semantics was first made mathematically precise by Kripke, for whom System K is named.

Although the kinds of modal logics we begin with would not normally be considered non-classical logics, they are *intensional* in so far as they are not fully truth-functional. A and B may have the same truth value without $\Box A$ and $\Box B$ thereby having the same truth value.

The reason for the foray is twofold:

1. To get our feet wet with possible worlds semantics, which forms the basis for the semantics for many non-classical logics;
2. To consider modally defined conditionals as alternatives to the material conditional.

C.2 Syntax

A *well-formed formula (wff)* is defined the same as in classical propositional logic (see p. 4), except we replace clause (vii) with:
(vii) if A is a wff, then so is $\Box A$;
(viii) if A is a wff, then so is $\Diamond A$; and
(ix) nothing that cannot be constructed by repeated applications of (i)–(viii) is a wff.

Syntactically, $\Box$ and $\Diamond$ behave like $\neg$; their scope is narrow unless parentheses are used. So, e.g., $\Box A \vee B$ is interpreted as $(\Box A) \vee B$ rather than $\Box(A \vee B)$.

Sometimes, only one of $\Box$ or $\Diamond$ is taken as a primitive sign, and the other is introduced as a definitional abbreviation. E.g., $\ulcorner \Diamond A \urcorner$ might be taken as an abbreviation of $\ulcorner \neg \Box \neg A \urcorner$; in that case, the relationships depicted by the modal square of opposition follow trivially. Priest takes both as primitive, however.

$\ulcorner A \strictif B \urcorner$ is often used to abbreviate $\ulcorner \Box(A \supset B) \urcorner$, and $\ulcorner A \strictiff B \urcorner$ for $\ulcorner \Box(A \equiv B) \urcorner$; more on these definitions in a later section.

C.3 Semantics

An interpretation of the language of system K, or *K-interpretation*, consists of three things:

1. A set of worlds W.

   (No restriction is placed upon what we consider the members of W to be; we write uppercase W for the entire set, and typically refer to the members as, e.g., w_0, w_1, w_2, etc., using lowercase w's.)

2. An *accessibility relation* R, taken formally as a subset of $W \times W$.

   (Formally, this is simply any arbitrary relation between worlds. Informally, we can understand this in terms of relative possibility. On certain understandings of possibility, for example, what is possible with respect to one world may be different from what is possible with respect to others. In worlds in which I am

married, it is possible for me to get divorced today. In those in which I am not, it is not. We write $w_0 R w_1$ to mean that w_1 is accessible from w_0, the upshot of which is that what is true at w_1 is considered "possible" at w_0.)

3. An evaluation function v which assigns to each propositional parameter either the value 1 (truth) or the value 0 (falsity) with respect to each world $w \in W$.

   Since v may assign 1 to p at some worlds and 0 at others, instead of writing simply $v(p) = 1$ as before, we write $v_w(p) = 1$ to mean that v makes p true at w (or $v_w(p) = 0$ as the case may be).

Formally, we may, following Priest, think of a K-interpretation as an ordered triple $\langle W, R, v\rangle$ (of the above).

An interpretation directly assigns truth values to propositional parameters, relative to worlds through its evaluation function v. It indirectly determines truth values for complex wffs, again relative to worlds, according to the following semantic rules:

$$v_w(\neg A) = \begin{cases} 1, & \text{if } v_w(A) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(A \supset B) = \begin{cases} 1, & \text{if } v_w(A) = 0 \text{ or } v_w(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(A \equiv B) = \begin{cases} 1, & \text{if } v_w(A) = v_w(B); \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(A \wedge B) = \begin{cases} 1, & \text{if } v_w(A) = 1 \text{ and } v_w(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(A \vee B) = \begin{cases} 1, & \text{if } v_w(A) = 1 \text{ or } v_w(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(\Diamond A) = \begin{cases} 1 & \text{if there is some } w' \in W, \text{ such that} \\ & \quad wRw' \text{ and } v_{w'}(A) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

$$v_w(\Box A) = \begin{cases} 1 & \text{if for every } w' \in W, \text{ if} \\ & \quad wRw', \text{ then } v_{w'}(A) = 1; \\ 0 & \text{otherwise.} \end{cases}$$

The first five rules here are nothing new; the world parameter changes nothing about the semantics for the operator. The rule for $\Diamond A$ makes it true at w just in case A is true at at least one world accessible to w. The rule for $\Box A$ states that it is true just in case A is true at every accessible world.

K-interpretations, however, remain flexible with regard to accessibility. There is nothing to require that a world must be accessible to itself or even have any other worlds accessible to it at all. Hence, K-interpretations leave open the possibility of $\Box A$ true at w without A being true at w, or even $\Box A$ being true at w without A being true at *any* world, provided that no world is accessible to w.

Such results make little sense if $\Box$ is interpreted as necessity of truth, but may make sense on other interpretations. E.g., on the deontic reading, unfortunately nothing is strange about the idea that something may be morally obligatory but not actually the case.

More "well-behaved" systems of modal logic result by adding restrictions on the notion of a K-interpretation to focus only on those with the desired properties of accessibility; more on that in the next chapter.

We continue to use the sign $\vDash$ for "semantic validity", except we must now account for truth values being relativized to worlds:

$\Delta \vDash A$ iff for all K-interpretations $\langle W, R, v\rangle$ and worlds $w \in W$, if for all members $B \in \Delta$, $v_w(B) = 1$ then $v_w(A) = 1$ as well.

(In other words, there can be no interpretation which makes every member of Δ true but also makes A false *at the same world*.)

$\vDash A$ iff $\varnothing \vDash A$, or equivalently, for every K-interpretation $\langle W, R, v\rangle$ and $w \in W$, $v_w(A) = 1$.

Some results include:

- For all wffs A, $\neg \Box A \vDash \Diamond \neg A$ and $\Diamond \neg A \vDash \neg \Box A$.
- For all wffs A, $\neg \Diamond A \vDash \Box \neg A$ and $\Box \neg A \vDash \neg \Diamond A$.
- For all wffs A, if $\vDash A$ then $\vDash \Box A$ (not to be confused with the generally invalid object-language scheme: $A \supset \Box A$).
- For all wffs A, $\vDash \Box(A \supset B) \supset (\Box A \supset \Box B)$. (This principle is sometimes itself called "K".)

C.4 *Tableaux for Modal K*

To accommodate the relativization of truth/falsity to worlds in the tableau method, at each step, rather than simply writing a wff, we write instead a wff followed by a comma and then a number. The number represents a world. We start with 0, but add worlds when applying

the rule for (true) $\Diamond$-statements. A branch is only closed when we get both A and $\neg A$ *for the same world* on that branch.

A tree branch may also include a line of the form "$n\ r\ m$", where n and m are numbers associated with worlds. This is to be taken to mean that the world associated with m is accessible the world represented by n, or, roughly that $w_n R w_m$.

The rules for the truth-functional propositional connectives $\neg$, $\lor$, $\land$, $\supset$ and $\equiv$ remain unchanged from the list on p. 6. Applying the rules does not increase nor decrease the number accompanying the formula.

We add rules for the modal operators and negations thereof:

Rule for negated modal formulæ: Change $\neg \Diamond A, n$ to $\Box \neg A, n$ and change $\neg \Box A, n$ to $\Diamond \neg A, n$.

Rule for $\Diamond$ formulæ: For $\Diamond A, n$ put both $n\ r\ m$ and A, m for some new number m not already used on the tree.

Rule for $\Box$ formulæ: For $\Box A, n$, for all m such that $n\ r\ m$ already occurs on the branch, put A, m on the branch. (This rule may need to be applied again if a new world is introduced later.)

(You may notice a similarity between the rules for $\Box$ and $\Diamond$ and the typical derivation rules for universal and existential quantification, respectively.) It is generally advisable to apply the rule for $\Diamond$ formulæ before that for $\Box$ formulæ.

Example:

$$\begin{array}{c}
\checkmark\ \neg(\Box(p \supset q) \supset (\Diamond p \supset \Diamond q)), 0 \\
\checkmark\ \Box(p \supset q), 0 \\
\checkmark\ \neg(\Diamond p \supset \Diamond q), 0 \\
\checkmark\ \Diamond p, 0 \\
\checkmark\ \neg \Diamond q, 0 \\
0\ r\ 1 \\
p, 1 \\
\checkmark\ \Box \neg q, 0 \\
\neg q, 1 \\
\checkmark\ p \supset q, 1
\end{array}$$

$$\begin{array}{cc}
\neg p, 1 & q, 1 \\
\times & \times
\end{array}$$

As before, if all branches of a tableau closes, it means there does not exist an interpretation satisfying the initial assumptions. In this case, the above tableau establishes that $\vdash \Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$, and hence $\models \Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$.

And again, as before, if a branch remains open it can be used to discover an interpretation in line with the initial suppositions. Thus, if we show $\Box p \land \Diamond q \nvdash \Box \Diamond(p \land q)$ we can find a counter-model showing that $\Box p \land \Diamond q \not\models \Box \Diamond(p \land q)$:

$$\begin{array}{c}
\checkmark\ \Box p \land \Diamond q, 0 \\
\checkmark\ \neg \Box \Diamond(p \land q), 0 \\
\checkmark\ \Box p, 0 \\
\checkmark\ \Diamond q, 0 \\
\checkmark\ \Diamond \neg \Diamond(p \land q), 0 \\
0\ r\ 1 \\
q, 1 \\
0\ r\ 2 \\
\checkmark\ \neg \Diamond(p \land q), 2 \\
p, 1 \\
p, 2 \\
\Box \neg(p \land q), 2
\end{array}$$

(Notice there is nothing to do with the final line here, since we do not have "$2\ r\ n$" for any world n.)

In general, the interpretation $\langle W, R, v \rangle$ corresponding to a given open branch is determined as follows:

1. Let W contain a world for each numbered world on the branch; in this case let $W = \{w_0, w_1, w_2\}$. (Do not include worlds that appear only on *other* branches.)
2. Let R hold between those worlds such that the appropriate line $n\ r\ m$ appears on the branch; in this case let $R = \{\langle w_0, w_1 \rangle, \langle w_0, w_2 \rangle\}$.
3. For each propositional parameter p and world w, let $v_w(p) = 1$ if p occurs on the branch with the number for world w, and let $v_w(p) = 0$ if $\neg p$ occurs with the number for world w. The other assignments are arbitrary. Here, we know that $v_{w_1}(p) = v_{w_2}(p) = v_{w_1}(q) = 1$; it does not matter what $v_{w_0}(p)$, $v_{w_0}(q)$, or $v_{w_2}(q)$ are, here.

The resulting interpretation will make each formula assumed true at the "0 world" true at the corresponding world.

Hence if a tableau for a list of formulas, $B_1, 0; , B_2, 0; \ldots, B_n, 0$ remains open, there exists an interpretation making all of them true at the same world.

Hence, if the tableau for $B_1, 0; B_2, 0; \ldots, B_n, 0; \neg A, 0$ remains open, the argument $B_1, B_2, \ldots, B_n \therefore A$ is invalid. By transposition, if the argument $B_1, B_2, \ldots, B_n \therefore A$ is valid, its tableau will close; this establishes completeness. (Soundness also holds, and is, if anything, more obvious.)

Interestingly, a small change will turn the above exmaple valid, i.e., if we consider whether $\Box p \wedge \Diamond q \vdash \Diamond(p \wedge q)$ instead.

$$
\begin{array}{c}
\checkmark\, \Box p \wedge \Diamond q, 0 \\
\checkmark\, \neg \Diamond(p \wedge q), 0 \\
\checkmark\, \Box p, 0 \\
\checkmark\, \Diamond q, 0 \\
\checkmark\, \Box \neg(p \wedge q), 0 \\
0 r 1 \\
q, 1 \\
p, 1 \\
\checkmark\, \neg(p \wedge q), 1
\end{array}
$$

$\neg p, 1$	$\neg q, 1$
$\times$	$\times$

We now show that $\Box(p \wedge q) \vdash \Box p \wedge \Box q$.

$$
\begin{array}{c}
\checkmark\, \Box(p \wedge q), 0 \\
\checkmark\, \neg(\Box p \wedge \Box q), 0
\end{array}
$$

$\checkmark\, \neg \Box p, 0$	$\checkmark\, \neg \Box q, 0$
$\checkmark\, \Diamond \neg p, 0$	$\checkmark\, \Diamond \neg q, 0$
$0 r 1$	$0 r 2$
$\neg p, 1$	$\neg q, 2$
$\checkmark\, p \wedge q, 1$	$\checkmark\, p \wedge q, 2$
$p, 1$	$p, 2$
$q, 1$	$q, 2$
$\times$	$\times$

Homework

Priest, §2.12, problem 2: choose 8 of (a)–(v) (pp. 34–35), but do at least two $\vdash$ and at least two $\nvdash$ problems.

C.5 *What are Possible Worlds?*

Many views exist in the philosophical literature.

Modal realism is the view that possible worlds are literally just that: worlds just like the actual world.

On this view, the trees and people in other possible worlds are trees and people every bit as much as we are (and those trees are). David Lewis, the most prominent proponent of this view, went so far as to hold that the only difference between this world and another one which makes this the actual one is that I am in this one.

Modal realism has advantages in its straightforward account of what worlds are and how they provide a non-reductive account of the truth and falsity of modal propositions. Disadvantages seem to include the intuitive implausibility, and difficulty in specifying what precisely is the difference between possibility and (distant) actuality.

Ersatzism is the view that possible worlds can be understood entirely in terms of *actual* objects (linguistic or otherwise) which can somehow be interpreted as *representing* other ways the world could be.

Many different forms of Ersatzism exist, but an example of such a view might include the view that possible worlds are maximally consistent sets of sentences.

Such a view is generally more in line with common sense, but may arguably fail to provide a non-circular account of modality, in presupposing a prior account of, e.g., consistency.

Modal actualism holds that there are indeed possible worlds, but they are not physical objects, but something quite different (e.g., abstract objects).

Combinatorialism is the view that a possible world is a recombination of objects in the actual world.

Combinatorialism *may* be a species of modal actualism depending on one's views on the nature of (re)combinations. (Priest seems to think it always is.) Combinatorialism faces difficulties with "alien" entities and properties seeming like possibilities, and so on. Other forms of actualism might include the view that possible worlds are sets of propositions (not understood merely linguistically), or similar. Such a view may have difficulties similar to those for Ersatzism.

Meinongianism about possible worlds is the view that possible worlds are non-existent objects (but objects nonetheless).

Modal fictionalism holds that possible worlds are fictions, and have the same kind of existence as other fictional entities, e.g., Sherlock Holmes and Hamlet.

Such a view often involves the claim that modal claims are false if interpreted literally, but that they explain why certain modal claims are taken to be true, or perhaps are true if one adds "according to the modal fiction ..." beforehand (or somesuch). Such views may have the merit of explaining why it is that it sometimes we seem to be able to *stipulate* what is true at other possible worlds, though in other ways the kinds of cases seem unlike.

Many of the above views raise epistemological problems about how we can do *know* anything about possible worlds, especially if we cannot causally interact with them; such worries may or may not be different from similar worries with other abstract objects.

[D] Other "Normal" Modal Logics

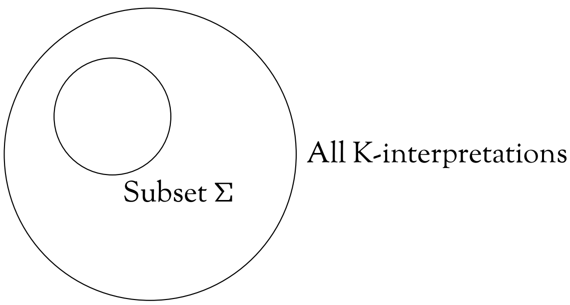
By "normal" modal logics, we mean those which are extensions of K, or, equivalently, those whose models can be considered a specific sub-category of K-interpretations.

The syntax of these systems is the same as K's syntax.

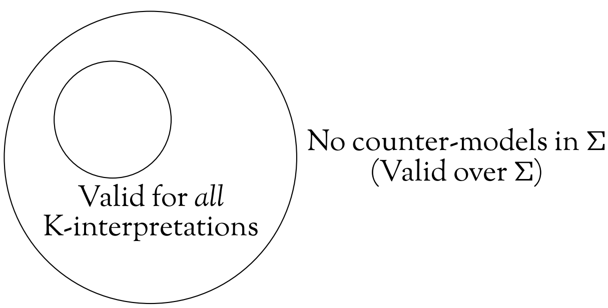
D.1 Semantics

Because these are extensions of K, their possible interpretations can be considered simply as subsets of K-interpretation, bearing in mind the inverse relationship:

Interpretations



Formulæ/Arguments



It is of course easier for a wff or argument to be valid for a subset of K-models than it is for it to be valid for all K-models absolutely, so *more* things will be valid in these systems than in K, but everything valid in system K will remain valid.

Here are some interesting subsets of K-interpretations:

A *D-interpretation* or *K_η-interpretation* is a K-interpretation in which the accessibility relation R is *extendable*, i.e., for all $w \in W$, there some $w' \in W$ such that wRw'.

A *T-interpretation* or *K_ρ-interpretation* is a K-interpretation in which the accessibility relation R is *reflexive*, i.e., for all $w \in W$ it holds that wRw.

A *KB-interpretation* or *K_σ-interpretation* is a K-interpretation in which the accessibility relation R is *symmetric*, i.e., for all $w, w' \in W$, if wRw' then $w'Rw$.

A *K4-interpretation* or *K_τ-interpretation* is a K-interpretation in which the accessibility relation R is *transitive*, i.e., for all $w, w', w'' \in W$, if wRw' and $w'Rw''$ then wRw''.

It is also interesting to look at intersections of these groups. (Notice first that T-interpretations are already a subset of D-interpretations, so there is no point in considering their intersection, as it is just the set of T-interpretations.)

A *B-interpretation* or *$K_{\rho\sigma}$-interpretation* is an interpretation whose accessibility relation is both symmetric and reflexive (thus both a KB-interpretation and a T-interpretation).

An *S4-interpretation* or *$K_{\rho\tau}$-interpretation* is an interpretation whose accessibility relation is both transitive and reflexive (thus both a K4-interpretation and a T-interpretation).

An *S5-interpretation* or *$K_{\rho\sigma\tau}$-interpretation* is an interpretation whose accessibility relation is reflexive, symmetric *and* transitive (thus all of the above).

We can define new species of validity in terms of any one of the above categories; in so doing of course we are also defining a new logical system. These logical systems are named as you might expect, D, T, KB, K4, B, S4, and S5. For example:

An argument $\Delta \therefore A$ is said to be *T-valid* iff there is no T-interpretation in which there is a world w such that $v_w(B) = 1$ for every $B \in \Delta$ but $v_w(A) = 0$. We may write this $\Delta \vDash_T A$.

Similarly we write simply $\vDash_T A$ to mean that A is true at all worlds in all T-interpretations.

We can similarly define variations of validity and $\vDash$ for each of the other systems. Often the subscript to $\vDash$ is omitted if it is obvious from context what it should be.

D.2 Characteristic Truths

At least some of the systems above were first studied as axiomatic systems in which they were named after certain axioms they adopted. Typically assuming the validity of these axioms is tantamount to restricting the allowable interpretations to those specified above. It is however, instructive, to list these basic principles which they render valid:

(D): $\Box A \supset \Diamond A$
(T): $\Box A \supset A$
(or equivalently: $A \supset \Diamond A$)
(B): $A \supset \Box \Diamond A$
(or equivalently: $\Diamond \Box A \supset A$)
(4): $\Box A \supset \Box \Box A$
(or equivalently: $\Diamond \Diamond A \supset \Diamond A$)
(5): $\Diamond A \supset \Box \Diamond A$
(or equivalently: $\Diamond \Box A \supset \Box A$)

(You might see (4) referred to, e.g., as either (K4) or (S4); even though it is true in both, and by itself does not require its interpretations to be S4 interpretations, (S4) was its original name in a system which already assured (T).)

It is worth taking a moment to consider why these principles are true in the corresponding sorts of models, and only those. For (D), consider that nothing can be trivially necessary; if A true in all accessible worlds (necessary), then it is true in at least one A (possible); this can only be if every world accesses at least one another. For (T), consider that each world accesses itself, so A must be true at it if it is true at all accessible worlds.

If were working in a traditional deductive system, we could make our systems complete simply by adopting the appropriate schemata from the above as freebies in a proof. (But we work with tableaux instead.)

D.3 Tableaux

The application of the semantics to create new tableaux rules for the various systems are fairly straightfoward.

D-rule: if a world number n appears on a branch, and no line of the form $n\,r\,m$ occurs, write in $n\,r\,j$ where j is a new world number. (Delay this rule as long as possible to avoid premature repetition to infinity.)

T-rule: if a world number n appears on a branch, write in $n\,r\,n$.

KB-rule: if $n\,r\,m$ appears on a branch, write in $m\,r\,n$.

K4-rule: if $n\,r\,m$ and $m\,r\,j$ both appear on a branch, write in $n\,r\,j$.

For combination systems such as B, S4 and S5, we just apply multiple of the above rules, as you would expect. Thus, e.g., for S4, we apply both the T-rule and K4-rule. We can use $B_1, \ldots, B_n \vdash_{\mathrm{D}} A$ to mean that the tableaux for $B_1, \ldots, B_n, \neg A$ closes when using the D-rule, and similarly for $\vdash_{\mathrm{T}}$, $\vdash_{\mathrm{B}}$, $\vdash_{\mathrm{S5}}$, etc., as you'd expect.

Let us now show that examples of the characteristic principles above are true in the relevant systems.

$$\begin{array}{c}
\underline{\vdash_{\mathrm{D}} \Box p \supset \Diamond p} \\
\checkmark \neg(\Box p \supset \Diamond p), 0 \\
\checkmark \Box p, 0 \\
\checkmark \neg \Diamond p, 0 \\
\checkmark \Box \neg p, 0 \\
0\,r\,1 \\
p, 1 \\
\neg p, 1 \\
\times
\end{array}$$

$$\begin{array}{c}
\underline{\vdash_{\mathrm{T}} \Box p \supset p} \\
\checkmark \neg(\Box p \supset p), 0 \\
0\,r\,0 \\
\checkmark \Box p, 0 \\
\neg p, 0 \\
p, 0 \\
\times
\end{array}$$

$$\begin{array}{c}
\underline{\vdash_{\mathrm{KB}} p \supset \Box \Diamond p} \\
\checkmark \neg(p \supset \Box \Diamond p), 0 \\
p, 0 \\
\checkmark \neg \Box \Diamond p, 0 \\
\checkmark \Diamond \neg \Diamond p, 0 \\
0\,r\,1 \\
\checkmark \neg \Diamond p, 1 \\
1\,r\,0 \\
\checkmark \Box \neg p, 1 \\
\neg p, 0 \\
\times
\end{array}$$

$$\begin{array}{c}
\underline{\vdash_{\mathrm{K4}} \Box p \supset \Box \Box p} \\
\checkmark \neg(\Box p \supset \Box \Box p), 0 \\
\checkmark \Box p, 0 \\
\checkmark \neg \Box \Box p, 0 \\
\checkmark \Diamond \neg \Box p, 0 \\
0\,r\,1 \\
\checkmark \neg \Box p, 1 \\
\checkmark \Diamond \neg p, 1 \\
1\,r\,2 \\
\neg p, 2 \\
0\,r\,2 \\
p, 2 \\
\times
\end{array}$$

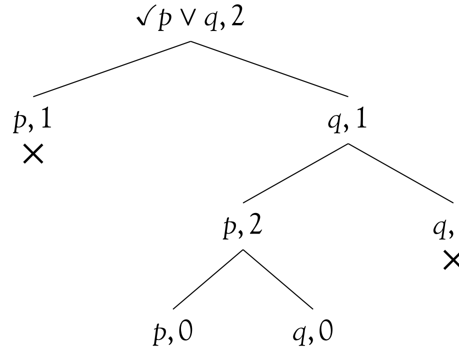
$$\underline{\vdash_{\mathrm{S5}} \Diamond p \supset \Box \Diamond p}$$

$\checkmark \neg(\Diamond p \supset \Box \Diamond p), 0$
$0\ r\ 0$
$\checkmark \Diamond p, 0$
$\checkmark \neg \Box \Diamond p, 0$
$0\ r\ 1$
$p, 1$
$1\ r\ 1$
$1\ r\ 0$
$\checkmark \Diamond \neg \Diamond p, 0$
$0\ r\ 2$
$\checkmark \neg \Diamond p, 2$
$2\ r\ 2$
$2\ r\ 0$
$1\ r\ 2$
$2\ r\ 1$
$\checkmark \Box \neg p, 2$
$\neg p, 0$
$\neg p, 1$
$\neg p, 2$
$\times$

On S5 tableaux, all worlds on the tableaux will access one another. Hence, there is little point in even writing the $n\ r\ m$. One may instead, simply always apply a line of the form $\Box A, n$ to arrive at A, m for every world number m.

Here we apply the quicker method:

$\checkmark \Box(p \vee q), 0$
$\checkmark \neg(\Box p \vee \Box q), 0$
$\checkmark \neg \Box p, 0$
$\checkmark \neg \Box q, 0$
$\checkmark \Diamond \neg p, 0$
$\checkmark \Diamond \neg q, 0$
$\neg p, 1$
$\neg q, 2$
$\checkmark p \vee q, 0$
$\checkmark p \vee q, 1$
$\checkmark p \vee q, 2$



We have open branches, and so this tableau establishes that $\Box(p \vee q) \nvdash_{S5} \Box p \vee \Box q$.

In certain circumstances, tableaux may be infinite. It is especially easy to imagine this happening for D-tableaux, since one may needs to add new worlds indefinitely in a branch does not close. (It may happen in other systems, however.) However, if an argument is valid in any of these systems, it will close in a finite number of steps, and hence, these systems are all complete in the relevant sense. It is usually easy to recognize what is going on when an infinite tree results, so that one does not need to do the impossible in writing out the complete tableaux.

Moreover, whether a branch is finite or infinite, an open branch corresponds to a counter-model, and the counter-model may be read off the branch just as for system K.

For the tree on the left, we consider the counter-model $\langle W, R, v\rangle$ such that $W = \{w_0, w_1, w_2\}$, the accessibility relation R holds universally between each pair of these worlds (in each direction), giving us an S5 model, and $v_{w_0}(p) = 1$, $v_{w_1}(p) = 0$, $v_{w_1}(q) = 1$, $v_{w_2}(p) = 1$, and $v_{w_2}(q) = 0$. (This validates us the left open branch.)

Homework

Priest, §3.10, problems 3–6 (p. 61). Keep in mind that Priest's K_ρ is T, his $K_{\rho\tau}$ is S4, and his K_v is S5.

D.4 Types of Modality

Why different systems? The answer is part historical, part because the systems were meant to capture different conceptions of modality.

D is so-named because it is meant to represent *deontic* logic, and its guiding principle is interpreted to mean that everything obligatory (or morally required) is permitted (morally allowable). In D, one often writes Op for $\Box p$ and Pp for $\Diamond p$. Stronger principles, such as (4) are usually seen as dubious here. Perhaps you ought to be punished, but it ought not be the case that you ought to be punished. Moral philosophers, however, may quibble about whether D is really the right logic for morality.

Traditional modal logics naturally employ something at least as strong as T, since naturally on most other interpretations what is necessary is true. Many of the other choices involve the behavior or iterated modal operators, which, depending on the conception of necessity involved, may be difficult to test one's intuitions about.

S5 is quite a popular system among philosophers to capture strong conceptions of modality, including analyticity, logical validity and metaphysical necessity. Here there is only one space of possible worlds, and anything true at any of them is possible, full stop, and anything false at any of them is false, full stop, and something has to be true

at all worlds absolutely to be necessary. Note that in S5 iterated modal operators always reduce to their rightmost operator, so $\Box\Diamond\Box\Box\Box\Diamond\Box A$ and $\Box A$ are equivalent, etc.

One *might* argue that something weaker is appropriate for physical or causal necessity. If something is guaranteed by the laws of nature, is it guaranteed by a law of nature that it is guaranteed by a law of nature? (Are the laws of nature themselves subject to laws?)

S4 has been suggested as appropriate for epistemic logic, where $\Box$ is interpreted as "*S* knows that ...". Then the principle $\Box p \supset \Box\Box p$ is tantamount to the KK-hypothesis that people only knows what they know that they know. However, the KK-hypothesis is not terribly popular among epistemologists, and S4 has the additional problem that all K-based systems are closed under entailment (e.g., $\Box p, \Box(p \supset q) \vDash_K \Box q$), whereas it is usually argued that knowledge is not so closed.

Other interpretations of $\Box$ worth exploring separately include "it is proven that..." and "it is determinately true that...". Arguments may be given either way as to whether or not a "normal" modal logic are appropriate for these.

In addition, Priest goes into detail at the end of the chapter about a tense logic which adopts not one but two modal-operator pairs, $\langle P\rangle, [P], \langle F\rangle, [F]$, where these mean, respectively *sometimes in the past*, *always in the past*, *sometimes in the future* and *always in the future*. Despite the complications that inevitably result from having two pairs of operators, the semantics for such systems can be given along the same lines as the systems we've been discussing, using Kripke-style worlds with different restrictions on accessibility, etc., making their discussion appropriate in this chapter. I didn't think it worth our time to delve into these in any detail.

[E]
Non-normal Modal Systems and Strict Implication

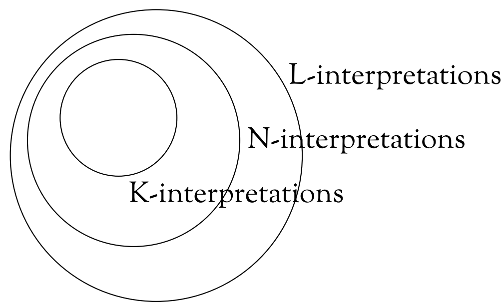
E.1 Introduction

The next group of modal systems we will examine are non-normal in the sense that the truth conditions for $\Box$ and $\Diamond$ they use are not, at least uniformly, taken as *truth in all accessible worlds* and *truth in at least one accessible world*. The "not uniformly" is important here, as they are sometimes given those semantics.

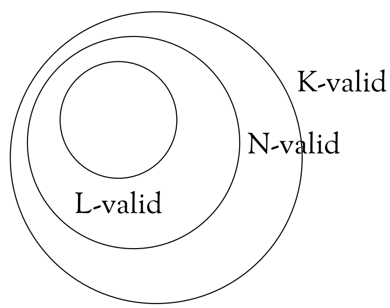
The main conceptual idea here is to divide possible worlds into two categories, "normal" and "non-normal". At normal worlds, the truth or falsity of $\Box$ and $\Diamond$ statements are interpreted with semantic rules in line with those given for system K; however, at non-normal modes, different rules, or possibly no rules at all, are employed instead.

We shall examine two base systems, N and L, and extensions of them. These systems are *weaker*, rather than stronger than K. Indeed, it is possible to regard K-interpretations as a subset of their interpretations, as we shall see. (Thus, T, S4, S5, etc., interpretations can also be thought of as subsets.) Bear in mind the *inverse* relationship between models and validity: the more interpretations over which an argument must preserve truth, the fewer arguments will succeed, and hence fewer things will be valid:

Interpretations



Formulæ/Arguments



We shall also explore extensions of N and L arrived at by placing restrictions on their accessibility relations, much like D, T, S4, S5, etc., are obtained from K. These *extensions* may not be strictly weaker than K, though they are not strictly stronger either. (They validate certain wffs or arguments K would not, and vice versa.)

The syntax of these systems is again the same as for K. Again, we can define $A \mathrel{-\!\!3} B$ as shorthand for $\Box(A \supset B)$. Historically these systems were explored primarily with an interest in the conditional $\mathrel{-\!\!3}$ they defined, called *strict implication*, suggested first by C. I. Lewis as a replacement for $\supset$ as a possible analysis of ordinary language conditionals. We begin with an examination of system N, in whose non-normal worlds "anything is possible". (The Lewis systems S2 and S3 are extensions of N.)

E.2 Semantics for N

An *N-interpretation* is an ordered quadruple $\langle W, N, R, v \rangle$, where:

- W is a set of worlds.
- N is a subset of W (the "normal" worlds).
- R is a two-place relation on W (the accessibility relation).
- v is a function assigning either 0 or 1 to each propositional parameter p at each world $w \in W$, again written $v_w(p) = 1$ or $v_w(p) = 0$.

(To make K-interpretations a subset of N-interpretations, we would *re*define a K-interpretation as an N-interpretation such that $N = W$. This would deviate technically from the definition given in the previous chapter, but not in any interesting or substantive way that would affect validity.)

The semantic rules for determining the truth values of complex states are unchanged for the non-modal operators. However, the rules for $\Box$ and $\Diamond$ are changed as follows:

$$v_w(\Box A) = \begin{cases} 1, & \text{if } w \in N \text{ and for all } w' \in W \text{ such that} \\ & \qquad wRw', \text{ it holds that } v_{w'}(A) = 1. \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the "otherwise" clause here includes *all* cases in which $w \notin N$, and so necessity claims are *always false* at non-normal worlds.

$$v_w(\Diamond A) = \begin{cases} 1, & \text{if } w \notin N, \\ 1, & \text{if } w \in N \text{ and there is a } w' \in W \text{ such that} \\ & \qquad wRw' \text{ and } v_{w'}(A) = 1. \\ 0, & \text{otherwise.} \end{cases}$$

So, at non-normal worlds, possibility statements are always true, even if what is claimed to be possible is not true at any world. This is what I meant earlier by claiming that for such words "everything is possible".

Here, we are getting closer to something non-classical. Notice that $\Diamond(p \wedge \neg p)$ could (must!) come out as true in non-normal worlds (although $p \wedge \neg p$ never comes out as true anywhere).

Validity is defined as truth or truth preservation in *normal* worlds only. (We could get an even weaker system by defining it as truth in all worlds whatever.)

$\Delta \vDash_{\mathrm{N}} A$ iff for every N-interpretation $\langle W, N, R, v \rangle$, for every world $w \in N$, if $v_w(B) = 1$ for every $B \in \Delta$, then $v_w(A) = 1$.

$\vDash_{\mathrm{N}} A$ iff for every N-interpretation $\langle W, N, R, v \rangle$, for every $w \in N$, $v_w(A) = 1$.

N can be extended much the same way as K is extended by considering only certain subsets of N-interpretations with conditions on accessibility.

An *S2-interpretation* or N_ρ*-interpretation* is an N-interpretation in which R is reflexive.

An *S3-interpretation* or $\mathrm{N}_{\rho\tau}$*-interpretation* in an N-interpretation in which R is reflexive and transitive.

An *S3.5 interpretation* or $\mathrm{N}_{\rho\sigma\tau}$*-interpretation* is an N interpretation in which R is reflexive, symmetric and transitive.

(Similarly for $\mathrm{N}_{\rho\sigma}$, etc.) A notion of $\vDash$ can be defined for each, again as truth (preservation) for normal worlds.

While some instances of necessitation (that if $\vDash_{\mathrm{N}} A$ then $\vDash_{\mathrm{N}} \Box A$) hold—e.g., both $\vDash_{\mathrm{N}} p \vee \neg p$ and $\vDash_{\mathrm{N}} \Box(p \vee \neg p)$—it does not hold generally in N. Notice that despite $\vDash_{\mathrm{N}} \Box(p \vee \neg p)$, we have $\nvDash_{\mathrm{N}} \Box\Box(p \vee \neg p)$. Non-normal worlds make $\Box(p \vee \neg p)$ false, so even a normal world will make $\Box\Box(p \vee \neg p)$ false if it accesses any non-normal worlds.

E.3 Tableaux for N

Tableaux are done for N similarly to how they are done for K, with one exception: because lines of the form $\Diamond A, n$ can be trivially true if n is non-normal, the rule for them is only applied when n is known to be normal. There are two ways to know that a world on a tableau is normal:

1. world 0 is always (assumed to be) normal, since validity is defined in terms of normal worlds so one assumes that the premises are true and conclusion false at a normal world;
2. if a statement of the form $\Box B, n$ occurs, n must be normal, because nothing is necessary at non-normal worlds.

Priest calls such words $\Box$*-inhabited worlds*.

Here we show that $\nvDash_{\mathrm{N}} \Box\neg\Diamond(p \wedge \neg p)$:

$$\begin{gathered} \checkmark\ \neg\Box\neg\Diamond(p \wedge \neg p), 0 \\ \checkmark\ \Diamond\neg\neg\Diamond(p \wedge \neg p), 0 \\ 0\,r\,1 \\ \checkmark\ \neg\neg\Diamond(p \wedge \neg p), 1 \\ \Diamond(p \wedge \neg p), 1 \end{gathered}$$

That is the complete tableau. Notice that although we applied the $\Diamond$ rule for world 0, which is known to be normal, we did not apply it to world 1. This shows that N is weaker than K, where we would get a new world from the final line, and a contradiction at that world.

When constructing a counter-model from an open tree branch, $w_0 \in N$, and otherwise a world is included in N only if it is $\Box$-inhabited. Other worlds are taken as non-normal.

Tableau for extensions of N are performed by simply adding the appropriate rules for introducing new $n\ r\ m$ lines exactly as for the corresponding extensions for K.

Here we show that $\vdash_{S2} \Box\Diamond(p \supset p)$

$$\begin{array}{c}
\checkmark\ \neg\Box\Diamond(p \supset p), 0 \\
0\ r\ 0 \\
\checkmark\ \Diamond\neg\Diamond(p \supset p), 0 \\
0\ r\ 1 \\
\checkmark\ \neg\Diamond(p \supset p), 1 \\
1\ r\ 1 \\
\checkmark\ \Box\neg(p \supset p), 1 \\
\checkmark\ \neg(p \supset p), 1 \\
p, 1 \\
\neg p, 1 \\
\times
\end{array}$$

Now we show that $\nvdash_{S3.5} \Diamond\Diamond p \supset \Diamond p$:

$$\begin{array}{c}
\checkmark\ \neg(\Diamond\Diamond p \supset \Diamond p), 0 \\
0\ r\ 0 \\
\checkmark\ \Diamond\Diamond p, 0 \\
\checkmark\ \neg\Diamond p, 0 \\
\checkmark\ \Box\neg p, 0 \\
0\ r\ 1 \\
\Diamond p, 1 \\
1\ r\ 1 \\
1\ r\ 0 \\
\neg p, 0 \\
\neg p, 1
\end{array}$$

This shows that S3.5 is weaker than S5, where this tableau would close. It did not here, because we do not know that world 1 is $\Box$-inhabited, and hence do not apply the rule to the line $\Diamond p, 1$ to get $1\ r\ 2$ and $p, 2$ and thence $0\ r\ 2$ and $\neg p, 2$ from the line $\Box\neg p, 0$. (This shows also that despite transitivity in S3 and S3.5, we still have things valid in S4 not valid in those systems.)

E.4 Systems L and S0.5

A weaker system than even N can be created by completely removing the semantic rules for $\Box$ and $\Diamond$ statements at non-normal worlds. Interpretations may simply "do as they please" for such statements at those worlds, in the same way that different interpretations may "do as they please" for atomic formulæ at various worlds.

We may define an *L-interpretation* as an ordered quadruple $\langle W, N, R, v\rangle$ where W, N and R are as for N-interpretations and v is a function such that $v_w(p) = 0$ or $v_w(p) = 1$ for every propositional parameter p and world $w \in W$, and $v_w(\Box A) = 0$ or $v_w(\Box A) = 1$ and $v_w(\Diamond A) = 1$ and $v_w(\Diamond A) = 1$ for every $w \in W - N$ and wff A.

v to be extended to assign truth-values to molecular formulæ at all worlds and modal formulæ at all normal worlds according to the usual rules, but the truth values for modal formulæ at non-normal worlds are simply stipulated by the interpretation.

With the above definition, we can consider an N-interpretation as an L-interpretation where $v_w(\Box A) = 0$ for all non-normal worlds, and $v_w(\Diamond A) = 1$ for all non-normal worlds (since the definition does not rule this out) for all formulæ A. K-interpretations may be defined as those where $\mathrm{N} = W$ as before.

L can be extended just like N and K by adding restrictions to accessibility. The system S0.5 created by E. J. Lemmon is another name for L_ρ, i.e., the system whose models are L-interpretations with a reflexive accessibility relation. (Priest named L after Lemmon.)

Tableaux are done as for N except that only world 0 is ever treated as normal. Even those where $\Box$ statements are true might be non-normal. Hence the rules for $\Diamond$ and $\Box$ statements are only applied when the world is 0. (In counter-models, all worlds are taken as non-normal except the one for 0.)

Despite reflexivity, $\nvdash_{S0.5} \Box(\Box p \supset p)$.

$$\begin{array}{c}
\checkmark\ \neg\Box(\Box p \supset p), 0 \\
0\ r\ 0 \\
\checkmark\ \Diamond\neg(\Box p \supset p), 0 \\
0\ r\ 1 \\
\checkmark\ \neg(\Box p \supset p), 1 \\
1\ r\ 1 \\
\Box p, 1 \\
\neg p, 1
\end{array}$$

We do not apply the $\Box$ rule to $\Box p, 1$, even though $1\ r\ 1$, since 1 may be non-normal, and make $\Box p$ true simply by fiat. (The above would close in S2, since 1 is clearly $\Box$-inhabited.)

Here we show that $\nvdash_{S0.5} \Box\Diamond(p \wedge q) \supset \Box\Diamond q$

$$
\begin{array}{c}
\checkmark \neg(\Box\Diamond(p \wedge q) \supset \Box\Diamond q), 0 \\
0\, r\, 0 \\
\checkmark \Box\Diamond(p \wedge q), 0 \\
\checkmark \neg\Box\Diamond q, 0 \\
\checkmark \Diamond\neg\Diamond q, 0 \\
0\, r\, 1 \\
\checkmark \neg\Diamond q, 1 \\
1\, r\, 1 \\
{}^*\Box\neg q, 1 \\
\checkmark \Diamond(p \wedge q), 0 \\
\Diamond(p \wedge q), 1 \\
0\, r\, 2 \\
\checkmark p \wedge q, 2 \\
2\, r\, 2 \\
p, 2 \\
q, 2 \\
\Diamond(p \wedge q), 2
\end{array}
$$

We do not apply any rule to the lines $\Box\neg q, 1$, $\Diamond(p \wedge q), 1$ or $\Diamond(p \wedge q), 2$, since modal statements may be given arbitrary truth values at worlds other than 0.

(Note, Priest himself would skip the line marked by *, since it involves applying a negation rule for modal operators at a world other than 0 (i.e., it comes from $\neg\Diamond q, 1$). This rule is usually a byproduct of the definition of $\Diamond$ as $\neg\Box\neg$, which Priest does not use, and which allows, in L-interpretations for which $v_w(\Diamond A) = 0$ even when $v_w(\Box\neg A) = 1$. Priest's treatment of S0.5 is thus a little different from the usual formulation of S0.5, though the difference hardly matters.)

There is little reason to consider systems adding transitivity or symmetry (etc.) to L, since such rules will produce no new validities. No new world numbers n will be introduced unless one gets $0\, r\, n$ to begin with, and nothing of the form $n\, r\, m$ where $n \neq 0$ matters at all for the rules.

Homework

Priest, §4.13, problems 2–4 (pp. 80–81). (If it wasn't obvious, the tree rule for $A \multimap B$ is to change it to $\Box(A \supset B)$.)

E.5 The Strict Conditional

How does $A \multimap B$ (recall that this is defined as $\Box(A \supset B)$) fare as an analysis of conditionals? Of course, it might matter which system we're looking at. Here, we assume something at least as strong as S0.5, so that we have $A \multimap B \models A \supset B$. The differences beyond this arguably matter little, since they mainly affect formulæ with iterated operators.

Some "improvements" over the material conditional:

$$
\begin{array}{c}
B \not\models A \multimap B \\
\neg A \not\models A \multimap B \\
\neg(A \multimap B) \not\models A \wedge \neg B \\
(A \wedge B) \multimap C \not\models (A \multimap C) \vee (B \multimap C) \\
(A \multimap B) \wedge (C \multimap D) \not\models (A \multimap D) \vee (C \multimap B)
\end{array}
$$

These deflate many of the worries we considered at the end of chapter 1.

Still there are issues to consider. Strict conditionals are always true when their antecedents are impossible or consequents are necessary:

$$
\begin{array}{r}
\neg\Diamond A \models A \multimap B \\
\Box B \models A \multimap B \\
\models (A \wedge \neg A) \multimap B \\
\models A \multimap (B \vee \neg B)
\end{array}
$$

We have these results regardless of relevance. Compare:

- If arithmetic can be recursively axiomatized in a complete and consistent way, then the events described in the *Harry Potter* books will come to pass.
- If I ate two mangos today, then Fermat's Last Theorem is true.

Priest makes quite a lot about the "explosion" of contradictions at the end of this chapter. He gives examples such as these:

1. One of Bohr's theories in physics was known to be inconsistent, but no one concluded from this that absolutely everything was a prediction it made.
2. In optical illusions, certain things may be regarded as impossible or contradictory, but this does not mean that one perceives absolutely everything.
3. Law and fiction are often inconsistent, yet this does not mean that everything is legal/illegal, or true in the fiction.

On the flip-side, however, he also considers arguments (due to C. I. Lewis) in favor of the results $\models (A \wedge \neg A) \multimap B$ and $\models A \multimap (B \vee \neg B)$:

1. For $\models (A \wedge \neg A) \multimap B$, Lewis gives the basic argument that from $A \wedge \neg A$, one gets both A and $\neg A$. From A one gets $A \vee B$, and from $A \vee B$ and $\neg A$ one gets B. This of course again uses disjunctive syllogism; Priest promises to come back to this.

2. For $\models A \multimap (B \vee \neg B)$, Lewis argues that from A one gets $(A \wedge B) \vee (A \wedge \neg B)$, from which one gets $A \wedge (B \vee \neg B)$ and then $B \vee \neg B$. Priest claims that the first step effectively smuggles in the conclusion.

That covers cases in which $A \multimap B$ is true when one might consider "if A then B" false. Although Priest doesn't discuss them in this chapter, there are also cases in which "if A then B" seems true, but $A \multimap B$ would appear to be false.

Notice that $\Box(A \supset B)$ is equivalent with $\neg\Diamond(A \wedge \neg B)$. Contrast, e.g.:

- If Romney wins the election, a Republican will be president next January.
- It is not possible that Romney will win the election and a Republican will not be president.

The second here seems perhaps a little *too* strong. There are remote possibilities... possibilities perhaps too "remote" to consider whereupon Romney wins, but then switches parties, etc.

The case seems even stronger with counterfactuals:

- If my daughter had forgotten to bring her lunch today, I would have brought it to her.
- It is not possible that my daughter could have forgotten her lunch and I would not have brought it to her.

A possible response might be to recall that we have not insisted on any particular strength of $\Box$ here: reading the possible here as "logical possibility" gets things wrong. Surely, there's a weaker sense of necessity according to which it really isn't possible that one could happen without the other, at least if the *if... then...* really holds. (A system weaker than S5, e.g., might allow it to be possibly possible that $A \wedge \neg B$ while $A \multimap B$ still holds.)

Related to these concerns involve the following sorts of arguments, which do come out as valid in the systems we have been studying:

$$A \multimap B \models \neg B \multimap \neg A \qquad \text{(transposition)}$$
$$A \multimap B, B \multimap C \models A \multimap C \qquad \text{(syllogism/transitivity)}$$
$$A \multimap B \models (A \wedge C) \multimap B \qquad \text{(antecedent strengthening)}$$

(The corresponding claims also hold for the material conditional.)

But compare:

- If you call me Ken instead of Kevin, I won't get mad. Therefore, if I get mad, you didn't call me Ken instead of Kevin.
- If I make the exam only one question long, you'll finish working on it in just a few minutes. If you finishing working on the exam in just a few minutes, you'll probably fail the exam. Therefore, if I make the exam only one question long, you'll probably fail the exam.
- If Romney wins the election, he'll be sworn in in January. Therefore, if Romney wins the election and dies of a heart attack the same night, he'll be sworn in in January.

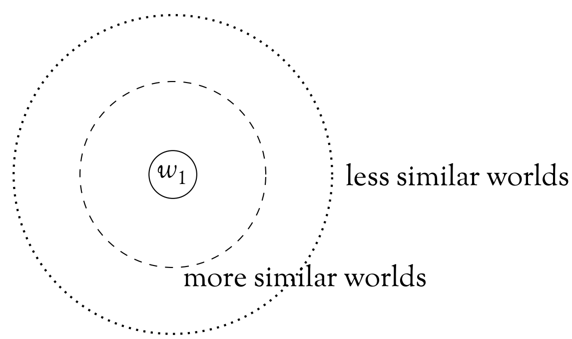
Dealing with these kinds of cases is a major motivation for conditional logic, which we turn to next.

[F] Conditional Logic

F.1 Introduction

The systems in this chapter derive from a similar philosophical analysis of subjunctive and counterfactual conditionals developed by Robert Stalnaker and David Lewis in the late 1960s and early 1970s. ("Lewis" in this chapter refers to David, not C. I.)

The usual interpretation of these systems employs a notion of a *(similarity) sphere of worlds*, with a given world at the center, and worlds "closer" or somehow "more similar" near the middle of the sphere, and more "distant" or "less similar" worlds on the periphery.



The conditional "if A then B" is interpreted as meaning roughly: in the closest (or most similar) possible worlds in which A holds, B holds as well.

Take "If McCain had won the election, a Republican would be president now". Sure, there are possible worlds where McCain won, but had switched parties before (or after), or in which there was a coup d'état in the past few years, etc., but those worlds are more distant—more unlike the actual world—than the world in which McCain

had simply won the election as a Republican and taken office. (Or so it is assumed here.)

F.2 Syntax

The language consists of propositional parameters and the operators $\supset$, $\vee$, $\wedge$, $\equiv$, $\neg$, $\Box$, $\Diamond$ and one new binary connective $>$ obeying the rule:

- If A and B are wffs, then $(A > B)$ is a wff.

(Not to be confused with "greater than"!) An alternative (earlier) notation here is $A \mathrel{\Box\!\!\rightarrow} B$. Another is $A \Rightarrow B$.

If we like, we may continue to use $A \dashv B$ to abbreviate $\Box(A \supset B)$.

F.3 Semantics for C

Obviously, here we will be working with a version of possible world semantics. To keep things simple, we presuppose that the correct logic for $\Box$ and $\Diamond$ in this language is S5, which removes the need for an accessibility relation governing them. Instead, we give simpler semantic rules for them:

$$v_w(\Box A) = \begin{cases} 1, & \text{if for every } w' \in W,\ v_{w'}(A) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(\Diamond A) = \begin{cases} 1, & \text{if for some } w' \in W,\ v_{w'}(A) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

(If we wanted another logic for $\Box$ and $\Diamond$, it is not difficult to imagine what changes would have to be made to the semantics below.)

In order to capture the semantics for $>$, however, we will need not one, but *many* (indeed, infinitely many) accessibility relations R_A, one for every formula A. Roughly:

$$w_0 R_A w_1$$

means that what is true at world w_1 matters when assessing whether or not statements of the form $(A > \ldots)$ are true at w_0. Formally, all that is required (at least at present) is that R_A is a relation between worlds, but intuitively the above might also be taken to mean that w_1 *is among the worlds most similar to* w_0 *where* A *holds*.

Priest likes to read $A > B$ as "if A, then, *ceteris paribus*, B". In that case $w_0 R_A w_1$ might also be read as meaning that the *ceteris paribus* clause for "if A" when assessed at w_0 is true at w_1.

A *C-interpretation* is a triple $\langle W, \{R_A : A \text{ is a wff}\}, v\rangle$, where:

- W is a set of worlds
- $\{R_A : A \text{ is a wff}\}$ is a set of accessibility relations, one for each wff A
- v is a function assigning either 1 or 0 to each propositional parameter at each world $w \in W$

As usual, we extend v to provide a truth value for every wff whatever at each world according to semantic rules. The rules for the truth-functional connectives are what you'd expect, the rules for $\Box$ and $\Diamond$ are the simplified versions given above, and the rule for $>$ is:

$$v_w(A > B) = \begin{cases} 1, & \text{if for all } w' \in W \text{ such that } wR_Aw', \\ & \quad \text{it holds that } v_{w'}(B) = 1. \\ 0, & \text{otherwise.} \end{cases}$$

We can define C-validity or $\models_C$ as one would expect in terms of truth preservation at all worlds in all interpretations. (All worlds are "normal" so we needn't worry about that complication.)

Priest adds some additional notation for discussing C-interpretations and their specialized accessibility relations:

- $[A]$ is used to refer to the set of worlds in which A is true, or $\{w : w \in W \text{ and } v_w(A) = 1\}$.
- $f_A(w)$ is used to refer to the set of worlds which matter for determining whether or not formulæ of the form $(A > \ldots)$ are true at w, or $\{w' : w' \in W \text{ and } wR_Aw'\}$.

With these notations, we can rewrite the rule for $A > B$:

$$v_w(A > B) = \begin{cases} 1, & \text{if } f_A(w) \subseteq [B], \\ 0, & \text{otherwise.} \end{cases}$$

F.4 Tableaux for C

Tableaux for C are built upon the method for doing tableaux for S5 in the "simplified way" as described on p. 15. In other words, we need not worry about accessibility lines $n\, r\, m$ when applying rules for $\Box$ and $\Diamond$ statements. However, we do need to write in lines of the form $n\, r_A\, m$ for tracking accessibility relations governing $(A > \ldots)$.

We need two new rules:

Rule for true > formulæ: For $A > B, n$, for each m for which we already have $n\ r_A\ m$ write B, m on the branch.

Rule for false > formulæ: For $\neg(A > B), n$, write both $n\ r_A\ m$ and $\neg B, m$ on the branch where m is a new world number.

These rules are quite similar to the old rules for $\Box$. In a way, it's as if we have a new necessity operator $\boxed{A}$ for each wff A, and $A > B$ means $\boxed{A}B$.

Some examples:

$$\underline{p > q \vdash_C p > \neg\neg q}$$

$$\begin{array}{c} \checkmark\, p > q, 0 \\ \checkmark\, \neg(p > \neg\neg q), 0 \\ 0\ r_p\ 1 \\ \checkmark\, \neg\neg\neg q, 1 \\ \neg q, 1 \\ q, 1 \\ \times \end{array}$$

$$\underline{\Box q \vdash_C p > q}$$

$$\begin{array}{c} \checkmark\, \Box q, 0 \\ \checkmark\, \neg(p > q), 0 \\ 0\ r_p\ 1 \\ \neg q, 1 \\ q, 0 \\ q, 1 \\ \times \end{array}$$

However:

$$\underline{p > s \nvdash_C (p \wedge q) > s}$$

$$\begin{array}{c} p > s, 0 \\ \checkmark\, \neg((p \wedge q) > s), 0 \\ 0\ r_{p\wedge q}\ 1 \\ \neg s, 1 \end{array}$$

Note that $0\ r_{p\wedge q}\ 1$ is not the same as $0\ r_p\ 1$, and so there is nothing to do with the top line, and this tableau is complete. Counter-models are read off in the usual way. We put a given world pair in the extension of R_A only if the appropriate $n\ r_A\ m$ line occurs on the branch.

However, C is, as is, still far too permissive regarding the accessibility relations R_A to capture the real intended meaning of $>$. Consider the following results:

$$\underline{\nvdash_C p > p}$$

$$\begin{array}{c} \checkmark\, \neg(p > p), 0 \\ 0\ r_p\ 1 \\ \neg p, 1 \end{array}$$

This tableau is complete here. Note that there is nothing so far to guarantee that a world that makes $A > B$ false make A true! For a similar reason:

$$p \multimap q \nvdash_C p > q$$

$$\begin{array}{c} \checkmark\, p \multimap q, 0 \\ \checkmark\, \neg(p > q), 0 \\ \checkmark\, \Box(p \supset q), 0 \\ 0\ r_p\ 1 \\ \neg q, 1 \\ \checkmark\, p \supset q, 0 \\ \checkmark\, p \supset q, 1 \end{array}$$

$$\begin{array}{ccc} \neg p, 1 & & q, 1 \\ \begin{array}{cc} \neg p, 0 & q, 0 \end{array} & & \times \end{array}$$

Another oddity is that it is never required that wR_Aw, which is strange when A is true at w. Consider then:

$$\underline{p \wedge \neg q \nvdash_C \neg(p > q)}$$

$$\begin{array}{c} \checkmark\, p \wedge \neg q, 0 \\ \checkmark\, \neg\neg(p > q), 0 \\ p, 0 \\ \neg q, 0 \\ p > q, 0 \end{array}$$

The same counter-model shows that $p > q \nvdash_C p \supset q$ and even $p > q, p \nvdash_C q$. This is clearly intolerable.

F.5 *The System* C$^+$

The system C$^+$, although still a rather weak conditional logic all things considered, fixes the most obvious difficulties with C.

A C$^+$*-interpretation* is a C-interpretation meeting the two constraints:

1. For any wff A and worlds w, w', if wR_Aw', then $v_{w'}(A) = 1$. (In other words, $f_A(w) \subseteq [A]$.)
2. For any wff A and world w, if $v_w(A) = 1$, then wR_Aw. (Or if $w \in [A]$ then $w \in f_A(w)$.)

The first constraint requires that A actually be true at those worlds that determine whether or not $A > B$, so no world can falsify $A > B$ without A being true there. The second constraint means that the world one is considering matters for determining whether or not $A > B$ at least if A is true there. Obviously, if A is true at w, the worlds most similar to w where A is true must at least include w itself!

C^+ validity, or $\vDash_{C^+}$, is defined as you would expect.

The method for doing tableaux for C^+ requires two changes from the method for C tableaux:

1. *Revised rule for false > formulæ:* From $\neg(A > B), m$, write three things: $m\ r_A\ n$, and A, n and $\neg B, n$, where n is a new world number.
2. *Rule for antecedents:* For each line of either the form $A > B, n$ or $\neg(A > B), n$, and for each world number m (whether $m = n$ or $m \neq n$) make *two* branches, one with $\neg A, m$, and another with *both* A, m and $m\ r_A\ m$.

(It may be wise to delay applying rule 2 here, as it may lead to many new branches quickly.)

Here we may contrast the results of some cases that were problematic for the simpler system C:

$$\vdash_{C^+} p > p$$

$\checkmark \neg(p > p), 0$
$0\ r_p\ 1$
$p, 1$
$\neg p, 1$
$\times$

$$p \wedge \neg q \vdash_{C^+} \neg(p > q)$$

$\checkmark p \wedge \neg q, 0$
$\checkmark \neg\neg(p > q), 0$
$p, 0$
$\neg q, 0$
$\checkmark p > q, 0$

- $\neg p, 0$ / $\times$
- $p, 0$ / $0\ r_p\ 0$ / $q, 0$ / $\times$

(What amounts to the same tableau will show that *modus ponens* holds for > in C^+.)

$$\neg\Diamond p \vdash_{C^+} p > q$$

$\checkmark \neg\Diamond p, 0$
$\checkmark \neg(p > q), 0$
$\checkmark \Box\neg p, 0$
$0\ r_p\ 1$
$p, 1$
$\neg q, 1$
$\neg p, 1$
$\times$

However:

$$p > q \nvDash_{C^+} \Box(p > q)$$

$\checkmark p > q, 0$
$\checkmark \neg\Box(p > q), 0$
$\checkmark \Diamond\neg(p > q), 0$
$\neg(p > q), 1$
$1\ r_p\ 2$
$p, 2$
$\neg q, 2$

- $\neg p, 2$ / $\times$
- $p, 2$ / $2\ r_p\ 2$
  - $\neg p, 0$
    - $\neg p, 1$
    - $p, 1$ / $1\ r_p\ 1$
  - $p, 0$ / $0\ r_p\ 0$ / $q, 0$
    - $\neg p, 1$ / (*)
    - $p, 1$ / $1\ r_p\ 1$

(We must branch for the antecedent rule for each of the world numbers 1, 2, 3 on the tableau. I did world 2 before 1 and 2 because I saw in advance that one of its branches would close.)

Let us consider the open branch marked with (*) as the basis for our counter-model. Here $W = \{w_0, w_1, w_2\}$. At w_0, $p > q$ is true, because p and q are both true at 0, and $w_0 R_p w_0$, or in other words, the world most similar to w_0 where p holds is w_0 itself, where q does hold. But $\Box(p > q)$ does not hold at w_0, nor any other world. (Recall that in S5, modal statements have the same truth value at every world.) This is because there is another world w_1 where $p > q$ is false, because, although $\neg p$ holds there, the world most similar to it where p holds is w_2, and at w_2, p is true and q is false.

F.6 *Stronger Constraints and System S*

Besides the constraints imposed on adequate interpretations and systems of worlds within C^+, the notion of "similarity spheres" of worlds motivates, at least arguably, a number of other constraints.

Consider:

3. If there are any worlds $w' \in W$ in which A is true (i.e., if A is *possible*), then R_A will always hold between a given world w and some other world, or in other words, if $[A] \neq \varnothing$ then $f_A(w) \neq \varnothing$.

The motivation for this condition can be appreciated by considering the leftmost open branch of the previous C^+ tableau. There $p > q$ held at world number 0, even though the model only contained three worlds, and at the only world in which p is true, world number 2, q was false. This did not lead to a branch closure because we did not have $0\ r_p\ 2$; world 2 was not considered "close enough" to world 0 to count for determining the truth or falsity of $p > q$ there. But surely, if there are *any* worlds in which p holds, however "unlike" a given world, then *at least one* of them must be considered "close enough" to count for assessing $p > q$ at that world.

4. If the worlds closest to w where A holds are all worlds where B is true, and the worlds closest to w where B holds are all worlds where A is true, then the worlds closest to w where A holds are the same as those closest to w where B holds, or more exactly, if $f_A(w) \subseteq [B]$ and $f_B(w) \subseteq [A]$ then $f_A(w) = f_B(w)$.

The motivation for this restriction can, in part, be appreviated by considering the odd fact that:

$$p > q \nvDash_{C^+} \neg\neg p > q$$

(This can be checked with a tableau if so desired.) The reason for this is basically that, in C^+, there is no relationship between R_p and $R_{\neg\neg p}$; different worlds may count for determining $p > \ldots$ statements and $\neg\neg p > \ldots$ statements, even though every world where p holds, $\neg\neg p$ also holds and vice versa. Hence, the *closest* worlds where p holds and the closest worlds where $\neg\neg p$ holds should be the same. The above restriction guarantees this (and more).

5. If B is true at any of the worlds closest to w where A is true, then those worlds are the closest to w where $A \wedge B$ is true, or more precisely, if $f_A(w) \cap [B] \neq \varnothing$ then $f_{A \wedge B}(w) \subseteq f_A(w)$.

Obviously, it cannot be that among the worlds closest to w where A is true there are some where B is true without those worlds being among the closest where $A \wedge B$ are both true, since obviously, $A \wedge B$ is true at those worlds. Without this, one has odd results such as:

$$(p \wedge q) > \neg r \nvDash_{C^+} p > \neg(q \wedge r)$$

Priest calls the system whose interpretations are those C-intepretations that obey not only restrictions 1 and 2 of C^+ intepretations, but also restrictions 3, 4 and 5, "System S". S is an proper extension of C^+.

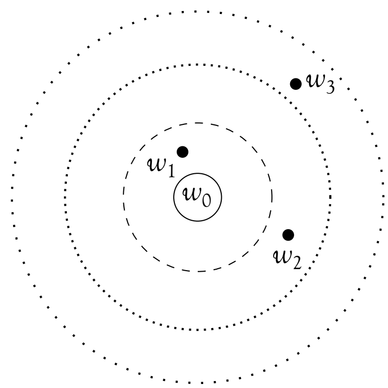
As a result of conditions 1 and 2, if A and B are true at the same worlds, then $A > C$ will be true just in case $B > C$ is true. One can think of the accessibility relations not so much as differing for different formulæ, but as differing for different propositions $[A]$, defined, e.g., as a set of worlds.

Priest does not give a tableau system for S, and indeed, no simple method is known for constructing them. However, one may attempt to construct a counter-model by describing a sphere of worlds in line with certain assumptions; a circle graph may help.

E.g., to show:

$$p > q, q > s \nvDash_S p > s$$

One may draw:



$w_0 : \neg p, \neg q, \neg s$
$w_1 : \neg p, q, s$
$w_2 : p, q, \neg s$
$w_3 : p, q, s$

Here the nearest world to w_0 where p holds is w_2, and there, q also holds, so $p > q$ is true. the nearest world to w_0 where q holds is w_1, and there s is true, so $q > s$, but at the nearest world where p holds, w_2, s does not hold, so $p > s$ is not true. (w_3 is more distant.)

The counter-model can fairly easily be read off the circle (or "sphere") diagram, and one can verify that it meets the condition imposed in system S. Without tableaux, however, there is no algorithm for constructing counter-models. (Beginning with a C^+ tableau and checking which open branches are compatible with S's restrictions may help, however.)

F.7 Stalnaker's C_2 and Lewis's C_1

Although formal systems capturing the analyses of conditionals offered by Lewis and Stalnaker often differ in slight (usually unimportant) ways, something at least close to their views can, in each case, be obtained by adding one further restriction to S.

Stalnaker's approach basically involves adding a *uniqueness* requirement for most similar worlds:

6. For any worlds w, w', w'' and wff A, if wR_Aw' and wR_Aw'' then $w' = w''$.

The system obtained by adding condition 6 is called C_2. Perhaps the most interesting feature of C_2 is that it validates the so-called *principle of Conditional Excluded Middle:*

$$\vDash_{C_2} (A > B) \lor (A > \neg B)$$

This enjoys at least some intuitive appeal, at least for a large class of cases. It seems likely that drinking a second cup of coffee this morning either would have made me sick or it wouldn't have, making it either true that if I had had the second cup, I would have been sick, or that if I had had the second cup, then (ceteris paribus) I wouldn't have been sick.

However, in other cases, this seems less plausible. My daughters are 4 and 6. If they were the same age, would they both be 4? Both be 6? Somewhere in the middle? So neither "if Ally and Emily were the same age, they'd be four" nor "if Ally and Emily were the same age, they would not be four" seem true.

Lewis, therefore, adopts a weaker principle:

7. If A is true at w, then for any world w' if $wR_A w'$ then $w = w'$.

If A is true at w, *then* there is a unique world, viz., w itself, closest to w where A holds.

The logic whose interpretations impose conditions 1–5 and 7 is called C_1. Note that since 6 along with 2 entails 7, so C_2 is an extension of C_1.

C_1, unlike S, has the following result:

$$A \land B \vDash_{C_1} A > B$$

Priest seems to find this result objectionable. Notice that it renders $A > B$ true when A and B are completely unrelated, as in "if $2 + 2 = 4$, then the dinosaurs are extinct" or "if you ate rice last Wednesday, you'll get sick today" (spoken to someone of whom those both happen to be true).

Priest notes also that both C_1 and C_2 validate:

$$\vDash ((A \lor \neg A) > A) \lor ((A \lor \neg A) > \neg A)$$

Consider:

- Either if I either wear a blue shirt tomorrow or don't, then I'll wear a blue shirt, or, if I either wear a blue shirt tomorrow or don't, then I won't wear a blue shirt tomorrow.

Suppose tomorrow comes and I in fact wear a blue shirt. Then the actual world is one where both $p \lor \neg p$ and p. But does this mean that p is true on the condition that $p \lor \neg p$. Even though it is not exactly similar to the actual world, the nearest world where I don't wear a blue shirt seems "close enough" to spoil the conditional $(p \lor \neg p) > p$.

Homework

Priest, §5.12 (pp. 101–102), problems 2, 3, 4 and 7.

F.8 *Further Philosophical Reflections*

We have already segued nicely into assessment of the $>$ as an analysis of the ordinary language conditional.

Like $\prec\!\!3$ we have (for at least some A, B, C):

$$B \nvDash A > B$$
$$\neg A \nvDash A > B$$
$$\neg(A > B) \nvDash A \land \neg B$$
$$\nvDash (A > B) \lor (B > C)$$
$$A > (B \lor C) \nvDash (A > B) \lor (A > C)$$
$$(A \land B) > C \nvDash (A > C) \lor (B > C)$$
$$(A > B) \land (C > D) \nvDash (A > D) \lor (C > B)$$

And unlike $\prec\!\!3$ we also have:

$$A > B \nvDash \neg B > \neg A$$
$$A > B, B > C \nvDash A > C$$
$$A > B \nvDash (A \land C) > B$$

To Priest's dismay, however, we still have:

$$\Box B \vDash_{C} A > B$$
$$\vDash_{C} A > (B \lor \neg B)$$
$$\neg \Diamond A \vDash_{C^+} A > B$$
$$\vDash_{C^+} (A \land \neg A) > B$$

(He also has worries about C_1 and C_2 as we have seen.)

Other philosophical worries involve the notion of similarity that is at the heart of (at least the stronger of) these conditional logics.

1. By the number of different true formulæ? But aren't there infinitely many? (Or just atomic parameters?)
2. Don't some differences matter more than other? Is it context or conversation dependent?
   - If the 1951 Yankees played the present day Yankees, they'd win by more than 5 runs.
   - If the USA threw its weapons into the sea tomorrow, there would be war; but if all the nuclear powers threw their weapons into the sea

tomorrow, there would be peace. (Compare reverse.)

3. Kit Fine's objection: A straightforward interpretation of "similarity" would seem not to work, because of e.g.:

   If Kennedy had pushed "the button" during the Cuban Missile crisis, something would have happened to prevent the nukes from detonating.

   But other interpretations may be circular.

[G] Intuitionist Logic

G.1 *Introduction*

We now examine a logical system that is unequivocally within the "non-classical" realm. It is the first such system to make waves among logicians, as it started out already having a philosophical movement behind it: intuitionism.

Intuitionism in the philosophy of mathematics rejects mathematical realism: that mathematical objects exist as real things independent of human thought, language and mathematical practice. Brouwer, the founder of intuitionism, conceived of mathematics as describing mental constructions, and held that nothing was true in mathematics unless a mental construction supported it.

Intuitionists typically see a very close connection between meaning, truth and proof (or some other notion of verification). As Priest puts, it is at least as natural to read the semantics of intuitionist logic as involving "proof conditions" rather than (or as their version of) "truth conditions".

Logical particles are interpreted with proof conditions depending on the proof conditions of their parts:

1. A proof of "A or B" is a proof of A or a proof of B.
2. A proof of "A and B" is a proof of A coupled with a proof of B.
3. A proof of "not A" is a proof that there is no proof of A.
4. A proof of "if A then B" is a method of transforming any proof of A into a proof of B.

Notice then that to prove "A or not A", one would need either a proof of A, or a proof that there can be no proof of A. Simply lacking a proof of A would not give you not-A. Hence "A or not A" is not intuitionistically valid in general.

Provability logic employing a classical base can be done using Kripke possible world semantics, and results usually in a logic similar to S4 or an extension thereof. The method can be used to provide a formal semantics for intuitionism in a classical framework, although it requires saying things intuitionists themselves would find objectionable. A possible world represents something like a current state of knowledge, and one might claim in the metalanguage that A is false in such a world without thereby validating the intuitionist "not A" there, since the intuitionist "not A" is interpreted as something stronger. An intuitionist might demur at this way of describing what is happening at the "world". It may be better simply to write "$v_w(A) = 0$".

G.2 *Syntax*

Because of the special way negations and conditionals are understood in intuitionist logic, we introduce new symbols ($\neg$ for negation and $\sqsupset$ for the conditional) so as not to confuse them with their classical counterparts. We do not use $\Box$ and $\Diamond$ as the "modal nature" of intuitionist logic is in a way already infused into the semantics generally. We do not bother with a sign for a biconditional; though one (e.g., $\sqsubset\sqsupset$) could be introduced as an abbreviation if we wished. Hence we offer the following definition:

A *well-formed formula (wff)* of intuitionist logic is defined recursively as follows:
(i) propositional parameters are wffs;
(ii) if A is a wff, so is $\neg A$;
(iii) if A and B are wffs, so is $(A \vee B)$;
(iv) if A and B are wffs, so is $(A \wedge B)$;
(v) if A and B are wffs, so is $(A \sqsupset B)$;
(vi) nothing that cannot be constructed by repeated applications of (i)–(vi) is a wff.

G.3 *Semantics*

An *I-interpretation* is a triple $\langle W, R, v\rangle$ where:

1. W is a set of worlds.
2. R is a *reflexive* and *transitive* accessibility relation holding between worlds in W.
3. v is a function which assigns to each propositional parameter p either 1 or 0 at each world $w \in W$; as usual we write this as $v_w(p) = 1$ or $v_w(p) = 0$.

4. v and R together obey the *hereditary condition:* for all worlds $w, w' \in W$, if $v_w(p) = 1$ and wRw', then $v_{w'}(p) = 1$ as well.

Notice that conditions 1–3 are exactly the conditions for being an S4 (or $K_{\rho\tau}$) interpretation. Indeed, it is possible in a way to regard I-interpretations as a subset of S4 interpretations: those obeying the hereditary condition. This is perhaps a tad misleading in that these interpretations are then *applied* to different languages, and v is extended to handle complex wffs with different rules. The parallel, however, is not without important or interesting consequences.

What's the motivation for the hereditary condition? The "possible worlds" for this semantics are better thought of as certain states of knowledge. What is accepted or proven at a given time is assigned 1 in that state. The accessible "worlds" or states represent possible future states of knowledge. However, once something has been proven or established, it stays proven in all later states.

If something is assigned 0 at a given state, that does not mean that it has been shown to be false or that its negation is acceptable. I.e., $v_w(p) = 0$ does not mean that $v_w(\neg p) = 1$. Similarly, $v_w(p) = 0$ does not mean that $v_w(p \sqsupset q) = 1$, since p may later be established without it leading to q. This requires we give new the semantic rules for $\neg$ and $\sqsupset$:

$$v_w(\neg A) = \begin{cases} 1, & \text{if for all } w' \in W, \text{ if } wRw', v_{w'}(A) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(A \sqsupset B) = \begin{cases} 1, & \text{if for all } w' \in W, \text{ if } wRw', \text{ then} \\ & \qquad \text{either } v_{w'}(A) = 0 \text{ or } v_{w'}(B) = 1. \\ 0, & \text{otherwise.} \end{cases}$$

The semantic rules for $\vee$ and $\wedge$ are the same as for classical logic (which is why we use the same signs).

Some observations:

1. In light of the hereditary condition, notice that a given propositional parameter p is in effect equivalent with what $\Box p$ would be if this language contained $\Box$.
2. Similarly, $\neg A$ is in effect $\Box \neg A$, and $A \sqsupset B$ is in effect $\Box(A \supset B)$.
3. Why shouldn't then the intuitionists have a version of $\vee$ that is equivalent to classical $\Box(A \vee B)$, and a version of $\wedge$ equivalent with $\Box(A \wedge B)$.

   In light of observation 1 above, this is unnecessary. Notice, e.g., that $\Box(\Box A \wedge \Box B)$ is equivalent with $\Box A \wedge \Box B$ in S4 anyway.

Validity or $\vDash_I$ is defined as one would expect as truth preservation at all worlds in all interpretations.

Result: The hereditary condition holds not only for propositional parameters but for all wffs A. That is, for all I-interpretations, and for all worlds $w, w' \in W$, if $v_w(A) = 1$ and wRw', then $v_{w'}(A) = 1$.

Let's go ahead and *prove* this in the metalanguage using (wff) induction. The base step holds by definition of an I-interpretation. For the induction step, we assume as inductive hypothesis that the condition holds for all formulæ B shorter than a given wff A, and proceed to prove that it holds for A as well. There are four cases to consider:

1. A takes the form $\neg B$ for some B. Suppose $v_w(A) = 1$, i.e., $v_w(\neg B) = 1$. By the semantic rule for negation this means that B has the value 0 at all worlds accessible to w. Consider some world w' wuch that wRw'. Obviously $v_{w'}(B) = 0$ due to the observation just made. But this does not yet establish $v_{w'}(\neg B)$. However consider any arbitrary w'' such that $w'Rw''$. By transitivity, wRw'' and so $v_{w''}(B) = 0$. Since w'' is arbitrary, $v_{w'}(\neg B) = 1$, i.e., $v_{w'}(A) = 1$.
2. A takes the form $B_1 \sqsupset B_2$. The fact that if $v_w(A) = 1$ and wRw', then $v_{w'}(A) = 1$ follows again from the semantic rule for $\sqsupset$ and transitivity; I omit the details, since they are similar to case 1 above.
3. A takes the form $B_1 \vee B_2$. Suppose $v_w(A) = 1$. By the semantic rule for $\vee$, either $v_w(B_1) = 1$ or $v(B_2) = 1$ (or both). In the first case, then by the *inductive hypothesis*, $v_{w'}(B_1) = 1$ as well for any w' such that wRw'. By the semantic rule for $\vee$, it follows that $v_{w'}(A) = 1$. The result follows similarly in the other case.
4. A takes the form $B_1 \wedge B_2$. The fact that if $v_w(A) = 1$ and wRw' then $v_{w'}(A) = 1$ follows by an argument using the inductive hypothesis and the semantic rule for $\wedge$. I again omit the details as they are similar to the third case above.

So not only does it hold for atomic propositional parameters that "once established, always established", it holds for complex wffs as well. In other words, the semantic value 1 is always "passed" along from one world in an interpretation to any accessible worlds. Notice, however, that 0 is not necessarily "passed" along, so worlds may still differ.

Suppose, however, that we consider an interpretation containing only one world. By reflexivity, it accesses itself. The hereditary condition holds trivially. Notice, that the semantic rule for $\neg$ would in effect require only falsity at the world of evaluation, and similarly for $\sqsupset$. I.e., $\neg$ would collapse to $\neg$ and $\sqsupset$ to $\supset$. We could call such an interpretation a *classical interpretation*. The wffs true at all classical interpretations would coincide exactly with the valid wffs of classical logic.

It follows that a wff or argument may not be intuitionistically valid unless the corresponding classical wff or argument is classically valid. Classical interpretations can be considered a subset of intuitionist interpretations, and classical logic may be considered an extension of intuitionist logic. The latter is strictly weaker than the former.

We can define other *intermediate* logical systems in between classical logic and intuitionist logic by placing further restrictions on I-interpretations which classical interpretations naturally obey. For example, Dummett's logical system LC can be modeled using *LC-interpretations*, defined as I-interpretations obeying a linear order, those in which for all $w, w' \in W$, either wRw' or $w'Rw$ or $w = w'$.

G.4 *Tableaux for Intuitionist Logic*

Tableaux for intuitionist logic are complicated by the fact that we can no longer represent $v_{w_n}(A) = 0$ by using an object language negation sign, $\neg A, n$, since $\neg A$ being true at world w_n requires more than simply $v_{w_n}(A) = 0$.

To remedy this, Priest prefixes every world number on a tableau with either "$-$" or "$+$", to mean that the semantic value of the formula on that line is either 0 or 1, respectively. This can be a bit confusing, especially since 0 and 1 are used as world numbers, and one must remember that "$+0$" means semantic value 1 at world 0, and "-0" means semantic value 0 at world 1.

Even the rules for $\vee$ and $\wedge$ are restated because this difference of approach. Notice also that there is no double negation rule; we simply apply the rule for $\neg$ twice.

To close a branch, one does not need to get both A and $\neg A$ at a world (although this would suffice). One only needs to get A having inconsistent semantic values at a world, i.e., $A, +n$ and $A, -n$ on the same branch.

Rules tantamount to the S4 rules for reflexivity and transitivity are also applicable. We add a rule for tf $p, +n$ and $n\,r\,m$, we get $p, +m$ as well.

Note finally that to test for validity of an argument $B_1, B_2, \ldots, B_n \therefore A$, or a formula A, we do not write $\neg A$ on the tableau, but rather $A, -0$, that A has semantic value 0 (while the premises have semantic value 1) at world 0.

($\wedge$ rules)	$\begin{array}{c} A \wedge B, +n \\ \downarrow \\ A, +n \\ B, +n \end{array}$	$\begin{array}{cc} \multicolumn{2}{c}{A \wedge B, -n} \\ A, -n & B, -n \end{array}$
($\vee$ rules)	$\begin{array}{cc} \multicolumn{2}{c}{A \vee B, +n} \\ A, +n & B, +n \end{array}$	$\begin{array}{c} A \vee B, -n \\ \downarrow \\ A, -n \\ B, -n \end{array}$
($\sqsupset$ rules)	$\begin{array}{cc} \multicolumn{2}{c}{A \sqsupset B, +n} \\ \multicolumn{2}{c}{n\,r\,m} \\ A, -m & B, +m \end{array}$	$\begin{array}{c} A \sqsupset B, -n \\ \downarrow \\ n\,r\,m \text{ (new } m) \\ A, +m \\ B, -m \end{array}$
($\neg$ rules)	$\begin{array}{c} \neg A, +n \\ n\,r\,m \\ \downarrow \\ A, -m \end{array}$	$\begin{array}{c} \neg A, -n \\ \downarrow \\ n\,r\,m \text{ (new } m) \\ A, +m \end{array}$
(other rules)	$\begin{array}{c} A, +n \\ A, -n \\ \downarrow \\ \times \end{array}$	$\begin{array}{c} A, (+/-)n \\ \downarrow \\ n\,r\,n \end{array}$
	$\begin{array}{c} p, +n \\ n\,r\,m \\ \downarrow \\ p, +m \end{array}$	$\begin{array}{c} n\,r\,m \\ m\,r\,j \\ \downarrow \\ n\,r\,j \end{array}$

Some examples:

$$\begin{array}{cc} \multicolumn{2}{c}{\underline{p \sqsupset q, p \vdash_I q}} \\ \multicolumn{2}{c}{\checkmark p \sqsupset q, +0} \\ \multicolumn{2}{c}{p, +0} \\ \multicolumn{2}{c}{q, -0} \\ \multicolumn{2}{c}{0\,r\,0} \\ p, -0 & q, +0 \\ \times & \times \end{array}$$

The hereditary property can be important:

$$\underline{p \lor q \vdash_{\mathrm{I}} \neg p \supset q}$$

$$\begin{array}{c}
\checkmark p \lor q, +0 \\
\checkmark \neg p \supset q, -0 \\
0\ r\ 0 \\
0\ r\ 1 \\
\checkmark \neg p, +1 \\
q, -1 \\
1\ r\ 1 \\
p, -1 \\
\begin{array}{cc}
\checkmark p, +0 & \checkmark q, +0 \\
p, +1 & q, +1 \\
\times & \times
\end{array}
\end{array}$$

It is easy to show that classic logic is a *proper* extension of intuitionist logic since, e.g., the reverse of the above argument is invalid! The tableau showing this, as many intuitionist tableau are, is infinite along certain branches. We see how this comes about due to transitivity:

$$\underline{\neg p \supset q \nvdash_{\mathrm{I}} p \lor q}$$

$$\begin{array}{c}
\checkmark \neg p \supset q, +0 \\
\checkmark p \lor q, -0 \\
0\ r\ 0 \\
p, -0 \\
q, -0 \\
\begin{array}{cc}
\begin{array}{c}
\checkmark \neg p, -0 \\
0\ r\ 1 \\
p, +1 \\
1\ r\ 1 \\
\begin{array}{cc}
\begin{array}{c}
\checkmark \neg p, -1 \\
1\ r\ 2 \\
p, +2 \\
2\ r\ 2 \\
0\ r\ 2 \\
\begin{array}{cc}
\begin{array}{c}
\checkmark \neg p, -2 \\
2\ r\ 3 \\
p, +3 \\
3\ r\ 3 \\
1\ r\ 3 \\
0\ r\ 3 \\
\begin{array}{cc}
\begin{array}{c}
\checkmark \neg p, -3 \\
\vdots
\end{array} & q, +3
\end{array}
\end{array} & q, +2
\end{array}
\end{array} & q, +1
\end{array}
\end{array} &
\begin{array}{c}
\checkmark q, +0 \\
\times
\end{array}
\end{array}
\end{array}$$

Here we see where the infinite repetition comes from. The rule for the false negation introduces a new world. Thanks to accessibility, that world is accessible to world 0, and hence we apply the rule for the conditional on the top line there, which gives us a fresh application for the rule for the false negation, and so on.

An infinite branch still counts as an open branch. (Since we have a finite open branches here as well, however, it may be easier to focus on those.)

Similarly, the law of excluded middle does not hold:

$$\underline{\nvdash_{\mathrm{I}} p \lor \neg p}$$

$$\begin{array}{c}
p \lor \neg p, -0 \\
0\ r\ 0 \\
p, -0 \\
\neg p, -0 \\
0\ r\ 1 \\
p, +1 \\
1\ r\ 1
\end{array}$$

We read counter-models off of open branches just as you would expect. In this case, we have an I-interpretation where $W = \{w_0, w_1\}$, $R = \{\langle w_0, w_0\rangle, \langle w_0, w_1\rangle, \langle w_1, w_1\rangle\}$, and $\nu_{w_0}(p) = 0$ and $\nu_{w_1}(p) = 1$. Note that this is acceptable since only semantic value 1 (truth) is hereditary; semantic value 0 is not.

Interestingly, double negation introduction holds, but double negation elimination does not:

$$\underline{p \vdash_{\mathrm{I}} \neg\neg p}$$

$$\begin{array}{c}
p, +0 \\
\neg\neg p, -0 \\
0\ r\ 0 \\
0\ r\ 1 \\
\neg p, +1 \\
1\ r\ 1 \\
p, -1 \\
p, +1 \\
\times
\end{array}$$

$$\underline{\neg\neg p \nvdash_{\mathrm{I}} p}$$

$$\begin{array}{c}
\neg\neg p, +0 \\
p, -0 \\
0\ r\ 0 \\
\neg p, -0 \\
0\ r\ 1 \\
p, +1 \\
1\ r\ 1 \\
\neg p, -1
\end{array}$$

$$
\begin{array}{c}
1\,r\,2 \\
p, +2 \\
2\,r\,2 \\
0\,r\,2 \\
\neg p, -2 \\
\vdots
\end{array}
$$

Here we have only one branch, and it is open but infinite. In such cases, one *may* be able to simplify the counter-model by identifying worlds where the same propositional parameters are true. Here, we need at least two words, since $v_{w_0}(p) = 0$ whereas $v_{w_1}(p) = 1$. There is, however, nothing preventing us from identifying worlds 1, 2 (and 3 and 4, etc., as we add more of the same).

Homework

Priest, §6.10 (pp. 117–118), problem 3 (any 4 of (a)–(h)) and problem 4 (any 3 of (a)–(e)).

G.5 *Philosophical Issues*

Assessing the philosophical merits of intuitionism is a large project, one we cannot fully address.

Those who prefer intuitionist logic often back this choice up with some kind of *anti-realist* philosophy, according to which what "there is" and "what is true about it" is not independent of human language, epistemic practices and conceptual scheming. In short, it is the denial of a kind of independent objective reality, either in general, or with respect to some particular domain. This is often coupled with the suggestion that what propositions are true cannot be independent of the means we have of establishing or verifying them.

As noted earlier, intuitionism first gained steam as a philosophy of mathematics, where many doubt the existence of a Platonic realist objective realm of mathematical entities, or even a language and thought-independent grounds for the truth of mathematical propositions.

Outside of the philosophy of mathematics, Michael Dummett has argued for the appropriateness of (something along the lines of) intuitionist logic on general semantic grounds. On this view, to grasp the meaning of a sentence requires the ability to show competence in ascertaining the conditions under which it is to be accepted and those under which it is to be rejected. But language has the capacity for forming grammatically correct sentences for which this capacity cannot be exercised. There is nothing then to their meaning that determines their "objective" truth or falsity, and for these, excluded middle will not hold.

There are many ways to respond to such arguments. One could simply deny the anti-realistic metaphysics which seems to underlie them. One might claim that even if means of establishing truth and truth are inextricably linked, but nonetheless believe that mere *possible* verification is enough for truth (though this may require adopting a realism about modality of some form).

Priest is of course also interested in the particular characteristics of the intuitionist conditional. A few features he likes:

$$
\begin{aligned}
(p \wedge q) \sqsupset s &\not\models (p \sqsupset s) \vee (q \sqsupset s) \\
(p \sqsupset q) \wedge (s \sqsupset t) &\not\models (p \sqsupset t) \vee (s \sqsupset q) \\
p \sqsupset (q \vee s) &\not\models (p \sqsupset q) \vee (p \sqsupset s) \\
\neg p \sqsupset \neg q &\not\models q \sqsupset p \\
\neg(p \sqsupset q) &\not\models p
\end{aligned}
$$

However, it also shares many features with strict and even material implication Priest dislikes:

$$
\begin{aligned}
q &\models p \sqsupset q \\
\neg p &\models p \sqsupset q \\
p \sqsupset q &\models \neg q \sqsupset \neg p \\
p \sqsupset q, q \sqsupset s &\models p \sqsupset s \\
&\models (p \wedge \neg p) \sqsupset q \\
&\models q \sqsupset \neg(p \wedge \neg p)
\end{aligned}
$$

Notice that in general we will have $\models A \sqsupset B$ when A is impossible or B necessary/valid/true-at-all-worlds. However, we cannot give as an example something like $\models q \sqsupset (p \vee \neg p)$ because $p \vee \neg p$ isn't valid. (We could give the last example in the list above instead, however.)

Of course, it was never our intention to explore intuitionism for the purpose of finding a conditional Priest likes, or even a conditional suitable for analyzing ordinary language conditionals.

One might also be interested in the ramifications of the parallel between I and S4. We can take this quite far. Indeed, take a group of formulas Δ of intuitionist logic, replace each propositional parameter with $\Box p$, each $\neg A$ with $\Box\neg A$ and each $A \sqsupset B$ with $\Box(A \supset B)$, resulting in a translated set Δ'. Perform the same operation on a conclusion A to obtain A' and one has the result:

$$\Delta \models_I A \text{ if and only if } \Delta' \models_{S4} A'$$

This is called the *McKinsey-Tarski translation*. Does this render intuitionist logic somehow redundant, or merely a notationally unique way of implementing S4?

[H] Many-valued Logic

H.1 *Many-valued logics generally*

In this chapter we focus on not just a single system, but a while family of logics, namely those that can be given a certain kind of semantics. The semantics here are simple in the sense that an interpretation does not require possible worlds, and may be reduced therefore to a single evaluation function v, which assigns truth values directly to propositional parameters and indirectly, in a rule-governed (recursive) and truth-functional way to complex formulæ.

These systems also have in common a syntax employing the signs $\vee$, $\wedge$, $\neg$, and $\supset$ as connectives. (From here on out, if we use $\ulcorner A \equiv B \urcorner$ we regard it as a mere abbreviation of $\ulcorner (A \supset B) \wedge (B \supset A) \urcorner$.)

Priest refers to the set of connectives $\{\vee, \wedge, \supset, \neg\}$ as $\mathcal{C}$.

Then we may define:

A *simple many-valued logic* is a structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\} \rangle$ where:

- $\mathcal{V}$ is a non-empty set of truth values.
- $\mathcal{D}$ is a subset of $\mathcal{V}$, called set of *designated* values. These are the truth values which must be preserved in any valid argument. (Informally, these may be regarded as the set of values which count as some kind of "truth".)
- $\{f_c : c \in \mathcal{C}\}$ is a set of functions governing the interpretation of the connectives $\vee$, $\wedge$, $\supset$, and $\neg$. (These will be three binary operations on $\mathcal{V}$ and one monadic operation on $\mathcal{V}$.)

The name here may be misleading, since standard binary classical logic counts as a simple many-valued logic under this definition, as would even a (silly) *one*-valued logic, where all wffs are evaluated the same.

For classical logic:

- $\mathcal{V}$ is the set $\{0, 1\}$. (Every wff is true or false.)
- $\mathcal{D}$ is the set $\{1\}$. (The value 1 is truth; it alone must be preserved in a valid argument.)
- $\{f_c : c \in \mathcal{C}\}$ is the set of classical truth functions assigned to $\vee$, $\wedge$, $\supset$ and $\neg$. (E.g., $\{\langle 0, 1\rangle, \langle 1, 0\rangle\}$ for $\neg$ and $\{\langle\langle 0, 0\rangle, 1\rangle, \langle\langle 0, 1\rangle, 1\rangle, \langle\langle 1, 0\rangle, 0\rangle, \langle\langle 1, 1\rangle, 1\rangle\}$ for $\supset$.

For any simple many-valued logic, an interpretation v is a function assigning a member of $\mathcal{V}$ to each propositional parameter p of the language. We write this truth value as $v(p)$. The function v is extended to give a value $v(A)$ in $\mathcal{V}$ for each complex wff according to the functions in $\{f_c : c \in \mathcal{C}\}$

For all such logics we may define validity as follows:

$\Delta \vDash A$ iff there is no interpretation v such that for all members B in Δ, $v(B) \in \mathcal{D}$ but $v(A) \notin \mathcal{D}$.

A simple many-valued is said to be *finitely many-valued* if the set $\mathcal{V}$ is finite.

Note that for any finitely-valued simple many-valued logic, the truth-table method may be used as an effective decision procedure to check for validity. Such tables will have n^m rows where n is the size of $\mathcal{V}$ and m is the number of propositional parameters in the wff or argument being checked, so this method is often tedious or impractical.

H.2 *Simple three-valued logics with gaps*

Philosophically, a truth value gap ought to be considered the lack of a truth value, but technically the lack of a truth value can be treated as if it were a third truth value. It is customary to designate this value as i (for *indeterminate*).

Here we examine systems in which $\mathcal{V}$ is $\{0, 1, i\}$.

Strong Kleene 3-valued logic, K_3 is the simple many-valued logic where $\mathcal{V} = \{0, 1, i\}$, and $\mathcal{D} = \{1\}$, and the truth functions governing the connectives are those given in the table below:

A	B	$A \vee B$	$A \wedge B$	$A \supset B$	$\neg B$
1	1	1	1	1	0
1	i	1	i	i	i
1	0	1	0	0	1
i	1	1	i	1	
i	i	i	i	i	
i	0	i	0	i	
0	1	1	0	1	
0	i	i	0	1	
0	0	0	0	1	

There is a certain intuitive justification to this table. When both inputs are classical, the output are the same as in classical logic. When an input is i the output is often i, with the exception of those cases in which the result would be same in classical logic regardless of the value, in which case the output is that value. E.g., if A is true, then $A \vee B$ is true regardless of the value of B, just as in classical logic. Indeed under a certain description one might say

that the "rules" governing these functions are the same as they always have been, only with a new "otherwise" clause, e.g., we have:

$$v(A \vee B) = \begin{cases} 1, & \text{if } v(A) = 1 \text{ or } v(B) = 1, \\ 0, & \text{if } v(A) = 0 \text{ and } v(B) = 0, \\ i, & \text{otherwise.} \end{cases}$$

One could regard the classical rule as the same, while only insisting that the "otherwise" clause is never enacted. Something similar could be said for the other connectives.

There is no wff A such that $\vDash_{K_3} A$ on its own. Do you see why?

There are of course, however, valid arguments. Some obvious ones include $A \vee B \vDash B \vee A$ and $A \supset B, A \vDash B$ and $\neg\neg A \vDash A$. These can be verified using truth tables. Here is a truth table establishing $\neg p \vee \neg q \vDash \neg(p \wedge q)$:

p	q	$\neg p \vee \neg q$	$\neg(p \wedge q)$
1	1	01 0 01	0 1 1 1
1	i	01 i i i	i 1 i i
1	0	01 1 10	1 1 0 0
i	1	i i i 01	i i i 1
i	i	i i i i i	i i i i
i	0	i i 1 10	1 i 0 0
0	1	10 1 01	1 0 0 1
0	i	10 1 i i	1 0 0 i
0	0	10 1 10	1 0 0 0

Here we look for a row in which the premise is true (looking at the column under $\vee$), or has a designated value, but in which the conclusion is not true/designated (i.e., has value 0 or i in the column under the $\neg$). In this case, we have not merely validity, but logical equivalence, since these final columns are identical.

Different systems can be obtained by tweaking the truth functions for the operators, especially when the input is i. Weak Kleene three-valued logic makes the output i whenever any input is i, though the resulting system is not terribly interesting.

Slightly more interesting is Łukasiewicz system Ł$_3$, which differs from K_3 with only one small change: the value of $A \supset B$ is 1 whenever $v(A) = v(B)$, including when $v(A)$ and $v(B)$ are both i.

This yields the generally desirable result:

$$\vDash_{Ł_3} A \supset A$$

So even when $v(A) = i$ we still have that $A \supset A$ is true outright, which seems intuitively plausible. Even if A is "indeterminate" in truth-value, there is no "possibility" that it is true in the antecedent but false in the consequent, and so the above would seem to hold either way. (Then again, we have $v(A \supset B) = 1$ when $v(A) = v(B) = i$ even when A and B are unrelated.)

Here we show that $\neg p \vee q \vDash_{Ł_3} p \supset q$:

p	q	$\neg p \vee q$	$p \supset q$
1	1	01 1 1	1 1 1
1	i	01 i i	1 i i
1	0	01 0 0	1 0 0
i	1	i i 1 1	i 1 1
i	i	i i i i	i 1 i
i	0	i i i 0	i i 0
0	1	10 1 1	0 1 1
0	i	10 1 i	0 1 i
0	0	10 1 0	0 1 0

Notice, there is no row here where $\neg p \vee q$ has value 1, but $p \supset q$ does not. However, these are not equivalent, and in fact, this table, read in the other direction shows that $p \supset q \nvDash_{Ł_3} \neg p \vee q$, in virtue of the middle row where $p \supset q$ is true but $\neg p \vee q$ has value i, which is not designated.

Note: for both K_3 and Ł$_3$ we have:

$$\nvDash p \vee \neg p$$

In Ł$_3$, however, there are more complex instances of $A \vee \neg A$ which are valid in virtue of one of the disjuncts being valid on its own, e.g.:

$$\vDash_{Ł_3} (p \supset p) \vee \neg(p \supset p)$$

H.3 Simple three-valued logics with gluts

Philosophically, a truth value glut is a combination of truth and falsity, rather than a third truth value; something has a "glut" of truth values when it is *both* true *and* false.

Again, technically, however, one can formulate a logical system that allows gluts by treating "both" (or "b" for short) as a third truth value.

Priest's system LP ("the logic of paradox") is the simple many-valued logic where $\mathcal{V} = \{0, 1, b\}$ and $\mathcal{D} = \{1, b\}$, and the truth-functions are just the same as in K_3, except swapping out b for i everywhere in the table. (In fact, Priest doesn't even bother to use a "b"; he just reinterprets the "i" in K_3's table as meaning "both".)

Notice that, then, as in K_3, there is a certain naturalness to the truth functions. One can again claim that a disjunction is true when either disjunct is true and false when both disjuncts are false. It's just that both of these clauses

are possible, and then the disjunction has both output truth values as well. Or more simply put:

$$v(A \vee B) = \begin{cases} 1, & \text{if either } A \text{ or } B \text{ is true,} \\ 0, & \text{if } A \text{ and } B \text{ are both false,} \\ b, & \text{if both of the above apply.} \end{cases}$$

The real technical difference between LP and K_3 lies not in their truth functions, but in making b a "designated" value, so that $\mathcal{D} = \{1, b\}$. Since b counts as a "kind" of truth, and valid arguments must preserve truth, this means that even when an argument has premises that are both true and false, the argument's conclusion must be true or both as well.

Consider *modus ponens*:

$p \supset q$	p	q
1 1 1	1	1
1 b b	1	b
1 0 0	1	0
b 1 1	b	1
b b b	b	b
b b 0	b	0
0 1 1	0	1
0 1 b	0	b
0 1 0	0	0

The problem here is the sixth row, where both premises have a designated value, b, put the conclusion is simply false, with value 0. This shows that modus ponens is invalid in LP. (For similar reasons, so is disjunctive syllogism.)

So here we see the difference from K_3, where modus ponens is valid. If we read the b's above as i's instead, the only relevant row is the top one, since only there do both premises have a K_3-designated value, and the conclusion has one as well.

Similarly, a wff A can be valid in LP even if it is sometimes false, provided that it is never *only* false, and always either true or both true and false. Indeed, somewhat surprisingly, the law of non-contradiction is valid in LP though not in K_3.

$$\vDash_{LP} \neg(A \wedge \neg A)$$

Notice that even when A is both true and false, so is $\neg A$ and hence, so is the conjunction and its negation as well. Since "both" counts as a kind of truth, the above is always true (despite being occasionally false as well).

LP is *paraconsistent*, i.e., in it contradictions do not explode, e.g.:

$$p \wedge \neg p \nvDash_{LP} q$$

This is *despite* the fact that LP's conditional makes contradictions imply anything:

$$\vDash_{LP} (p \wedge \neg p) \supset q$$

When p is both true and false and q simply false, this conditional is also both true and false, and hence, has a designated value. Notice that without modus ponens, having this conditional will still not make a contradiction explode, since one would never be able to discharge the antecedent.

To obtain a "ponens-able" conditional in a simple three-valued logic with gluts, it makes sense to modify the truth-function for the conditional so that $A \supset B$ is false only when the antecedent is true (or both) and the consequent is not only true. The resulting logic (due to Anderson and Belnap) is called RM_3. Its truth table for $\supset$ is as follows:

A	B	$A \supset B$
1	1	1
1	b	0
1	0	0
b	1	1
b	b	b
b	0	0
0	1	1
0	b	1
0	0	1

Notice that there is no row here where A and $A \supset B$ have designated values but B does not; hence MP is valid in RM_3.

Here is a chart comparing some results regarding the conditional in the four example systems we have looked at:

Result?	K_3	$Ł_3$	LP	RM_3
$q \vDash p \supset q$	Y	Y	Y	N
$\neg p \vDash p \supset q$	Y	Y	Y	N
$(p \wedge q) \supset s \vDash (p \supset s) \vee (q \supset s)$	Y	Y	Y	Y
$(p \supset q) \wedge (s \supset t) \vDash (p \supset t) \vee (s \supset q)$	Y	Y	Y	Y
$\neg(p \supset q) \vDash p$	Y	Y	Y	Y
$p \supset s \vDash (p \wedge q) \supset s$	Y	Y	Y	Y
$p \supset q, q \supset s \vDash p \supset s$	Y	Y	N	Y
$p \supset q \vDash \neg q \supset \neg p$	Y	Y	Y	Y
$\vDash p \supset (q \vee \neg q)$	N	N	Y	N
$\vDash (p \wedge \neg p) \supset q$	N	N	Y	N

As you may have guessed, the number of Y's above leads Priest to the conclusion that none of these systems employs a conditional adequate for analyzing ordinary language conditionals (though he thinks RM_3 fares the best). Again, however, it is doubtful that any of these systems were designed to do so (especially when it comes to subjunctive conditionals).

Indeed, a generic argument can be given that no *finitely-valued* simple many-valued logic can do justice to the conditional. To validate $A \supset A$ (and hence $A \equiv A$), as desired, such a logic will have to assign $A \supset B$ a designated value when $v(A) = v(B)$. But then assuming a well-behaved $\vee$, if one disjoins all the equivalences of the form $p \equiv q$ for $n+1$ distinct propositional parameters, where n is the size of $\mathcal{V}$, the result must be true, since two such parameters must be given the same truth value. But it would be permissible to understand each of these parameters as making a completely unrelated claim, which should not imply or be implied by any of the others.

Homework

Priest, §7.14 (p. 140), problem 2. This involves proving that the example systems mentioned above, and any other "normal" simple many-valued logic (those that have truth values 0 and 1 among others, in which 1 is designated but 0 is not, and in which the outputs of the truth functions for 0 and 1 as inputs match the classical rules) are all sub-logics of classical logic, or that everything valid in these systems is classically valid as well. Hint: to prove this, consider proving the contraposition, i.e., that if an argument is classically *invalid*, then it is invalid in any normal simple many-valued logic as well.

H.4 Supervaluations and Subvaluations

Consider for example that in both K_3 and $Ł_3$:

$$\nvDash p \vee \neg p$$

However, depending on one's reasons for accepting truth value gaps, one might want to maintain that this law is always true. Consider, e.g., future contingents. One might wish to claim that "it will rain tomorrow" is, as of now, neither true nor false; it lacks a truth value. Nevertheless, the disjunction "either it will rain tomorrow or it won't" is true, even now. The thought is that *however it turns out*, one of the disjunctions will *become* true, and since it does not matter which one, the disjunction is already true.

We can make this logically precise with the notions of *resolutions* and *supervaluations*. For the time being, we focus only on normal simple many-valued logics in which $\mathcal{V} = \{0, 1, i\}$.

An interpretation v' is said to be a *resolution* of another interpretation v iff for all p, if $v(p) = 1$ then $v'(p) = 1$ and if $v(p) = 0$ then $v'(p) = 0$ and if $v(p) = i$ then either $v'(p) = 1$ or $v'(p) = 0$.

This is abbreviated $v \leq v'$.

Resolutions are in effect classical "reinterpretations" of non-classical interpretations.

If v is an interpretation, then *the supervaluation of* v, written v^+ is a function from wffs to values in $\mathcal{V}$ such that:

$$v^+(A) = \begin{cases} 1, & \text{if for all } v' \text{ such that } v \leq v', \\ & \quad v'(A) = 1, \\ 0, & \text{if for all } v' \text{ such that } v \leq v', \\ & \quad v'(A) = 0, \\ i, & \text{otherwise.} \end{cases}$$

We can, using these definitions, define two *new* conceptions of validity.

An argument is *resolutely valid*, written $\Delta \vDash^r A$ just in case, for every interpretation v, for every resolution of v, v' ($v \leq v'$), if $v'(B) \in \mathcal{D}$ for every $B \in \Delta$, then $v'(A) \in \mathcal{D}$ as well.

An argument is *supervalid*, written $\Delta \vDash^s A$ just in case, for every interpretation v, if $v^+(B) \in \mathcal{D}$ for every $B \in \Delta$, then $v^+(A) \in \mathcal{D}$ as well.

Result: Assuming a normal simple three-valued logic like K_3 or $Ł_3$ where $\mathcal{V} = \{0, 1, i\}$ and $\mathcal{D} = \{1\}$, then $\Delta \vDash^r A$ iff $\Delta \vDash A$ is true in classical logic. (The same holds for $\vDash^s$ for K_3.)

(In such logics, $\vDash^s$ and $\vDash^r$ only come apart if you define a multi-conclusion relation $\Delta \vDash \Gamma$ that holds when the truth of every member of Δ requires the truth of at least one member of Δ. We shall not bother with such details.)

For normal simple three-valued logics with *gluts* instead of gaps such as LP and RM_3, one can define in precisely the same way the notions of *resolutions* and *subvaluations*, simply replacing the i's in the above definitions with b's. This gives us new validities, written $\vDash_r$ and $\vDash_s$. For LP, we have that $\Delta \vDash_r A$ iff $\Delta \vDash A$ is classically valid, and at least in the case of single-premise arguments $A \vDash_s B$ iff $A \vDash B$ is classically valid.

(Note, Priest uses an uppercase S for $\vDash_s$ instead. I avoid this to minimize confusion with conditional logic system S.)

Interest in these new conceptions of validity may vary depending on one's reasons for accepting gaps and gluts.

H.5 Quasi-modal extensions

Sometimes simple three-valued logics are extended to add two additional operators $\Box$ and $\Diamond$, which despite their

modal overtones are interpreted truth-functionally as follows:

A	$\Box A$	$\Diamond A$
1	1	1
i	0	1
0	0	0

(Or the same, with b replacing i for glutty systems.)

One might read $\Box A$ in this context as "determinately, A". Assuming a K_3-like treatment of $\neg$, we have the usual equivalence of $\Diamond A$ and $\neg\Box\neg A$, so $\Diamond A$ might be read as saying that A is not determinately untrue.

Adding these operators, however, to the sorts of systems we have been studying does not yield anything equivalent to the kinds of modal logics examined earlier the course. Indeed, they have some funny results. In gappy systems, one will typically get $A \vDash \Box A$ and in glutty systems one will get $\Diamond A \vDash A$. Other strange results include $\Diamond A, \Diamond B \vDash \Diamond(A \wedge B)$, which would not hold in traditional modal logics.

Indeed, it can be proven that none of the modal systems studied earlier in the course, for which Kripke semantics is appropriate, is equivalent to any finitely-valued simple many-valued logic, including intuitionist logic. (Nearly all logics using this same syntax, however, is equivalent to an infinitely valued simple many-valued logic, on the other hand.)

H.6 Philosophical Considerations

Some reasons that might be put forth for accepting truth value gaps include:

1. Anti-realist views equating truth with establishability, as in intuitionism.
2. Logical and semantic paradoxes. (See examples under gluts, below.)
3. Denotation failure, as in: "The King of France is married" or "The largest integer is prime."
4. Incomplete fictions, as in "Holmes had three maiden aunts."
5. Claims made about actual objects when assessed in possible worlds/situations in which they do not exist, e.g., "Socrates is a philosopher" assessed in a world consisting only a single metal sphere.
6. Claims about the future ("future contingents"), e.g., "The human first child born in the 22nd century will be male."
7. Claims made involving borderline cases of a vague predicate, such as "Kris is bald."
8. Category mistakes, e.g., "The number four is hungry."
9. Cases of semantic underdetermination. "Kevin is a relative of (a sibling of, etc.) Kevin."

Reasons for accepting gluts include:

1. Inconsistent laws or regulations.
2. Inconsistencies in fiction.
3. Logical and semantic paradoxes. E.g.:
    - The Liar Paradox: Consider: "This sentence is false." Is that sentence true or false?
    - The Strengthened Liar Paradox: "This sentence is either false, or neither true nor false." (Seems to work against gappy response.)
    - Russell's paradox: Let R be the set of all sets that are not members of themselves. Is R a member of itself?
4. Claims made about something in the instant of change. "As of this instant, Obama is president" spoken of at the precise moment of inauguration.
5. Vague claims, perhaps.
6. Claims in certain dialectical philosophies, assuming you accept them. (Hegel might suggest: "Pure reason is its own proper object.")
7. Cases of semantic *over*determination, perhaps. "This design is art."

Of course, all these arguments are controversial, and deserve further scrutiny. Some *themes* involved in responses include:

- Distinguishing epistemological and metaphysical issues.
- Distinguishing propositions or states of affairs from sentences.
- Providing full analyses of complex statements.
- Purposefully avoiding vagueness or semantically undetermined language in favor of simpler, more precise, more exact, more useful formal languages.
- Introducing intensional operators which precede what appear to be inconsistent claims: "according to the fiction p" does not contradict "according to the fiction $\neg p$", etc.

[I] First Degree Entailment

I.1 Introduction and Syntax

First Degree Entailment (FDE) is a logic with its origins in the work of Anderson and Belnap on relevance logic. FDE as initially formulated is not itself a relevance logic, though it can be used to form the basis for a family of different such logics, as we shall see later in the semester.

By itself, it is very close to the many-valued systems with gaps and gluts explored in the previous chapter. Indeed, one way of providing a semantics for FDE is as a four-valued logic where $\mathcal{V} = \{0, 1, b, n\}$ with "b" for "both" and "n" for "neither". In fact, both K_3 and LP are extensions of FDE. We shall also look at two other ways of providing semantics for FDE, which are equivalent in the sense that they define coextensive conceptions of validity.

Besides propositional parameters, FDE employs only the three operators $\neg$, $\wedge$ and $\vee$, though one may introduce $\supset$ and $\equiv$ by definition as usual, with $\ulcorner A \supset B \urcorner$ shorthand for $\ulcorner \neg A \vee B \urcorner$ and $\ulcorner A \equiv B \urcorner$ shorthand for $\ulcorner (A \supset B) \wedge (B \supset A) \urcorner$.

I.2 Relational Semantics

Another way of providing a semantics for FDE, with which we begin, is to replace talk of a one-valued evaluation function ν, such that $\nu(A)$ is always one value for every wff A, with talk of a *relation* between wffs and truth values, ρ. This relation may hold between a wff and more than one truth value at the same time. This allows us to speak only of two truth values, 0 and 1, for truth and falsity. In the metalanguage, instead of $\nu(A) = 1$, we now write:

$$A \, \rho \, 1$$

To mean that A is true ("relates to truth"), and:

$$A \, \rho \, 0$$

To mean that A is false ("relates to false"). The advantage of this is that we are not restricted from holding that A may be related to both, or neither, truth values, as we would be if ρ were a function that delivered exactly one value for every argument.

An *FDE interpretation* is a relation ρ which either holds or doesn't hold (or both) between each propositional parameter and the truth values 1 and 0.

(If we let $\mathcal{P}$ be the set of propositional parameters, we may more technically claim that ρ is any subset of $\mathcal{P} \times \{1, 0\}$. Then $p \, \rho \, 1$ means $\langle p, 1 \rangle \in \rho$, etc.)

Different interpretations behave differently with regard to propositional parameters, but are all extended in the same way to cover complex wffs, as follows:

$$\begin{aligned}
&A \vee B \, \rho \, 1 \text{ iff } A \, \rho \, 1 \text{ or } B \, \rho \, 1;\\
&A \vee B \, \rho \, 0 \text{ iff } A \, \rho \, 0 \text{ and } B \, \rho \, 0;\\
&A \wedge B \, \rho \, 1 \text{ iff } A \, \rho \, 1 \text{ and } B \, \rho \, 1;\\
&A \wedge B \, \rho \, 0 \text{ iff } A \, \rho \, 0 \text{ or } B \, \rho \, 0;\\
&\neg A \, \rho \, 1 \text{ iff } A \, \rho \, 0;\\
&\neg A \, \rho \, 0 \text{ iff } A \, \rho \, 1.
\end{aligned}$$

These rules work exactly as in classical logic, though of course the assumption that always either $A \, \rho \, 0$ or $A \, \rho \, 1$ and the assumption that not both of these are both dropped, affecting whether the relation may hold for neither or both for complex wffs as well.

$\Delta \vDash_{FDE} A$ means that for every FDE interpretation ρ if $B \, \rho \, 1$ for every $B \in \Delta$ then $A \, \rho \, 1$ as well. (I.e., if every member of Δ relates to true or is true, then so does/is A.)

Notice that although a wff is false just in case its negation is true, one must still distinguish the falsity of A from the non-truth or untruth of A.

I.3 Tableaux for FDE

To test for validity, one must attempt to find an interpretation in which the premises have truth, or relate to truth, but the conclusion is untrue, i.e., does not relate to truth. For reasons somewhat similar to intuitionism, one cannot simply write $\neg A$ to represent A's untruth, since $\neg A$'s truth requires only that A be false, not that it lack truth.

Hence, lines on a tableaux will take the form $\ulcorner A, + \urcorner$ or $\ulcorner A, - \urcorner$, which mean, respectively that $A \, \rho \, 1$ and not-$(A \, \rho \, 1)$.

(Note, we can use $\neg A, +$ to mean that $A \, \rho \, 0$ since $\neg A \, \rho \, 1$ just in case $A \, \rho \, 0$.)

So one begins a tableaux for testing whether or not $B_1, B_2, \ldots, B_n \vdash A$ with the lines:

$$\begin{gathered}
B_1, +\\
B_2, +\\
\vdots\\
B_n, +\\
A, -
\end{gathered}$$

Because both $A \ \rho\ 1$ and $A \ \rho\ 0$ are allowed in FDE interpretations, getting both $A, +$ and $\neg A, +$ will not close a branch (and similarly for getting both $A, -$ and $\neg A, -$). To close a branch, one needs both $A, +$ and $A, -$ for the same wff A, meaning that A both has, and lacks, truth (whether or not it is false).

The FDE tableaux rules are as follows:

$A \vee B, +$ branching to $A, +$ and $B, +$	$A \vee B, -$ $\downarrow$ $A, -$ $B, -$	$\neg(A \vee B), +$ $\downarrow$ $\neg A \wedge \neg B, +$	$\neg(A \vee B), -$ $\downarrow$ $\neg A \wedge \neg B, -$
$A \wedge B, +$ $\downarrow$ $A, +$ $B, +$	$A \wedge B, -$ branching to $A, -$ and $B, -$	$\neg(A \wedge B), +$ $\downarrow$ $\neg A \vee \neg B, +$	$\neg(A \wedge B), -$ $\downarrow$ $\neg A \vee \neg B, -$
$\neg\neg A, +$ $\downarrow$ $A, +$	$\neg\neg A, -$ $\downarrow$ $A, -$	$A, +$ $A, -$ $\downarrow$ $\times$	

Note we have separate rules for negations depending on what is being negated. For negations of conjunctions and disjunctions, the rule is to change them into their equivalent forms using DeMorgan's Laws, keeping the $+$ or $-$ the same.

There is no special rule to apply for a propositional parameter or its negation (as in classical logic).

The other rules are fairly self-explanatory.

Here are some examples.

$$\underline{p \wedge (q \vee r) \vdash_{\text{FDE}} (p \wedge q) \vee (p \wedge r)}$$

$$\begin{array}{c}
\checkmark\, p \wedge (q \vee r), + \\
\checkmark\, (p \wedge q) \vee (p \wedge r), - \\
p, + \\
\checkmark\, q \vee r, + \\
\checkmark\, p \wedge q, - \\
\checkmark\, p \wedge r, - \\
\begin{array}{cc}
q, + & r, + \\
\begin{array}{cc} p, - & q, - \\ \times & \times \end{array} & \begin{array}{cc} p, - & r, - \\ \times & \times \end{array}
\end{array}
\end{array}$$

However,

$$\underline{\neg(\neg p \wedge q), \neg\neg q \nvdash_{\text{FDE}} p}$$

$$\begin{array}{c}
\checkmark\, \neg(\neg p \wedge q), + \\
\checkmark\, \neg\neg q, + \\
p, - \\
\checkmark\, \neg\neg p \vee \neg q, + \\
q, + \\
\begin{array}{cc}
\begin{array}{c} \checkmark\, \neg\neg p, + \\ p, + \\ \times \end{array} & \neg q, +
\end{array}
\end{array}$$

Here the right branch remains open, despite the fact that we have both q and $\neg q$ true on that branch. The branch would only close if one of these were required to both have truth and lack truth, but although q is false here, it does not lack truth.

We read counter-models in the following way: for each propositional paramter p, we make $p \ \rho\ 1$ if "$p, +$" occurs on the branch, and make $p \ \rho\ 0$ if "$\neg p, +$" appears on the branch. In the above open branch, we make $q \ \rho\ 0$ and $q \ \rho\ 1$. (It must not be the case that $p \ \rho\ 1$, though, for the above, it does not matter whether or not $p \ \rho\ 0$; this is arbitrary. To adopt a simple procedure, we simply make ρ not hold between wffs and truth values unless we are forced to do so.)

FDE is non-explosive, thus paraconsistent, e.g.:

$$\underline{p \wedge \neg p \nvdash_{\text{FDE}} q}$$

$$\begin{array}{c}
\checkmark\, p \wedge \neg p, + \\
q, - \\
p, + \\
\neg p, +
\end{array}$$

The branch remains open. The counter-model simply makes p both true and false, and q neither true nor false (though it would also work to make it false only).

Classical arguments for explosion fail in part due to the invalidity of disjunctive syllogism:

$$\underline{p \vee q, \neg p \nvdash_{\text{FDE}} q}$$

$$\begin{array}{c}
\checkmark\, p \vee q, + \\
\neg p, + \\
q, - \\
\begin{array}{cc}
p, + & \begin{array}{c} q, + \\ \times \end{array}
\end{array}
\end{array}$$

The left branch remains open. The counter-model is the same as in the previous example: p is both true and false and q is either neither, or only false.

Note that with $A \supset B$ defined as $\neg A \vee B$, *modus ponens* and *modus tollens* are simply forms of disjunctive syllogism, and so are also invalid in FDE.

Priest defends the failure of disjunctive syllogism, claiming that it seems "patently wrong" as soon as one accepts gluts. He dismisses the intuitive validity of the argument form as an overextension of our expectations of consistency as the norm, much like the intuition that proper subsets must have a smaller cardinality, which is untrue for infinite sets, might be a byproduct of our greater familiarity with the finite.

I.4 Relationship with Other Many-Valued Systems

It is perhaps obvious that one could give an equivalent semantics for FDE whereupon one would make use not of a relations to truth-values, but an evaluation function with four truth-values, giving us a simple many-valued logic where $\mathcal{V} = \{0, 1, n, b\}$ ("b" for both, "n" for neither) and $\mathcal{D} = \{0, b\}$, and these truth functions for the connectives:

A	B	$A \vee B$	$A \wedge B$	$\neg B$
1	1	1	1	0
1	b	1	b	b
1	n	1	n	n
1	0	1	0	1
b	1	1	b	
b	b	b	b	
b	n	1	0	
b	0	b	0	
n	1	1	n	
n	b	1	0	
n	n	n	n	
n	0	n	0	
0	1	1	0	
0	b	b	0	
0	n	n	0	
0	0	0	0	

Negation reverses 0 and 1 and leaves b and n untouched. To remember the functions for $\wedge$ and $\vee$, Priest suggests consulting the following diagram:

$$\begin{tikzcd}
& 1 & \\
b \arrow[ur] & & n \arrow[ul] \\
& 0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The output of $\wedge$ is the highest spot on the diagram from which one can reach both inputs by traversing only up along arrows.

The output of $\vee$ is the lowest spot on the diagram from which one can reach both inputs by traversing only down along arrows.

Three things are worth remarking here:

- The outputs match the outputs in the systems LP, K_3 and classical logic when the inputs are values used in those systems respectively.
- One never gets a b as output unless at least one input is b.
- One never gets a n as output unless at least one input is n.

Because of these facts, LP interpretations, K_3 interpretations and classical interpretations can all be considered subclasses of FDE interpretations. A K_3 interpretation will be an interpretation ν such that $\nu(p) = 1$, $\nu(p) = 0$ or $\nu(p) = n$ (treating n as the same as K_3's value i) for every propositional parameter p. (One will never get b as value for any wff on such an interpretation.)

As usual, when one system's interpretations are a subset of another's, the resulting logic is an extension of that logic. Hence, LP, K_3 and classical logic are all extensions of FDE. Nothing is valid in FDE which is not also valid in those systems as well. (It is strictly weaker than all of them; LP and K_3 are each strictly weaker than classical logic, as we've seen.)

The parallel semantics goes in the other direction as well. That is, one can provide a *relational* semantics for LP, K_3 and classical logic as well, using ρ-relations rather than ν-functions. From this vantage point:

A *K_3 (relational) interpretation* is an FDE interpretation ρ that obeys *exclusion*, i.e., for no propositional parameter p, both $p\ \rho\ 1$ and $p\ \rho\ 0$.

A *LP (relational) interpretation* is an FDE interpretation ρ that obeys *exhaustion*, i.e., for every propositional parameter p, either $p\ \rho\ 1$ or $p\ \rho\ 0$.

A *classical (relational) interpretation* is an FDE interpretation that obeys both exhaustion and exclusion.

Tableaux for these systems may therefore be constructed by slight modifications to the procedure for FDE. In particular, we add new rules for the closure of branches:

$$\begin{array}{cc}
\underline{\text{Exclusion rule:}} & \underline{\text{Exhaustion rule:}} \\
A, + & A, - \\
\neg A, + & \neg A, - \\
\downarrow & \downarrow \\
\times & \times
\end{array}$$

K_3 tableaux employ the exclusion rule; LP tableaux use the exhaustion rule, and a new method (equivalent to the old method) for doing classical tableaux results from employing both rules in addition to FDE rules.

If you revisit the FDE tableaux presented earlier for the invalidity of explosion and disjunctive syllogism, it is easy to see that they would be closed if the exclusion rule were enforced (though not by the exhaustion rule). Hence, these are K_3-valid.

Here we show:

$$\underline{\vdash_{\mathrm{LP}} (q \wedge \neg q) \supset p}$$

$$\begin{array}{c}
\checkmark (q \wedge \neg q) \supset p, - \\
\checkmark \neg(q \wedge \neg q) \vee p, - \\
\checkmark \neg(q \wedge \neg q), - \\
p, - \\
\checkmark \neg q \vee \neg\neg q, - \\
\neg q, - \\
\checkmark \neg\neg q, - \\
q, - \\
\times
\end{array}$$

This tableaux would remain open in FDE, since in it, $\neg q, -$ and $q, -$ would not close the branch.

Relational semantics may also be employed for $Ł_3$ and RM_3, though in those systems, it would be inappropriate to define $A \supset B$ as an abbreviation for $\neg A \vee B$. One would have, instead, to take $\supset$ as a new primitive connective, and give it its own semantic rule.

In $Ł_3$:

$A \supset B\ \rho\ 1$ iff either $A\ \rho\ 0$ or $B\ \rho\ 1$ or
(neither A nor B bear ρ to 0 or 1)
$A \supset B\ \rho\ 0$ iff $A\ \rho\ 1$ and $B\ \rho\ 0$

In RM_3:

$A \supset B\ \rho\ 1$ iff either not-($A\ \rho\ 1$) or
not-($B\ \rho\ 0$) or
both A and B bear ρ to both 0 and 1
$A \supset B\ \rho\ 0$ iff $A\ \rho\ 1$ and $B\ \rho\ 0$

To do tableaux for these systems we can build on the procedure for FDE but we also need rules for conditionals and their negations. Unfortunately, these would sometimes branch three-ways, given the three conditions for $A \supset B\ \rho\ 1$.

For $Ł_3$ we would employ the *exclusion rule* as well as the following new rules:

$$\begin{array}{c}
A \supset B, + \\
\begin{array}{ccc}
\neg A, + & B, + & \begin{array}{c} A \vee \neg A, - \\ B \vee \neg B, - \end{array}
\end{array}
\end{array}
\qquad
\begin{array}{c}
A \supset B, - \\
\begin{array}{cc}
\begin{array}{c} A, + \\ B, - \end{array} & \begin{array}{c} \neg A, - \\ \neg B, + \end{array}
\end{array}
\end{array}$$

$$\begin{array}{c}
\neg(A \supset B), + \\
\downarrow \\
A, + \\
\neg B, +
\end{array}
\qquad
\begin{array}{c}
\neg(A \supset B), - \\
\begin{array}{cc}
A, - & \neg B, -
\end{array}
\end{array}$$

For RM_3 the rules are the *exhaustion rule* as well as the rules for $A \supset B, -$, $\neg(A \supset B), +$ and $\neg(A \supset B), -$ from $Ł_3$. However, the rule for $A \supset B, +$ is changed to the following:

$$\begin{array}{c}
A \supset B, + \\
\begin{array}{ccc}
A, - & \neg B, - & \begin{array}{c} A \wedge \neg A, + \\ B \wedge \neg B, + \end{array}
\end{array}
\end{array}$$

As our example, we show:

$$\underline{\neg q \supset \neg p \vdash_{RM_3} p \supset q}$$

$$\begin{array}{c}
\checkmark \neg q \supset \neg p, + \\
\checkmark p \supset q, - \\
\begin{array}{ccc}
\begin{array}{c}
\neg q, - \\
\begin{array}{cc}
\begin{array}{c} p, + \\ q, - \\ \times \end{array} & \begin{array}{c} \neg p, - \\ \neg q, + \\ \times \end{array}
\end{array}
\end{array}
&
\begin{array}{c}
\checkmark \neg\neg p, - \\
p, - \\
\begin{array}{cc}
\begin{array}{c} p, + \\ q, - \\ \times \end{array} & \begin{array}{c} \neg p, - \\ \neg q, + \\ \times \end{array}
\end{array}
\end{array}
&
\begin{array}{c}
\checkmark \neg q \wedge \neg\neg q, + \\
\checkmark \neg p \wedge \neg\neg p, + \\
\neg q, + \\
\checkmark \neg\neg q, + \\
\neg p, + \\
\checkmark \neg\neg p, + \\
q, + \\
p, + \\
\begin{array}{cc}
\begin{array}{c} p, + \\ q, - \\ \times \end{array} & \begin{array}{c} \neg p, - \\ \neg q, + \\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$

Notice here that the first and fourth branches close due to the exhaustion rule; the other four branches close due to the normal closure rule.

We have examined two equivalent ways of doing semantics for FDE: one may treat it as a four-valued logic in which every wff has exactly one of those four values, or one may treat it as a logic in which more than or fewer than one of the two truth value 1 and 0 relate to the same formula.

A third method makes use of possible worlds, and assigns each wff wff either the value 1 or the value 0 (and not both or neither) at each world. However, it changes the semantic rule for negation so that the value of $\neg A$ at w does not depend on the value of A at w, but rather at some *other* world, the *star world* of w, written w^*. Each world is paired with one and only one star world, and the star world of its star world is itself.

Routley star semantics is in some ways less intuitive than the other kinds of semantics, but technically, it differs less than the other ways from other kinds of interpretations examined, e.g., for modal logic earlier in the semester.

A *Routley structure* is an 3-tuple $\langle W, *, v\rangle$ where:

- W is a set of worlds;
- $*$ is a function mapping each member $w \in W$ to members of W such that $w^{**} = w$.
- v is a function mapping each propositional parameter p to either 0 or 1 at each world $w \in W$, written as always $v_w(p) = 1$ or $v_w(p) = 0$.

The evaluation function v is extended to assign values to complex wffs according to the semantic rules:

$$v_w(A \vee B) = \begin{cases} 1, & \text{if } v_w(A) = 1 \text{ or } v_w(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(A \wedge B) = \begin{cases} 1, & \text{if } v_w(A) = 1 \text{ and } v_w(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_w(\neg A) = \begin{cases} 1, & \text{if } v_{w^*}(A) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the rules for $\wedge$ and $\vee$ are the classical ones, but the rule for $\neg$ makes it a sort of intensional operator, where truth of negation ("falsity") depends not on having semantic value at the world of evaluation w, but rather at the *star world* of w.

This chart may help explain how these three ways of doing semantics accomplish the same thing. The rows represent "corresponding phenomena" in the different semantics:

Many-valued semantics	Relational semantics	Routley semantics
$v(A) = 1$	$A\ \rho\ 1$	$v_w(A) = 1$
	not-($A\ \rho\ 0$)	$v_{w^*}(A) = 1$
$v(A) = 0$	$A\ \rho\ 0$	$v_w(A) = 0$
	not-($A\ \rho\ 1$)	$v_{w^*}(A) = 0$
$v(A) = b$	$A\ \rho\ 1$	$v_w(A) = 1$
	$A\ \rho\ 0$	$v_{w^*}(A) = 0$
$v(A) = n$	not-($A\ \rho\ 1$)	$v_w(A) = 0$
	not-($A\ \rho\ 0$)	$v_{w^*}(A) = 1$

Notice for example that if A has value 1 at w and value 0 at w^*, then both A and $\neg A$ will be evaluated as having value 1 at w; this is in effect, this semantics' version of a "true contradiction".

We can define validity in the usual way as the preservation of value 1 at all worlds in all Routley structures; the resulting validities are just those of FDE under its other characterizations. Perhaps it is clear why this is the case.

A method for doing tableaux that is more convenient to use when attempting to discover counter-models in this semantics would involve using world numbers along with $+$'s and $-$'s, as we did for intuitionist logic. We write "$n^\#$" for the star world of the world numbered n. (For the moment, we will have no need ever to use numbers except "0" and "$0^\#$"; but later we shall look at modifications of FDE that require this.) When stating the rules, $\overline{n}$ either adds or removes the # depending upon whether or not n already has a # on it. Thus, $\overline{0} = 0^\#$ and $\overline{0^\#} = 0$.

The rules are greatly reduced in number:

$$\begin{array}{c} A \vee B, +n \\ \swarrow \quad \searrow \\ A, +n \qquad B, +n \end{array} \qquad\qquad \begin{array}{c} A \vee B, -n \\ \downarrow \\ A, -n \\ B, -n \end{array}$$

$$\begin{array}{c} A \wedge B, +n \\ \downarrow \\ A, +n \\ B, +n \end{array} \qquad\qquad \begin{array}{c} A \wedge B, -n \\ \swarrow \quad \searrow \\ A, -n \qquad B, -n \end{array}$$

$$\begin{array}{c} \neg A, +n \\ \downarrow \\ A, -\overline{n} \end{array} \qquad \begin{array}{c} \neg A, -n \\ \downarrow \\ A, +\overline{n} \end{array} \qquad \begin{array}{c} A, +n \\ A, -n \\ \downarrow \\ \times \end{array}$$

Counter-models may be read off of open branches in precisely the way one would expect; here of course "$A, -n$" does not just mean lack of value 1, but actually having value 0 at world numbered n.

As our example, we show that a version of disjunctive syllogism is still invalid:

$$\underline{\neg p \vee \neg q, q \nvdash_{\text{FDE}} \neg p}$$

$$\checkmark\, \neg p \vee \neg q, +0$$
$$q, +0$$
$$\checkmark\, \neg p, -0$$
$$p, +0^{\#}$$

$$\checkmark\, \neg p, +0 \qquad \checkmark\, \neg q, +0$$
$$p, -0^{\#} \qquad q, -0^{\#}$$
$$\times \qquad\qquad$$

(Of course, since this way of doing the semantics and tableaux for FDE is equivalent to the other ways, this should not be surprising.)

Here, in our counter-model for the open branch, we simply give q the value 1 at world w_0 and value 0 at its star world, $w_{0^{\#}}$, and p must have value 1 at the star world.

However, here we show that:

$$\underline{p \vee \neg q, p \vee \neg r \vdash_{\text{FDE}} p \vee \neg(q \vee r)}$$

$$\checkmark\, p \vee \neg q, +0$$
$$\checkmark\, p \vee \neg r, +0$$
$$\checkmark\, p \vee \neg(q \vee r), -0$$
$$p, -0$$
$$\checkmark\, \neg(q \vee r), -0$$
$$\checkmark\, q \vee r, +0^{\#}$$

$$p, +0 \qquad \checkmark\, \neg q, +0$$
$$\times \qquad q, -0^{\#}$$

$$p, +0 \qquad \checkmark\, \neg r, +0$$
$$\times \qquad r, -0^{\#}$$

$$q, +0^{\#} \qquad r, +0^{\#}$$
$$\times \qquad \times$$

The reason for the focus on (the admittedly less intuitive at first) Routley star semantics will become clearer in later chapters.

Homework

Priest, §8.10 (pp. 161–162) problems 1 (any six of (a)–(l)), problem 2 (same six as in problem 1) and problem 6 (any two of the same six).

[J]

FDE-based Systems with Strict Implication

FDE as formulated in previous chapter is a very weak logic, weaker even than LP. Especially due to the lack of a ponens-able conditional, very little reasoning can be captured in it. A natural move then is to supplement FDE with a stronger conditional, such as a strict conditional. Such systems will naturally go hand-in-hand with many-valued possible worlds semantics.

J.1 The System K_4

The system K_4 represents perhaps the simplest way to accomplish this. This should not be confused with modal system K4, and indeed, the relationship between possible worlds in it is more like S5, where every world is treated as accessible to any other so that an accessibility relation is not needed in the interpretations. The subscript on K_4 indicates the four truth-values, much like the subscript on, e.g., $Ł_3$.

K_4 has the same syntax as FDE, except adding a new binary connective for the conditional $\rightarrow$. (We do not use $\Box$ or $\Diamond$, hence the usual defined sign $⥽$ cannot be used.) However, bear in mind that $\rightarrow$ is not a material conditional.

K_4 can be given a relational semantics with the following definition:

> An K_4 *interpretation* is a structure $\langle W, \rho \rangle$ where W is a set of worlds, and ρ is a *three* place relation between propositional parameters, worlds and the values 0 and 1.

We write $p\ \rho_w\ 1$ to mean that p is true (relates to truth) at world w. The relation ρ is extended to cover complex wffs according to the rules:

$A \vee B\ \rho_w\ 1$ iff either $A\ \rho_w\ 1$ or $B\ \rho_w\ 1$
$A \vee B\ \rho_w\ 0$ iff both $A\ \rho_w\ 0$ and $B\ \rho_w\ 0$
$A \wedge B\ \rho_w\ 1$ iff both $A\ \rho_w\ 1$ and $B\ \rho_w\ 1$
$A \wedge B\ \rho_w\ 0$ iff either $A\ \rho_w\ 0$ or $B\ \rho_w\ 0$
$\neg A\ \rho_w\ 1$ iff $A\ \rho_w\ 0$
$\neg A\ \rho_w\ 0$ iff $A\ \rho_w\ 1$
$A \rightarrow B\ \rho_w\ 1$ iff for every $w' \in W$ such that $A\ \rho_{w'}\ 1$ it is also the case that $B\ \rho_{w'}\ 1$
$A \rightarrow B\ \rho_w\ 0$ iff there is some $w' \in W$ such that $A\ \rho_{w'}\ 1$ and $B\ \rho_{w'}\ 0$.

Notice these semantic rules, except for the new ones for

the conditional are the same as in FDE. Validity is defined as truth preservation at all worlds in all interpretations (as in FDE, modulo the addition of worlds).

Tableaux can be done for K_4 by augmenting the FDE rules given on p. 37. We will also need world numbers to go along with the $+$ and $-$'s; the FDE rules never change the world number. We need four new rules, for true and untrue conditionals and their negations:

$$\begin{array}{c} A \to B, +n \\ \swarrow \quad \searrow \\ A, -m \quad B, +m \end{array}$$

(for every world number m on the branch)

$$\begin{array}{c} A \to B, -n \\ \downarrow \\ A, +m \\ B, -m \end{array}$$

(where m is a new world number)

$$\begin{array}{c} \neg(A \to B), +n \\ \downarrow \\ A, +m \\ \neg B, +m \end{array}$$

(where m is a new world number)

$$\begin{array}{c} \neg(A \to B), -n \\ \swarrow \quad \searrow \\ A, -m \quad \neg B, -m \end{array}$$

(for every world number m on the branch)

Branches close when we get both $A, +n$ and $A, -n$ (with the same wff and world number).

First we'll show that *modus ponens* holds for $\to$:

$$\begin{array}{c} \underline{p \to q, p \vdash_{K_4} q} \\ \checkmark\, p \to q, +0 \\ p, +0 \\ q, -0 \\ \swarrow \quad \searrow \\ p, -0 \quad q, +0 \\ \times \qquad \times \end{array}$$

Modus tollens, on the other hand is still invalid:

$$\begin{array}{c} \underline{p \to q, \neg q \nvdash_{K_4} \neg p} \\ \checkmark\, p \to q, +0 \\ \neg q, +0 \\ \neg p, -0 \\ \swarrow \quad \searrow \\ p, -0 \quad q, +0 \end{array}$$

Counter-models are read off branches just as in FDE, with the additional complication of multiple worlds to consider. Here, the left open branch here represents a single-world model in which p is neither true nor false and q is false. *Trivially*, the conditional is true because there are no worlds in which p is true, so therefore none in which p is true but q is false. The right branch is compatible with an interpretation in which p is either neither true nor false, or in which p is only true, and q is both true and false. In that case, q is true at every world in which p is, and q is also false, but this does not mean that p has to be false anywhere.

Another example:

$$\begin{array}{c} \underline{p \to (q \to s) \vdash_{K_4} (p \wedge q) \to s} \\ \checkmark\, p \to (q \to s), +0 \\ \checkmark\, (p \wedge q) \to s, -0 \\ \checkmark\, p \wedge q, +1 \\ s, -1 \\ p, +1 \\ q, +1 \\ \swarrow \quad \searrow \\ p, -1 \qquad \checkmark\, q \to s, +1 \\ \times \qquad\qquad \swarrow \quad \searrow \\ \qquad\qquad q, -1 \quad s, +1 \\ \qquad\qquad \times \qquad \times \end{array}$$

The argument does not work in the other direction, however. Simply let $W = \{w_0, w_1\}$. At w_0 let p be true and q be untrue (it does not matter what s is). And at w_1 let p be untrue, q be true and s be untrue. There are no worlds then where $p \wedge q$ is true, so it would be trivially true that $(p \wedge q) \to s$. However, p can be true at world w_0 without it being the case that all worlds where q is true, s is also true (as w_1 shows).

Some results in K_4 that Priest likes:

$$\nvDash_{K_4} (p \wedge \neg p) \to q$$
$$\nvDash_{K_4} p \to (q \vee \neg q)$$
$$p \nvDash_{K_4} q \to p$$
$$\neg q \nvDash_{K_4} q \to p$$
$$\neg(p \to q) \nvDash_{K_4} p$$

But there are those he dislikes, e.g.:

$$\text{if } \vDash_{K_4} A \text{ then } \vDash_{K_4} B \to A$$
$$\text{e.g. } \vDash_{K_4} q \to (p \to p)$$
$$\text{if for all } w \in W, \text{not-}(A\ \rho_w\ 1) \text{ then } A \to B\ \rho_w\ 1$$

Against the validity of $q \to (p \to p)$, Priest asks us to consider:

- If every instance of the law of identity failed, if cows were black, cows would be black.

To correctly deal with such conditionals, Priest suggests a return to non-normal worlds, or worlds where the laws of logic are different. (I find his terminology here highly misleading, but I'll discuss that later.) This brings us to our next system.

J.2 The System N_4

System N_4 basically stands to K_4 the way that non-normal modal logic L stands to K. (Priest confesses in a footnote to the second edition that L_4 would have been a better name since N_4 is more like L than N.)

The syntax is the same as K_4.

The chief difference in the semantics is that there is a separation of normal and non-normal worlds. At normal worlds, conditionals receive their truth values according to the same semantic rules as in K_4. At non-normal worlds, "anything goes", i.e., an interpretation is free to make any conditional true, false, both or neither as it sees fit. To make this precise:

An N_4 *interpretation* is a triple $\langle W, N, \rho \rangle$ where:

- W is a set of worlds;
- N is a subset of W, the *normal* worlds;
- ρ is a relation that holds between propositional parameters, worlds and truth values 0 and 1, or between conditional statements $A \to B$, members of $W - N$ and the truth values 0 and 1.

It is therefore up to the interpretation to assign truth and/or falsity to conditionals directly at non-normal worlds. Every way of doing so results in a different interpretation. The relation ρ is extended to cover other complex wffs, and conditionals at normal worlds according to the semantic rules of K_4.

Validity is defined as truth-preservation at *normal worlds* only.

Notice that K_4 interpretations can be considered a subset of N_4 interpretations for the same reason that K interpretations can be considered a subset of L interpretations (those where $W = N$). Hence, K_4 is an extension of N_4, and nothing is valid in N_4 which is not also valid in K_4.

Tableaux are done just as they are for K_4 with the exception that rules for conditionals and their negations, $A \to B, +n$, $A \to B, -n$, $\neg(A \to B), +n$ and $\neg(A \to B), -n$ are only applied at world 0. (This resembles how L-tableaux are changed from K-tableaux.)

Here we show that this change solves Priest's earlier worry:

$$\begin{array}{c}
\nvdash_{N_4} q \to (p \to p) \\
\hline
q \to (p \to p), -0 \\
q, +1 \\
p \to p, -1
\end{array}$$

The tableau ends here. We do not apply any rule to $p \to p, -1$ since it is not world 0. We may consider w_1 to be non-normal and simply make $p \to p$ untrue by *fiat*. Notice that in K_4 we would have to create a new world where $p, +2$ and $p, -2$ and the branch would close.

K_4 valid arguments and wffs without nested conditionals, however, will remain valid in N_4.

$$\begin{array}{c}
\neg(p \to q) \vdash_{N_4} \neg(p \to (q \wedge p)) \\
\hline
\checkmark\ \neg(p \to q), +0 \\
\checkmark\ \neg(p \to (q \wedge p)), -0 \\
p, +1 \\
\neg q, +1 \\
\begin{array}{cc}
p, -1 & \checkmark\ \neg(q \wedge p), -1 \\
\times & \checkmark\ \neg q \vee \neg p, -1 \\
 & \neg q, -1 \\
 & \neg p, -1 \\
 & \times
\end{array}
\end{array}$$

Interestingly, N_4 counts as a *relevant logic* under the following definition:

A propositional logic employing a conditional $\to$ is said to be *relevant logic* iff there are no valid conditionals $A \to B$ where A and B do not share any propositional parameters in common.

In U.S. English, relevant logics are often called *relevance* logics instead. Here, I stick to Priest's more Australian vocabulary,

Sketch of proof that N_4 is a relevant logic: Let A and B have no propositional parameter in common. Now consider an interpretation with two worlds, w_0 and w_1, and let w_0 be normal and w_1 be non-normal. Let every propositional parameter, and every conditional in A be *both* true *and* false at w_1, but let every conditional and every propositional parameter in B be neither true nor false. One can prove by induction that A is both true and false at w_1, and so true, and that B is neither true nor false at w_1 and so untrue. Because there is a world where A is true and B is not true, $A \to B$ is not true at any normal world, including w_0. Hence $A \to B$ is not valid.

J.3 The Systems K_* and N_*

We saw in the last chapter that the relational semantics and the star semantics for FDE are equivalent. However, if the star semantics of FDE are modified to accommodate a strict conditional $\rightarrow$ the results differ from K_4 and N_4.

A *K_* interpretation* is a Routley Structure $\langle W, *, \nu \rangle$, where

- W is a set of worlds;
- $*$ is a function on worlds such that for all $w \in W$ it holds that $w^{**} = w$;
- ν is a function that assigns either 0 or 1 to each propositional parameter at each world in W

The evaluation function ν is extended to cover complex wffs just as in the star semantics for FDE with the additional clause:

$$\nu_w(A \rightarrow B) = \begin{cases} 1, & \text{if for all } w' \in W, \text{ if } \nu_{w'}(A) = 1 \\ & \qquad \text{then } \nu_{w'}(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

If we wish to accommodate non-normal worlds, we can define:

A *N_* interpretation* is a structure $\langle W, N, *, \nu \rangle$, where:

- W is a set of worlds, and N is a subset of W (the "normal worlds");
- $*$ is a function on W obeying $w^{**} = w$;
- ν is a function which assigns 1 or 0 to every propositional parameter at every world, and also assigns 1 or 0 to every conditional $A \rightarrow B$ at every world in $W - N$.

Validity would then be defined as truth preservation at normal worlds; K_* is clearly an extension of N_* just as K_4 is an extension of N_4.

Tableaux rules are the same as those for the star semantics method for FDE (see p. 40) and adding these rules for the conditional:

$$\begin{array}{c} A \rightarrow B, +n \\ \swarrow \quad \searrow \\ A, -m \quad B, +m \end{array}$$

(for every world m where either m or $\overline{m}$ appears on the branch; e.g., you must apply it to both 0 and $0^\#$.)

$$\begin{array}{c} A \rightarrow B, -n \\ \downarrow \\ A, +m \\ B, -m \end{array}$$

(where m is a new world number)

For K_* you apply these rules at every world; for N_* you only apply them when $n = 0$.

To see that these systems differ from K_4 and N_4 we can show that in them, *modus tollens* is valid:

$$\underline{p \rightarrow q, \neg q \vdash_{N_*} \neg p}$$

$$\begin{array}{c} \checkmark\, p \rightarrow q, +0 \\ \checkmark\, \neg q, +0 \\ \checkmark\, \neg p, -0 \\ q, -0^\# \\ p, +0^\# \\ \swarrow \quad \searrow \\ p, -0^\# \quad q, +0^\# \\ \times \qquad \times \end{array}$$

Notice that, unlike in K_4 one cannot "get away with" $p \rightarrow q$ being trivially true because p is not true anywhere while at the same time making $\neg p$ untrue, since this means p has to be true at the star world of 0.

There are also arguments that are valid in K_4 and N_4 invalid in these systems such as this one:

$$\underline{p \wedge \neg q \nvdash_{K_*} \neg(p \rightarrow q)}$$

$$\begin{array}{c} \checkmark\, p \wedge \neg q, +0 \\ \checkmark\, \neg(p \rightarrow q), -0 \\ p, +0 \\ \checkmark\, \neg q, +0 \\ q, -0^\# \\ p \rightarrow q, +0^\# \\ \swarrow \qquad \searrow \\ p, -0 \qquad\qquad q, +0 \\ \times \qquad\qquad \swarrow \quad \searrow \\ \qquad\qquad p, -0^\# \quad q, +0^\# \\ \qquad\qquad\qquad\qquad \times \end{array}$$

Every world where p is true, q is true as well; however $p \wedge \neg q$ does not require q to be false at the same world p is true, but instead that q be false at the star world of the world where p is true. Hence we can have p be false at that star world, leaving us with an open branch. I leave it to you to establish that the above is valid in N_4 (and *a fortiori* K_4).

The conditional $\rightarrow$ of K_* and N_* works in effect like a conditional one could define using the conditional of K_4 and N_4 as $(A \rightarrow B) \wedge (\neg B \rightarrow \neg A)$. Indeed, if one defines $A \Rightarrow B$ in K_4/N_4 thusly, then an argument is valid in K_*/N_* just in case the corresponding argument is valid in K_4/N_4 with $\Rightarrow$ replacing $\rightarrow$.

Homework

Priest, §9.11 (pp. 185–186). Problem 2 (any three of (a)–(m)), problem 3 (any three of (a)–(e)), problem 4 (same choices as for problem 2) and problem 5 (any three of earlier choices).

J.4 Non-normal Worlds and the Logically Impossible

Priest is clearly in favor of semantics employing non-normal worlds, and suggests we can consider them to be those worlds in which the laws of logic are different, or as "logically impossible worlds".

This interpretation seems misleading to me. Firstly, notice that in the gluts-and-gaps type systems we have been studying, even in *normal worlds*, the same wff can be true and false, or neither true nor false. It seems that even in those worlds, what we *ordinarily* think of as logically necessary is already out the window.

Priest, however, sees the "laws of logic" as mainly consisting of conditionals; they tell us what follows from what, and thinks the "normal" laws of logic are basically the paraconsistent ones of, e.g., K_3, but also thinks we can evaluate "what ifs" where the laws were different, e.g., "if the law of double negation did not hold, then…", and so non-normal worlds are needed to provide a proper semantics for such conditionals.

He sees worlds where the laws of logic are different as no more objectionable than worlds where the laws of nature are different, which contemporary philosophers are usually happy to consider. Perhaps these are not "possible worlds", but that is not grounds for dismissing them.

Those who take "possible worlds" as maximally consistent sets of wffs can certainly also countenance "impossible worlds" as, e.g., inconsistent or less than maximal sets of sentences. Modal realists, however, may be less than enthusiastic. Contrast Priest's attitude that "we seem to envisage such worlds when we evaluate conditionals such as 'if intuitionist logic were correct…" ' with Wittgenstein's remark in the *Tractatus* that we cannot say of an illogical situation what it would be like. Speaking personally, I find it difficult to truly "envisage" worlds with different logics; what I think of when evaluate such conditionals has rather more to do with what inferences I imagine myself accepting rather than those I accept now.

I do, however, believe that if possible worlds semantics were to be used in, e.g., providing a semantics for things such as belief states or other propositional attitudes, "impossible situations" may need to be countenanced, since it seems possible to have conflicting beliefs (and certainly conflicting desires and fears). Perhaps possible world semantics is not best for these, however.

Priest notes that might consider modifying K_4 and N_4 by introducing a special world @ for "the actual world", and adopt exclusion and exhaustion principles for *that world only*, and then, perhaps, redefine validity as truth preservation at @ in all interpretations. This would give us something much closer to classical logic for all wffs except those involving embedded conditionals. I rather like the idea, but Priest claims instead that this approach "would require justification by some novel considerations".

J.5 Constructible Negation and Connexive Logic

By starting with a FDE-base, and employing a possible world semantics with an accessibility relation, we can form logics quite similar to intuitionistic logic. Here we add to the syntax of FDE a conditional $\sqsupset$ and offer the following definitions:

An I_4 *interpretation* is a triple $\langle W, R, \rho\rangle$ where:

- W is a set of worlds;
- R is a transitive and reflexive accessibility relation between worlds;
- ρ is a relation that relates each propositional parameter to either 0 or 1, neither or both at each world, which obeys the following hereditary conditions:
  if $p\ \rho_w\ 1$ and wRw' then $p\ \rho_{w'}\ 1$
  if $p\ \rho_w\ 0$ and wRw' then $p\ \rho_{w'}\ 0$

The second of the hereditary conditions may appear suspect in anything like intuitionist logic. However, in relational semantics, what corresponds to the old feature of having semantic value 0 in our old intuitionist semantics is lacking the ρ relation 1. Bearing ρ to 0 here has more to do with truth of negation than lacking truth.

The relation ρ is extended to cover complex wffs in I_4 just as in K_4 for $\vee$, $\wedge$ and $\neg$. (We do not need a special $\neg$ with special semantics; the second hereditary condition will fix this.) For the conditional we have:

$A \sqsupset B\ \rho_w\ 1$ iff for all $w' \in W$ such that wRw' if $A\ \rho_{w'}\ 1$ then $B\ \rho_{w'}\ 1$.
$A \sqsupset B\ \rho_w\ 0$ iff $A\ \rho_w\ 1$ and $B\ \rho_w\ 0$

I_4 is called the *logic of constructible negation* because in it, a conditional $A \sqsupset B$ will only be false at w if A and $\neg B$

hold there; one can, as it were, provide or "construct" a counterexample to the conditional in the world of evaluation.

Tableaux are done as in K_4 except here we need accessibility lines governing conditionals, so the rules are as follows:

$$\begin{array}{c} A \supset B, +n \\ n\,r\,m \\ \swarrow \quad \searrow \\ A, -m \quad B, +m \end{array} \qquad \begin{array}{c} A \supset B, -n \\ \downarrow \\ n\,r\,m \text{ (new } m) \\ A, +m \\ B, -m \end{array}$$

$$\begin{array}{c} \neg(A \supset B), +n \\ \downarrow \\ A, +n \\ \neg B, +n \end{array} \qquad \begin{array}{c} \neg(A \supset B), -n \\ \swarrow \quad \searrow \\ A, -n \quad \neg B, -n \end{array}$$

There are also the usual reflexivity and transitivity rules plus the hereditary rules:

$$\begin{array}{c} p, +n \\ n\,r\,m \\ \downarrow \\ p, +m \end{array} \qquad \begin{array}{c} \neg p, +n \\ n\,r\,m \\ \downarrow \\ \neg p, +m \end{array}$$

The resulting system is equivalent to intuitionist logic for negation-free arguments and wffs. However, it differs where negation is involved. Indeed we have, for example, double negation elimination:

$$\underline{\vdash_{I_4} \neg\neg p \supset p}$$

$$\begin{array}{c} \checkmark\, \neg\neg p \supset p, -0 \\ 0\,r\,0 \\ 0\,r\,1 \\ \checkmark\, \neg\neg p, +1 \\ p, -1 \\ 1\,r\,1 \\ p, +1 \\ \times \end{array}$$

A logic somewhat closer to I's treatment of negation would come by adopting the *exclusion* principle, i.e., by ruling out those interpretations with gluts. This gives us a logic called I_3, where one would adopt the rule:

$$\begin{array}{c} A, +n \\ \neg A, +n \\ \downarrow \\ \times \end{array}$$

With results invalid in I_4 such as:

$$\underline{\vdash_{I_3} (p \wedge \neg p) \supset q}$$

$$\begin{array}{c} \checkmark\, (p \wedge \neg p) \supset q, -0 \\ 0\,r\,0 \\ 0\,r\,1 \\ \checkmark\, p \wedge \neg p, +1 \\ q, -1 \\ 1\,r\,1 \\ p, +1 \\ \neg p, +1 \\ \times \end{array}$$

Another interesting system—called W for Wansing—results by changing the semantic rule for false conditionals to the following:

$$A \supset B\ \rho_w\ 0 \text{ iff } A \supset \neg B\ \rho_w\ 1$$

Or equivalently:

$$A \supset B\ \rho_w\ 0 \text{ iff for all } w' \in W \text{ where } wRw', \text{ if } A\ \rho_{w'}\ 1 \text{ then } B\ \rho_{w'}\ 0$$

The tableaux rules for negated conditionals simplify to:

$$\begin{array}{c} \neg(A \supset B), +n \\ \downarrow \\ A \supset \neg B, +n \end{array} \qquad \begin{array}{c} \neg(A \supset B), -n \\ \downarrow \\ A \supset \neg B, -n \end{array}$$

This yields two interesting results regarding conditionals, named after the philosophers who endorsed them. The first is known as (Boethius):

$$\underline{\vdash_{W} (p \supset q) \supset \neg(p \supset \neg q)} \qquad \text{(Boethius)}$$

$$\begin{array}{c} \checkmark\, (p \supset q) \supset \neg(p \supset \neg q), -0 \\ 0\,r\,0 \\ 0\,r\,1 \\ \checkmark\, p \supset q, +1 \\ \checkmark\, \neg(p \supset \neg q), -1 \\ 1\,r\,1 \\ \checkmark\, p \supset \neg\neg q, -1 \\ 1\,r\,2 \\ p, +2 \\ \checkmark\, \neg\neg q, -2 \\ 2\,r\,2 \\ 0\,r\,2 \\ q, -2 \\ \swarrow \quad \searrow \\ p, -2 \quad q, +2 \\ \times \qquad \times \end{array}$$

(Boethius) is also known as *the principle of conditional non-contradiction*, in line with the principle of conditional excluded middle we considered when discussing Conditional Logic system C_2.

The second is known as (Aristotle):

$$\frac{\vdash_{W} \neg(p \sqsupset \neg p)}{} \qquad \text{(Aristotle)}$$

$$\begin{array}{c}
\checkmark\, \neg(p \sqsupset \neg p), -0 \\
0\, r\, 0 \\
p \sqsupset \neg\neg p, -0 \\
0\, r\, 1 \\
p, +1 \\
\checkmark\, \neg\neg p, -1 \\
1\, r\, 1 \\
p, -1 \\
\times
\end{array}$$

A propositional logic with a conditional → (or ⊐, ⊃, >, etc.) is called a *connexive logic* if every instance of $\neg(A \to \neg A)$ and $(A \to B) \to \neg(A \to \neg B)$ is valid.

Clearly, W is a connexive logic. The principles (Aristotle) and (Boethius) have a certain intuitive appeal when it comes to thinking about conditionals. This gives W a rather unique status among systems we've looked at with different conditionals.

Notice, moreover, that the corresponding wffs $\neg(p \supset \neg p)$ and $(p \supset q) \supset \neg(p \supset \neg q)$ are not classically valid. (These are false when p is false.) Hence, W is also unique in not being a sub-logic of classical logic.

Indeed, W is inconsistent! (It is paraconsistent, however, so this is not as bad as it would otherwise be.) We have for example, both of the following:

$$\vDash_{W} \neg((p \wedge \neg p) \sqsupset \neg(p \wedge \neg p))$$
$$\vDash_{W} (p \wedge \neg p) \sqsupset \neg(p \wedge \neg p)$$

The former is an instance of (Aristotle), hence valid, and the latter can also easily be checked with a tableau:

$$\begin{array}{c}
\checkmark\, (p \wedge \neg p) \sqsupset \neg(p \wedge \neg p), -0 \\
0\, r\, 0 \\
0\, r\, 1 \\
\checkmark\, p \wedge \neg p, +1 \\
\checkmark\, \neg(p \wedge \neg p), -1 \\
1\, r\, 1 \\
p, +1 \\
\neg p, +1 \\
\checkmark\, \neg p \vee \neg\neg p, -1 \\
\neg p, -1 \\
\neg\neg p, -1 \\
\times
\end{array}$$

Despite this oddity, Priest considers W a very "natural" implementation of connexive logic.

Homework

Priest, §9.11 (p. 186): problem 10 (any four of (a)–(h)).

[K] Other Systems of Relevant Logic

K.1 Introduction

Two systems, N_4 and N_*, of the previous chapter count as relevant logics. In chap. 9, Priest also briefly mentions *filter logics* which hold a conditional $A \to B$ to be valid just in case $A \supset B$ is classically valid along with one other "filtering" condition (perhaps simply that A and B share a parameter), but does not go into detail.

In all relevant logics, the goal is to provide an understanding of a conditional where $A \to B$ is true when (and only when) A is appropriately related to B. Often, systems such as N_* are taken as too weak for this purpose.

In chap. 10, Priest discusses so called "mainstream" systems of relevant logic, which are all built on a "base" system called B for *base system* (not to be confused with the the modal system B which Priest would call $K_{\rho\sigma}$ instead). Arguably, B is not itself very interesting, but various elaborations of it are. This system is given a Routley Star-style semantics somewhat similar to N_*, but where truth-conditions for conditionals at non-normal worlds are not completely lawless, but constrained by a less stringent rule than the truth-conditions at normal worlds.

K.2 Semantics for B

A *B interpretation* is a structure $\langle W, N, R, *, v\rangle$, where:

- W is a set of worlds;
- N is a subset of W, the *normal worlds*, and for all $w \in W$, $w \in N$ iff ($Rww'w''$ iff $w' = w''$ for all $w', w'' \in W$);
- R is a three-place (ternary) relation among worlds in W (or technically $R \subseteq W^3$);
- * is function on worlds in W such that for all $w \in W$, $w^{**} = w$;
- v is a function assigning either 0 or 1 to each propositional parameter p at each word $w \in W$; as usual we write $v_w(p) = 1$ or $v_w(p) = 0$.

The relation R here plays a role similar to the role played by accessibility relations in modal logic. Of course, matters are complicated by its being a three-place relation. It is difficult to intuitively describe what $Rw_0w_1w_2$ means, but it is something like, *what's true at w_0 allows the information at w_1 to be recognized at w_2*, but even this is probably

misleading. Technically, what's required is that $Rw_0w_1w_2$ means that whenever $A \to B$ is true at w_0 and A is true at w_1 then B is true at w_2.

v is extended to cover complex wffs according to rules similar to those in K$_*$ and N$_*$ for everything but the conditional, where R comes in to play:

$$v_w(A \to B) = \begin{cases} 1, & \text{for all for all } w', w'' \in W \text{ such that} \\ & \qquad Rww'w'', \text{ if } v_{w'}(A) = 1 \\ & \qquad \text{then } v_{w''}(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that a normal world is in effect defined as follows:

w is a *normal world* iff ($Rww'w''$ iff $w' = w''$ for all $w', w'' \in W$).

which means

- For all $w' \in W$, $Rww'w'$. And
- It is never true that $Rww'w''$ unless $w' = w''$.

This means that $\to$ is in effect a strict conditional at normal worlds; $A \to B$ will be true there just in case at every world where A is true, B is true as well. I.e.:

$$\text{if } w \in N, \text{ then } v_w(A \to B) = \begin{cases} 1, & \text{if for all } w' \in W, \\ & \qquad \text{if } v_{w'}(A) = 1 \\ & \qquad \text{then } v_{w'}(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

B is in between K$_*$ and N$_*$ in strength. Notice that if every world is normal, i.e., $N = W$, then we in effect have a K$_*$-interpretation, so everything that is B-valid is K$_*$-valid as well. Moreover, each B interpretation can be considered an N$_*$ interpretation—just in let the N$_*$ interpretation assign its values to conditionals at non-normal worlds according to the rule above, invoking some R not in the model. Hence everything that is N$_*$-valid is B-valid.

K.3 Tableaux for B

Tableaux are done using the Routley Star semantics FDE tree rules as described on p. 40 (also used for K$_*$ and N$_*$). However, we now will need accessibility lines to track the ternary relation R, now written $r\,n\,m\,j$ where this means, basically, $Rw_nw_mw_j$. In addition to the FDE rules we also need a new rule for conditionals (true and false) and a rule for representing the normality of world 0 (which is the only world we'll ever know to be normal). This gives us $r\,0\,n\,n$ for any n such that n or $\overline{n}$ is on the branch. This is called the *normality rule*:

$$\begin{array}{c} A \to B, +n \\ r\,n\,m\,j \\ \swarrow \quad \searrow \\ A, -m \quad B, +j \end{array} \qquad \begin{array}{c} A \to B, -n \\ \downarrow \\ r\,n\,m\,j \\ A, +m \\ B, -j \end{array} \qquad \begin{array}{c} \text{(normality)} \\ \downarrow \\ r\,0\,n\,n \\ \text{for all } n \text{ such that} \\ n \text{ or } \overline{n} \text{ is on branch} \end{array}$$

Note for the middle rule, for false conditionals, m and j are new, and distinct unless $n = 0$ in which case $m = j$, but it is still new.

Some examples:

$$\underline{p \to q, q \to s \vdash_{\mathrm{B}} p \to s}$$

$$\begin{array}{c} \checkmark p \to q, +0 \\ \checkmark q \to s, +0 \\ \checkmark p \to s, -0 \\ r\,0\,0\,0, r\,0\,0^{\#}\,0^{\#} \\ r\,0\,1\,1, r\,0\,1^{\#}\,1^{\#} \\ p, +1 \\ s, -1 \\ \swarrow \qquad \searrow \\ \begin{array}{c} p, -1 \\ \times \end{array} \qquad \begin{array}{c} q, +1 \\ \swarrow \quad \searrow \\ \begin{array}{c} q, -1 \\ \times \end{array} \quad \begin{array}{c} s, +1 \\ \times \end{array} \end{array} \end{array}$$

Although *modus ponens* is valid in B, we nonetheless have:

$$\underline{\nvdash_{\mathrm{B}} (p \wedge (p \to q)) \to q}$$

$$\begin{array}{c} \checkmark (p \wedge (p \to q)) \to q, -0 \\ r\,0\,0\,0, r\,0\,0^{\#}\,0^{\#} \\ r\,0\,1\,1, r\,0\,1^{\#}\,1^{\#} \\ \checkmark p \wedge (p \to q), +1 \\ q, -1 \\ p, +1 \\ p \to q, +1 \end{array}$$

Because we do not have anything of the form $r\,1\,m\,j$, the tableau ends here. Can you see why this is invalid, even though MP is valid? It is because $p \to q$ can be true at a non-normal world like w_1 for arbitrary reasons. It does not require q to be true there even when p is.

Counter-models are read off branches in the normal way.

Here is an example of a formula valid in in B but not valid in N_*:

$$\underline{\vdash_B ((p \to q) \land (p \to s)) \to (p \to (q \land s))}$$

$$\checkmark ((p \to q) \land (p \to s)) \to (p \to (q \land s)), -0$$
$$r\,0\,0\,0, r\,0\,0^\#\,0^\#$$
$$r\,0\,1\,1, r\,0\,1^\#\,1^\#$$
$$\checkmark (p \to q) \land (p \to s), +1$$
$$\checkmark p \to (q \land s), -1$$
$$\checkmark p \to q, +1$$
$$\checkmark p \to s, +1$$
$$r\,1\,2\,3$$
$$p, +2$$
$$\checkmark q \land s, -3$$
$$r\,0\,2\,2, r\,0\,2^\#\,2^\#$$
$$r\,0\,3\,3, r\,0\,3^\#\,3^\#$$

$$\begin{array}{cccc} p,-2 & q,+3 & & \\ \times & & & \\ & p,-2 & s,+3 & \\ & \times & & \\ & & q,-3 & s,-3 \\ & & \times & \times \end{array}$$

K.4 A Traditional Deductive System for B

Relevant logics are usually studied in the form of more traditional, axiomatic, deductive systems. Especially because tableaux get more and more annoying for extensions of B, Priest breaks down and presents his first traditional deductive system for B.

A *deduction* of wff A from Δ is a finite ordered series of wffs $B_1, B_2, \ldots, B_n$ such that for each B_i $(1 \leq i \leq n)$, B_i is either (a) a member of of Δ, (b) an instance of an axiom schemata (an axiom), (c) the result of applying an *inference rule* to previous members of the series, and A is B_n.

A *proof* of wff A is deduction of A from $\varnothing$.

We write $\Delta \vdash A$ if there is a proof of A from Δ. (This is the more usual use of this sign, though Priest also uses it for tableaux proofs as well.)

We write $\vdash A$ as shorthand for $\varnothing \vdash A$. If this holds, A is called a *theorem*.

Note that with these definitions there are no indirect or conditional proofs. That makes this form of deduction *axiomatic* rather than a "natural deduction" system, though the boundary between the two kinds of system is rather unclear. It is worth noting, however, that the system would not be equivalent if conditional proof (and/or indirect proof) were added; indeed, I believe it would collapse into classical logic if these were added.

An *axiom of B* is any instance of the following schemata:

(A1)	$A \to A$
(A2)	$A \to (A \lor B)$
	$B \to (A \lor B)$
(A3)	$(A \land B) \to A$
	$(A \land B) \to B$
(A4)	$A \land (B \lor C) \to ((A \land B) \lor (A \land C))$
(A5)	$((A \to B) \land (A \to C)) \to (A \to (B \land C))$
(A6)	$((A \to C) \land (B \to C)) \to ((A \lor B) \to C)$
(A7)	$\neg\neg A \to A$

(It would be easy enough to show these are all valid in B using tableaux if need be.)

Note that these are schemata, and so every instance of them is an axiom. There are therefore infinitely many axioms. As an instance of (A1) we have, e.g., not just $p \to p$ and $q \to q$ but also $\neg\neg(p \lor \neg s) \to \neg\neg(p \lor \neg s)$.

An *inference rule of B* is any of the following:

MP	From A and $A \to B$ infer B.
Conj	From A and B infer $A \land B$.
Pre[fixing]	From $A \to B$ infer $(C \to A) \to (C \to B)$.
Suf[fixing]	From $A \to B$ infer $(B \to C) \to (A \to C)$.
Trans	From $A \to \neg B$ infer $B \to \neg A$.

(Those are my abbreviations. Priest numbers these instead.)

Again, it would easy enough to check that the corresponding arguments are valid (e.g., $A \to \neg B \vDash_B B \to \neg A$) using tableaux.

This deductive system is sound and complete for B.

Prefixing and Suffixing are the most "interesting" differences between B and N_*, where those are invalid. Hence, B and its extensions are sometimes called *affixing* relevant logics.

Here is a simple deduction for $p \to q, q \to s \vdash_B p \to s$:

(1) $p \to q$	Pr
(2) $q \to s$	Pr
(3) $(q \to s) \to (p \to s)$	1 Suf
(4) $p \to s$	2, 3 MP

You can also do a deduction in a "schematic form", and then think of the result as a "theorem schema" or "derived rule" (depending on the details) which could be invoked for anything of the appropriate form, e.g.:

$$\vdash_B A \to \neg\neg A$$

(1) $\neg A \to \neg A$ (A1)
(2) $A \to \neg\neg A$ 1 Trans

The first line here is not strictly speaking an instance of (A1), but is a "sub-schema"—any instance of it is an instance of (A1). The last line here is a theorem schema, which, once proven, can be used as if it were an axiom. (One could always repeat the steps above if need be to get the result.) We'll call this (DN).

Similarly if we went through the deduction of $p \to q, q \to s \vdash p \to s$ and replaced "p" with "A", "q" with "B", throughout, one would have general result that if you had $A \to B$ and $B \to C$ you could deduce $A \to C$. You can then use this as if it were an inference rule, since you could always go through all the intermediate steps instead. This is called a *derived rule*. I'll call this one HS for Hypothetical Syllogism, and use it in extensions of B.

K.5 Extensions of B

The axiomatic system for B might be extended by adding one or more of the following additional axiom schemata:

(A8) $(A \to \neg B) \to (B \to \neg A)$
(A9) $(A \to B) \to ((B \to C) \to (A \to C))$
(A10) $(A \to B) \to ((C \to A) \to (C \to B))$
(A11) $(A \to (A \to B)) \to (A \to B)$

Only some of the possible resulting systems have names, but just to mention a couple that do:

DW = B + (A8)
TW = B + (A8), (A9) and (A10)

Of course, since these are extensions of B, and B is complete with respect to its semantics as is, obviously there are instances of the above that are not valid in the semantics we outlined earlier. However, their adoption would be justifiable by adding restrictions on the ternary R accessibility relation in B-interpretations (much like adopting different modal axioms such as (S4) or (S5) amounts to placing restrictions on accessibility in K-interpretations).

Conditions (C8)–(C11) are the restrictions which validate (A8)–(A11) respectively:

(C8) For all $w, w', w'' \in W$ if $Rww'w''$ then $Rww''^*w'^*$.

(C9) For all $w_1, w_2, w_3, w_4 \in W$, if there is a $w \in W$ such that both Rw_1w_2w and Rww_3w_4 then there is a $w' \in W$ such that Rw_1w_3w' and $Rw_2w'w_4$.

(C10) For all $w_1, w_2, w_3, w_4 \in W$, if there is a $w \in W$ such that both Rw_1w_2w and Rww_3w_4 then there is a $w' \in W$ such that Rw_2w_3w' and $Rw_1w'w_4$.

(C11) For all $w_1, w_2, w_3 \in W$, if $Rw_1w_2w_3$ then there is a $w \in W$ such that both Rw_1w_2w and Rww_2w_3.

None of these are particularly obvious or easy to remember. Priest discusses how the tableaux method can be modified to include corresponding rules, but it hardly seems worth it. Many of the resulting tableaux are infinite and annoyingly complex. The axiomatic deductive systems seem much easier to wrap one's head around (for a human, anyway).

Fairly obviously, some of these give you "conditional" versions of what we already had as rules. (A8)–(A10) then would make the Pre, Suf and Trans rules redundant. Because we do not have conditional proof, these axiom versions are stronger than the rule versions. (Suppose e.g., you had something of one of these forms as antecedent of another conditional; having the rule version would not help.) It is harmless to maintain Pre, Suf and Trans as redundant rules, or else we can consider them to be derived rules.

K.6 Content Inclusion

To correctly characterize the semantics for other notable extensions of B, we need to add another component to an interpretation. This takes the form of a relation on worlds $\sqsubseteq$, so that $w \sqsubseteq w'$ basically means that the information in w is a subset of the information in w'. (This is a tad misleading, but close enough for now.)

We now *re*-define a B interpretation:

A *B-interpretation* is a structure $\langle W, N, R, *, \sqsubseteq, v\rangle$, where $W, N, R, *, v$ are as before, and, $\sqsubseteq$ is a binary relation on members of W (i.e., a subset of $W \times W$) such that:

1. If $w \sqsubseteq w'$ and $v_w(p) = 1$ then $v_{w'}(p) = 1$.
2. If $w \sqsubseteq w'$ then $w'^* \sqsubseteq w^*$.
3. If $w \sqsubseteq w'$ and $Rw'w_1w_2$ then ($w \in N$ and $w_1 \sqsubseteq w_2$) or ($w \notin N$ and Rww_1w_2).

Note, every B-interpretation in the old definition of B-interpretation can be considered a B-interpretation in this new definition where $\sqsubseteq$ is simply the identity relation between worlds.

Together the three conditions on $\sqsubseteq$ make the first condi-

tional generalize over all formulæ, i.e., we have:

$$\text{If } w \sqsubseteq w' \text{ and } v_w(A) = 1 \text{ then } v_{w'}(A) = 1.$$

It is then easy to see how $\sqsubseteq$ represents a kind of subset relation. It is written in a squarish shape because of the relationship to truth in intuitionist logic, which is always passed on through accessibility thanks to the hereditary condition. (Here, truth is passed on from w to w' when $w \sqsubseteq w'$.) Notice however that were are using $\sqsubseteq$ as part of the *metalanguage*, and we will not be using it in the object language.

We can now describe further extensions of B by adding restraints to $\sqsubseteq$ and its relation to R, N, v, etc. Here are several conditions we might consider:

(C12) If $Rww'w''$ then there is a w_1 such that $w_1 \sqsubseteq w$ and $Rw'w_1w''$.

(C13) If $w \in N$, $w^* \sqsubseteq w$.

(C14) If $w \in N$, then $w^* \sqsubseteq w$; but if, instead, $w \in W - N$, then Rww^*w.

(C15) If $Rww'w''$ then $w \sqsubseteq w''$.

(C16) If $Rww'w''$ then either $w \sqsubseteq w''$ or $w' \sqsubseteq w''$.

Clearly, these conditions are not all independent; (C15) implies (C16), for example.

Accepting any of these conditions is tantamount, for all intents and purposes, of regarding one of the following axiom schemata as valid, respectively:

(A12) $A \to ((A \to B) \to B)$
(A13) $A \lor \neg A$
(A14) $(A \to \neg A) \to \neg A$
(A15) $A \to (B \to A)$
(A16) $A \to (A \to A)$

By accepting some or others of these, we get many different systems. Here are some that have names. (Priest gives a chart showing their relative strengths on p. 203.)

BX = B + (A13)
DWX = B + (A8) + (A13)
TWX = B + (A8) + (A9) + (A10) + (A13)
T = B + (A8) + (A9) + (A10) + (A11) + (A14)
RW = B + (A8) + (A9) + (A10) + (A12)
R = B + (A8) + (A9) + (A10) + (A11) + (A12)
RWK = B + (A8) + (A9) + (A10) + (A12) + (A15)
RM = B + (A8) + (A9) + (A10) + (A11) + (A12) + (A16)

Of course, there are many others that do not have names. RW and RWK are also called C and CK. Priest avoids these names to avoid confusion with the conditional logic C. Some of these aren't technically relevant logics any more (RWK and RM), because they are too strong and allow, for instance, $\vDash_{RWK} q \to (p \to p)$ and $\vDash_{RM} (p \land \neg p) \to \neg(q \land \neg q)$.

Again, Priest develops some very complicated tableaux systems for these, but they seemed too complicated to me to be worth implementing, especially bearing in mind the relatively simple axiomatic deductive systems that are available.

K.7 *System R*

Of the systems in the previous chart, system R is the best known and most studied. It has (A1)–(A12) as its axiom schemata, and MP and Conj as inference rules. (The rules Pre, Suf and Trans are also available as derived rules.) It was first developed as a deductive system; semantics were added later. Even with semantics, it is often hard to determine what is valid or not in R. The system is known to be *undecidable*, that is, there is no mechanical technique for determining whether or not a given wff or argument is valid. (Tableaux are usually infinite.)

Here are some deductions in R. First we'll establish the derived rule Perm[utation]:

$$\underline{A \to (B \to C) \vdash_R B \to (A \to C)} \qquad \text{(Perm)}$$

(1) $A \to (B \to C)$	Pr
(2) $(A \to (B \to C)) \to (((B \to C) \to C) \to (A \to C))$	(A9)
(3) $((B \to C) \to C) \to (A \to C)$	1, 2 MP
(4) $B \to ((B \to C) \to C)$	(A12)
(5) $B \to (A \to C)$	3, 4 HS

Strictly speaking, this deduction is more than five lines long, because we invoked the derived rule HS. We could insert the corresponding lines from that deduction.

Next we'll show the theorem schema (CM) (*consequentia mirabilis*):

$$\underline{\vdash_R (A \to \neg A) \to \neg A} \qquad \text{(CM)}$$

(1) $(A \to \neg A) \to (A \to \neg A)$	(A1)
(2) $A \to ((A \to \neg A) \to \neg A)$	1 Perm
(3) $((A \to \neg A) \to \neg A) \to (A \to \neg(A \to \neg A))$	(A8)
(4) $(A \to ((A \to \neg A) \to \neg A)) \to (A \to (A \to \neg(A \to \neg A)))$	3 Pre
(5) $A \to (A \to \neg(A \to \neg A))$	2, 4 MP
(6) $(A \to (A \to \neg(A \to \neg A))) \to (A \to \neg(A \to \neg A))$	(A11)
(7) $A \to \neg(A \to \neg A)$	5, 6 MP
(8) $(A \to \neg A) \to \neg A$	7 Trans

Finally we'll show that the law of excluded middle holds in R (even without (A13) as an axiom schema):

$$\underline{\vdash_R A \vee \neg A} \qquad \text{(EM)}$$

(1)	$A \rightarrow (A \vee \neg A)$	(A2)
(2)	$(A \vee \neg A) \rightarrow \neg\neg(A \vee \neg A)$	(DN)
(3)	$A \rightarrow \neg\neg(A \vee \neg A)$	1, 2 HS
(4)	$\neg(A \vee \neg A) \rightarrow \neg A$	3 Trans
(5)	$\neg A \rightarrow (A \vee \neg A)$	(A2)
(6)	$\neg(A \vee \neg A) \rightarrow (A \vee \neg A)$	4, 5 HS
(7)	$\neg(A \vee \neg A) \rightarrow \neg\neg(A \vee \neg A)$	2, 6 HS
(8)	$(\neg(A \vee \neg A) \rightarrow \neg\neg(A \vee \neg A)) \rightarrow \neg\neg(A \vee \neg A)$	(CM)
(9)	$\neg\neg(A \vee \neg A)$	7, 8 MP
(10)	$\neg\neg(A \vee \neg A) \rightarrow (A \vee \neg A)$	(A7)
(11)	$A \vee \neg A$	9, 10 MP

One disadvantage of focusing on deductions is that they can only be used to show that something is valid. They cannot be used to show that something is invalid. The natural way to show something invalid would be to find a counter-model, e.g., with a tableaux. But for R there is at least one other method, which is to compare it to the simple three-valued paraconsistent logic, RM_3, discussed on p. 33. It is relatively easy to check that every axiom and inference rule of R is valid in RM_3, and so R is a subtheory of RM_3. If something is R-valid, it is also RM_3-valid. Hence, if something is RM_3-*invalid*, it is also R-invalid. Hence, an RM_3 tableau (much easier than R tableaux, despite three way branching), or RM_3 truth table, showing something invalid in RM_3 also shows it invalid in R.

Homework

Priest, §10.11 (p. 218), problems 2 and 4.

K.8 Relevant Ceteris Paribus *Conditionals*

Because systems built on B (including R) validate such rules as hypothetical syllogism, contraposition, antecedent strengthening, etc., their conditionals are arguably inappropriate for subjunctive or counterfactual conditionals. Recall from our discussion of conditional logics that Priest thinks of such conditionals as involving a *ceteris paribus* clause. However, he criticized such logics (C^+ and stronger) for validating $A > B$ whenever A is impossible or B is necessary.

We are finally in a position to describe a conditional that neither validates syllogism, contraposition, antecedent strengthening, etc., nor $A > B$ when A is not relevant to B, buy combining the approaches of conditional logic and relevant logic.

Here we briefly sketch two systems C_B and C^+_B, similar to conditional logics C and C^+ but using relevant logic B as its underlying system. We add the connective $>$ so that $A > B$ means (if A, then *ceteris paribus*, B); we maintain $A \rightarrow B$ for conditionals without such a clause.

To our formal semantics, we add another set of accessibility relations $\{R_A : A \text{ is a wff}\}$, one for each wff, to a B-interpretation. Then wR_Aw' means that the ceteris paribus clause for assessing conditionals of the form "$A > \ldots$" at w holds at w', or that w' is among those worlds that matter for determining the truth of "$A > \ldots$" at w. As in our chapter on conditional logic, we write $f_A(w)$ for the set of worlds w' for which wRw', and $[B]$ for the set of worlds w where $v_w(B) = 1$. Then:

$$v_w(A > B) = \begin{cases} 1, & \text{if } f_A(w) \subseteq [B]; \\ 0, & \text{otherwise.} \end{cases}$$

For C^+_B we assume:

1. for any $w \in N$, if $f_A(w) \subseteq [A]$, and
2. if $w \in N$ and $w \in [A]$ then $w \in f_A(w)$.

Tableaux rules like those of C^+ are added to those for B:

$$\begin{array}{c} A > B, +n \\ n\, r_A\, m \\ \downarrow \\ B, +m \end{array} \qquad \begin{array}{c} A > B, -n \\ \downarrow \\ n\, r_A\, m \\ [A, +m] \ (\text{if } n = 0) \\ B, -m \end{array} \qquad \begin{array}{c} \cdot \\ \swarrow \quad \searrow \\ A, -0 \qquad A, +0 \\ \qquad\quad 0\, r_A\, 0 \end{array}$$

(The last rule is applied for every antecedent A where $A > \ldots$ or $\neg(A > \ldots)$ appears on the branch.)

$$\underline{\nvdash_{C^+_B} (p \wedge \neg p) > q}$$

$$\begin{array}{c}
\checkmark (p \wedge \neg p) > q, -0 \\
r\,0\,0\,0, r\,0\,0^{\#}\,0^{\#} \\
0\, r_{p\wedge\neg p}\, 1 \\
\checkmark p \wedge \neg p, +1 \\
q, -1 \\
r\,0\,1\,1, r\,0\,1^{\#}\,1^{\#} \\
p, +1 \\
\checkmark \neg p, +1 \\
p, -1^{\#} \\
\swarrow \qquad\qquad \searrow \\
\begin{array}{c} \checkmark p \wedge \neg p, -0 \\ \swarrow \quad \searrow \\ \begin{array}{cc} p, -0 & \begin{array}{c} \checkmark \neg p, -0 \\ p, +0^{\#} \end{array} \end{array} \end{array}
\qquad
\begin{array}{c} \checkmark p \wedge \neg p, +0 \\ 0\, r_{p\wedge\neg p}\, 0 \\ p, +0 \\ \checkmark \neg p, +0 \\ p, -0^{\#} \end{array}
\end{array}$$

[L]
Fuzzy Logic

L.1 *The Sorites Paradox and System Ł*

Consider the following argument:

1. For all n, if n grains of wheat do not make a heap, $n+1$ grains of wheat do not make a heap.
2. One grain of wheat does not make a heap.
3. Therefore, 10000 grains of wheat do not make a heap.

The premises of this argument intuitively seem true. But the conclusion seems false. If one were to deny a premise, it would likely be the first premise. But this commits you to claiming that there is some specific number n—27 say—such that n grains of wheat do not make a heap, but $n+1$ (28) grains do. But any choice of n seems arbitrary.

Similar puzzles can be constructed with any other vague predicate, "bald", "old", "skinny", "high up", "green [vs. blue-green vs. blue]", etc. These problems are called *sorites paradoxes*.

Epistemicists claim that there is some specific boundary that can be drawn in all such cases, but that it is impossible for us to know what that boundary is (because we cannot discriminate between cases where the boundary has been crossed and when it hasn't). Supervaluationists claim either that there is a range with indeterminate truth-value, or many different acceptable disambiguations of vague predicates, but that in non-borderline cases, predication of the predicate or its opposite is warranted due to its holding under all resolutions, as in sec. 8.4.

A final response is to claim that truth and falsity themselves come in degrees, rather than being all or nothing.

A *fuzzy logic* is an infinitely valued many-valued logic in which the possible truth-values, $\mathcal{V}$ make up a continuum in the interval $[0, 1]$, i.e., all real numbers $\varepsilon \in \mathbb{R}$ such that $0 \leq \varepsilon \leq 1$.

The thought is that something with truth-value 0 is wholly false, something with truth-value 1 is wholly true, and for those in between, something with value 0.75 is "more true" than something with value 0.25, and so on.

The most well known fuzzy logic is Ł, developed by Łukasiewicz along with Tarski. The propositional version has propositional parameters and the connectives $\neg$, $\vee$, $\wedge$ and $\rightarrow$.

L.2 *Semantics of Ł*

An *Ł-interpretation* is a function v assigning a member of $\mathcal{V}$ to each propositional parameter p of the language. (As usual, we write this value as $v(p)$.)

v is extended to cover complex wffs as follows:

$$
\begin{aligned}
v(\neg A) &= 1 - v(A)\\
v(A \vee B) &= \text{the maximum of } v(A) \text{ and } v(B)\\
v(A \wedge B) &= \text{the minimum of } v(A) \text{ and } v(B)\\
v(A \rightarrow B) &= v(A) \ominus v(B), \text{ where:}
\end{aligned}
$$

$$
x \ominus y = \begin{cases} 1, & \text{if } x \leq y;\\ 1-(x-y), & \text{otherwise (i.e. if } x > y). \end{cases}
$$

Notice that if the inputs are the classical 0 or 1, the outputs match the classical outputs. Also notice that if consider the value i from the gappy three-valued system $Ł_3$ to be value 0.5, the outputs also match their outputs in $Ł_3$ for inputs allowed there.

What should count as a "designated value"? Only 1? But isn't 0.999999 "close enough" to count as truth? Maybe even 0.75? or lower? Perhaps this is context dependent. For each $\varepsilon \in \mathcal{V}$, we can define a species of validity:

- $\Delta \vDash_{\varepsilon} A$ iff there is no interpretation v such that $v(B) \geq \varepsilon$ for all $B \in \Delta$, but $v(A) < \varepsilon$.

These context-dependent senses of validity allow us to define two context-independent ones:

- *$Ł_{\aleph}$-validity:* $\Delta \vDash_{Ł_{\aleph}} A$ iff $\Delta \vDash_1 A$ (i.e., $\Delta \vDash_{\varepsilon} A$ where $\varepsilon = 1$.)
- *Ł-validity:* $\Delta \vDash A$ iff for all $\varepsilon \in \mathcal{V}$, $\Delta \vDash_{\varepsilon} A$.

The *greatest lower bound* of a set of numbers $\mathcal{X}$ is the largest number ε such that for all $x \in \mathcal{X}$, $\varepsilon \leq x$.

(Notice that for a finite set, its greatest lower bound will just be its minimum.)

Result: $\Delta \vDash A$ iff, for all interpretations v, (the greatest lower bound of truth-values $v(B)$ of members B of Δ) $\leq v(A)$.

Result: $B_1, \ldots, B_n \vDash A$ iff for all interpretations v, $v((B_1 \wedge \ldots \wedge B_n) \rightarrow A) = 1$.

Result: $B_1, \ldots, B_n \vDash A$ iff $\vDash_{Ł_{\aleph}} (B_1 \wedge \ldots \wedge B_n) \rightarrow A$.

Because of this last result, an examination of Ł-validity is interchangeable with an examination of the $Ł_{\aleph}$-validity of conditionals, which it is often more practical to examine directly.

L.3 Axiomatization of $Ł_\aleph$

Priest presents two different ways of axiomatizing $Ł_\aleph$. I'll skip right to the second, which arrives at an axiomatization of $Ł_\aleph$ simply by adding a single axiom schema (A17) to RWK from the last chapter. The inference rules are MP and Conj. The axiom schemata are:

(A1)	$A \to A$
(A2)	$A \to (A \vee B)$
	$B \to (A \vee B)$
(A3)	$(A \wedge B) \to A$
	$(A \wedge B) \to B$
(A4)	$A \wedge (B \vee C) \to ((A \wedge B) \vee (A \wedge C))$
(A5)	$((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(A6)	$((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(A7)	$\neg\neg A \to A$
(A8)	$(A \to \neg B) \to (B \to \neg A)$
(A9)	$(A \to B) \to ((B \to C) \to (A \to C))$
(A12)	$A \to ((A \to B) \to B)$
(A15)	$A \to (B \to A)$
(A17)	$((A \to B) \to B) \to (A \vee B)$

The numbering here is set up to match the axioms in relevant logic. (Hence the lack of (A10), (A11), etc.) We could go through these and for each, show that it is valid algebraically. E.g., suppose for *reductio* that there were an instance of (A15) that were invalid. Then for some A, B, it would hold that:

$$v(A \to (B \to A)) \neq 1$$

Then

$$v(A) > v(B \to A)$$

This means that

$$v(A) > (v(B) \ominus v(A))$$

Because $v(A)$ is greater than it, it cannot be that $v(B) \ominus v(A) = 1$, and hence:

$$v(B) \ominus v(A) = 1 - (v(B) - v(A))$$

It then follows that:

$$v(A) > 1 - (v(B) - v(A)) \text{ i.e.}$$
$$v(A) > 1 - v(B) + v(A)$$

From which it follows that

$$0 > 1 - v(B) \text{ and so}$$
$$v(B) > 1$$

This is impossible. Hence, every instance of (A15) is valid. Similar techniques show the rest to be valid. Conj and MP clearly preserve validity as well. E.g., note that $A \to B$ has value 1, and A has value 1, then B must also have value 1.

With the validity of the axioms established we can use them to establish other validities. Notice that $Ł_\aleph$ is an extension of B, and so its derived rules, etc., may be used.

$\underline{\vdash_{Ł_\aleph} (A \wedge B) \to \neg(\neg A \vee \neg B)}$

(1) $(A \wedge B) \to A$	(A3)
(2) $(A \wedge B) \to B$	(A3)
(3) $A \to \neg\neg A$	(DN)
(4) $B \to \neg\neg B$	(DN)
(5) $(A \wedge B) \to \neg\neg A$	1, 3 HS
(6) $(A \wedge B) \to \neg\neg B$	2, 4 HS
(7) $\neg A \to \neg(A \wedge B)$	5 Trans
(8) $\neg B \to \neg(A \wedge B)$	6 Trans
(9) $(\neg A \to \neg(A \wedge B)) \wedge (\neg B \to \neg(A \wedge B))$	7, 8 Conj
(10) $((\neg A \to \neg(A \wedge B)) \wedge (\neg B \to \neg(A \wedge B))) \to$ $((\neg A \vee \neg B) \to \neg(A \wedge B))$	(A6)
(11) $(\neg A \vee \neg B) \to \neg(A \wedge B)$	9, 10 MP
(12) $(A \wedge B) \to \neg(\neg A \vee \neg B)$	11 Trans

However, the missing axiom of R, i.e., (A11), is not universally valid in $L_\aleph$. Consider:

$$(p \to (p \to q)) \to (p \to q)$$

Consider an interpretation in which $v(p) = 0.9$ and $v(q) = 0.6$. Then $v(p \to q) = v(p) \ominus v(q) = 1-(0.9-0.6) = 0.7$. And $v(p \to (p \to q)) = 1 - (0.9 - 0.7) = 0.8$, and so the value of the whole wff is $1-(0.9-0.8)$, i.e., 0.9, which is not 1. Hence $Ł_\aleph$ is not an extension of R.

For the same interpretation, the wff:

$$(p \wedge (p \to q)) \to q$$

also has truth-value 0.9. Hence it is not $Ł_\aleph$-valid either.

Consider now:

$$(p \wedge \neg p) \to q$$

Consider the interpretation in which $v(p) = 0.5$ and $v(q) = 0$; then $v((p \wedge \neg p) \to q) = 0.5$, and so the above is invalid.

Recall that $B_1, \ldots, B_n \vDash_Ł A$ just in case, for the associated conditional, $\vDash_{Ł_\aleph} (B_1 \wedge \ldots \wedge B_n) \to A$. Hence, system Ł is paraconsistent, $p \wedge \neg p \nvDash_Ł q$. However, Ł is not relevant. We have, for example:

$$\vdash_{Ł_\aleph} (p \to p) \to (q \to (p \to p)) \qquad \text{(instance of (A15))}$$

And so (because $\vDash_L p \to p$):

$$\vDash_Ł q \to (p \to p)$$

L.4 *Modus Ponens in Ł and the Sorites Paradox*

We've seen that in $Ł_\aleph$:

$$\nvDash_{Ł_\aleph} (p \wedge (p \rightarrow q)) \rightarrow q$$

It then follows that, for Ł:

$$p, p \rightarrow q \nvDash_{Ł} q$$

And hence, *modus ponens* is straight-up invalid in Ł. This may seem undesirable but is essential for the fuzzy logic response to the sorites paradox.

Modus ponens is of course needed to arrive at the conclusion of a *sorites* argument. This is perhaps more evident if we use the "forced-march" formulation of the paradox:

1. If one grain of wheat does not make a heap, two grains of wheat do not make a heap.
2. If two grains of wheat do not make a heap, three grains of wheat do not make a heap.
3. If three gains of wheat do not make a heap, four grains of wheat do not make a heap.

   ⋮

9999. If 9999 grains of wheat do not make a heap, 10000 grains of wheat do not make a heap.
10000. One grain of wheat does not make a heap.
10001. Therefore, 10000 grains of wheat do not make a heap.

Here we have simply replaced the first premise of the old formulation with instances. The argumentation of the paradox then reduces to 9999 applications of *modus ponens*. If *modus ponens* is invalid, one may accept the truth of all of the premises and still deny the conclusion. Consider the list of antecedents and consequents, along with the truth-value to which one might assign them:

One grain of wheat does not make a heap. (1)

Two grains of wheat do not make a heap. (1)

Three grains of wheat do not make a heap. (0.95)

⋮

Twenty seven grains of wheat do not make a heap. (0.52)

Twenty eight grains of wheat do not make a heap. (0.48)

⋮

9999 grains of wheat do not make a heap. (0)

10000 grains of wheat do not make a heap. (0)

In Ł, the truth-value of $A \rightarrow B$ is basically inversely proportional to how far the truth-value of B "drops off" when comparing it to the truth-value of A. Throughout this list, there are many places where the truth-value drops off a little, but never very much, and so the truth-value of all the conditionals in the forced-march argument will be very close to 1. Hence, unless one is dealing with $Ł_1$ (= $Ł_\aleph$), or other really high value of ε, they will all have a designated value, but, again *modus ponens* will have to be rejected as only "valid-like" or "valid-esque" but not valid.

Homework

Priest, §11.10 (p. 239), problems 3 and 4.

L.5 *Fuzzy Relevant Logic*

The $\rightarrow$ of Ł has a number of features Priest dislikes:

$$A \vDash B \rightarrow A$$
$$\neg B \vDash B \rightarrow A$$
$$(A \wedge B) \rightarrow C \vDash (A \rightarrow C) \vee (B \rightarrow C)$$
$$(A \rightarrow B) \wedge (C \rightarrow D) \vDash (A \rightarrow D) \vee (C \rightarrow B)$$
$$\neg(A \rightarrow B) \vDash A$$

To remedy this, Priest discusses how to "fuzzify" relevant logic, describing a system FB ("fuzzy-B"), where the basic procedure could also be used to generate fuzzified versions of other relevant systems (e.g., FR for "fuzzy-R", etc.)

An *FB-interpretation* is a structure $\langle W, N, R, *, \nu \rangle$ (or also with a $\sqsubseteq$ if you prefer) where W is a set of worlds, R a ternary accessibility relation, as in B, N the subset of W which are normal worlds (as defined for B), and * the normal Routley star function, and ν is a function assigning a value in the range $[0, 1]$ to every propositional parameter p at each world $w \in W$.

The function ν is extended to cover complex wffs built with $\vee$, $\wedge$ and $\neg$ in just the way it is in Ł, except with the complication of possible worlds. For conditionals, we have:

$\nu_w(A \rightarrow B) =$ the greatest lower bound of all values $\nu_{w'}(A) \ominus \nu_{w''}(B)$ for those w', w'' such that $Rww'w''$.

Validity is defined as follows:

$\Delta \vDash_{\text{FB}} A$ iff for every *normal world* w of every interpretation v, the greatest lower bound of values $v_w(B)$ for the members B of Δ is less than or equal to $v_w(A)$.

For those interpretations that assign only the classical values 1 or 0, the value for a complex wff A in FB will match that of the corresponding B-interpretation. Hence, FB is a sub-logic of B, and therefore relevant.

Hence we have:

$$\text{if } \Delta \vDash_{\text{FB}} A \text{ then } \Delta \vDash_{\text{B}} A$$

We also have:

$$\text{if } \vDash_{\text{B}} A \text{ then } \vDash_{\text{FB}} A$$

This can be established as follows: the axioms of the deductive system for B are all FB-valid (which can be checked algebraically), and the inference rules preserve validity, and therefore the theorems of B are FB-valid as well. However, it is *not* the case that:

$$\text{if } \Delta \vDash_{\text{B}} A \text{ then } \Delta \vDash_{\text{FB}} A$$

where premises are involved. An example is fuzzy modus ponens:

$$p, p \rightarrow q \vDash_{\text{B}} q \text{ but } p, p \rightarrow q \nvDash_{\text{FB}} q$$

Modus ponens fails in FB for more or less the same reason as in Ł. Let $W = N = \{w\}$, let $w^* = w$, let $Rwww$, and let $v_w(p) = 0.9$ and $v_w(q) = 0.6$. Then $v_w(p \rightarrow q) = 0.7$ and so the greatest lower bound of $v_w(p)$ and $v_w(p \rightarrow q)$ is also 0.7, but this not less than or equal to $v_w(q)$.

Priest does not develop a tableaux or deductive system for FB, and notes that it is an open question whether or not one can be constructed.

We still have:

$$p \rightarrow s \vDash_{\text{FB}} (p \wedge q) \rightarrow s$$

$$p \rightarrow q \vDash_{\text{FB}} \neg q \rightarrow \neg p$$

$$(\text{although } p \rightarrow q, q \rightarrow s \nvDash_{\text{FB}} p \rightarrow s)$$

So to get what Priest calls "a theory of enthymematic fuzzy relevant conditionals" one would have to modify FB in the way B was modified to yield, e.g., C^+_{B}, to yield, C^+_{FB}. However, Priest leaves this to our imagination.

Finally, Priest adds an additional section on other fuzzy logics, i.e., those with $\mathcal{V}$ is the set of real numbers in the interval $[0, 1]$ but make use of different truth functions for $\vee$, $\wedge$, $\rightarrow$, etc., possibly also adding a stronger conjunction, $\circ$. While technically interesting, I didn't think there was anything sufficiently philosophically compelling about these systems to be worth our time.

[M] Classical First-Order Logic

We are finally in a position to segue from discussing propositional or sentential non-classical logic to those involving quantification, terms and predicates. It behooves us to begin by laying out classical first-order quantified logic by way of comparison. (We simplify slightly by skipping discussion of function terms.)

M.1 Syntax

A *constant* is the letter k, written with a numerical superscript, and thus a member of the list: $k^1, k^2, k^3, \ldots$.

Arbitrary constants will be referred to with the letters a, b, c, etc. Strictly speaking, these are metalinguistic variables *for* object-language constants, but we will usually ignore that fussiness.

A *variable* is the letter v, written with a numerical superscript, and thus a member of the list: $v^1, v^2, v^3, \ldots$.

Arbitrary variables will be referred to with the letters x, y, z, etc.

A *term* is a variable or constant.

Arbitrary terms will be referred to with the letters t, t_1, t_2, t', etc. These are not parts of the object language but are used to make claims about the object language which would remain true whether t is taken to be or replaced by a constant *or* a variable.

A *predicate* is the letter P written with a numerical subscript ≥ 1, and a numerical superscript, and thus a member of one of the series: $P^0_1, P^1_1, P^2_1, P^3_1, \ldots, P^0_2, P^1_2, P^2_2, P^3_2, \ldots, P^0_3, P^1_3, P^2_3, P^3_3, \ldots$.

The superscript is performing its usual function of ensuring an infinite supply.

The subscript, however, indicates how many terms the predicate is to be applied to. This is typically left off when it is clear from context what it must be. E.g., in $\forall x(P^2 x \supset P^2 x)$, clearly the subscript on P^2 is 1. Arbitrary predicates are referred to be the letters F, G, P, Q, S, etc. (Priest uses only P, Q, S, but the use of F, G is too deeply ingrained in me to forgo.)

A *connective* is one of the signs $\wedge$, $\vee$, $\supset$, $\equiv$, or $\neg$.

A *quantifier* is one of the signs $\forall$ or $\exists$.

$\forall$ is the *universal* quantifier.

Priest does not call $\exists$ the existential quantifier, but rather the *particular* quantifier. (The reason for this is explained in a later chapter.)

An *atomic formula* is a string of the form $P_n t_1 \ldots t_n$, where P_n is a predicate with subscript n and $t_1, \ldots, t_n$ are terms.

A *well-formed formula* (or *wff*) is defined recursively:

a) Atomic formulæ are wffs.
b) If A is a wff, so is $\neg A$.
c) If A and B are wffs, then so is $(A \wedge B)$.
d) If A and B are wffs, then so is $(A \vee B)$.
e) If A and B are wffs, then so is $(A \supset B)$.
f) If A and B are wffs, then so is $(A \equiv B)$.
g) If A is a wff and x is a variable, then $\forall x\, A$ is a wff.
h) If A is a wff and x is a variable, then $\exists x\, A$ is a wff.

Just as above, A and B are used for arbitrary wffs; Δ, Γ and Σ are used for sets thereof.

An occurrence of a variable x in a wff A is said to be *bound* if it occurs within a sub-wff of A of the form $\forall x \ldots x \ldots$ or $\exists x \ldots x \ldots$ (or A itself is of that form). Otherwise, the occurrence of x is said to be *free*.

A wff A is said to be *closed* iff it contains no free variables; otherwise it is *open*.

E.g. "Fx" is open; "$\exists x\, Pxa$" is closed.

Priest writes $A_x(t)$ for the wff that results when term t replaces all *free* occurrences of x in A.

E.g., if A is "$(Fx \vee \neg Fx) \to \forall x\, Fx$", then $A_x(a)$ is "$(Fa \vee \neg Fa) \to \forall x\, Fx$".

M.2 Semantics

A *classical interpretation* is an ordered pair $\langle D, \nu \rangle$, where:

- D is a non-empty set, called *the domain of quantification*; this is what the interpretation interprets the quantifiers as "ranging over".
- ν is a function such that $\nu(c) \in D$ for each constant c in the language, and $\nu(P_n) \subseteq D^n$ where P_n is a predicate and n is its subscript.

So an interpretation provides an "interpretation" of the meaning of each constant and predicate: some member of the domain, in case of a constant, and some extension (i.e., set of ordered n-tuples) for each n-place predicate.

The evaluation function ν is extended so that it assigns a classical truth-value 1 or 0 to every *closed* wff as follows. First, we either assume, or extend the language, so that every member d of D has a constant that represents it, k_d; thus, in the metalanguage, we write $\nu(k_d) = d$. Then ν is extended recursively as follows:

a) $\nu(P_n c_1 \ldots c_n) = \begin{cases} 1, & \text{if } \langle \nu(c_1), \ldots, \nu(c_n) \rangle \in \nu(P_n); \\ 0, & \text{otherwise.} \end{cases}$

b) $\nu(A \wedge B) = \begin{cases} 1, & \text{if } \nu(A) = \nu(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$

c) $\nu(A \vee B) = \begin{cases} 1, & \text{if } \nu(A) = 1 \text{ or } \nu(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$

d) $\nu(A \supset B) = \begin{cases} 1, & \text{if } \nu(A) = 0 \text{ or } \nu(B) = 1; \\ 0, & \text{otherwise.} \end{cases}$

e) $\nu(A \equiv B) = \begin{cases} 1, & \text{if } \nu(A) = \nu(B); \\ 0, & \text{otherwise.} \end{cases}$

f) $\nu(\forall x\, A) = \begin{cases} 1, & \text{if for all } d \in D, \nu(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}$

g) $\nu(\exists x\, A) = \begin{cases} 1, & \text{if for some } d \in D, \nu(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}$

If Δ is a set of closed wffs, and A is a closed wff, then $\Delta \vDash A$ iff for every interpretation $\langle D, \nu \rangle$, if $\nu(B) = 1$ for every member $B \in \Delta$, then $\nu(A) = 1$ as well.

M.3 Classical tableaux

The tableaux method is the same as for classical propositional logic, including all the same rules for the connectives. The rules for the quantifiers are fairly easy to guess if one compares them to the rules for modal operators:

$$\begin{array}{cccc} \neg \forall x\, A & \neg \exists x\, A & \exists x\, A & \forall x\, A \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \exists x \neg A & \forall x \neg A & A_x(c)\ (\text{new } c) & A_x(a)\ (\text{old } a\text{'s}) \end{array}$$

The final rule there, for $\forall$-statements, must be applied to *every* constant on the branch, and must be reapplied whenever a new constant is introduced. It also must be applied at least once, so if there is no constant on the branch, we choose one.

It is therefore wise to delay applying the rule for $\forall$-statements

Hence, tableaux may be infinite, and the method does *not* represent a decision procedure. Indeed, first-order predicate logic has been proven to be undecidable by Church, as a corollary of Gödel's incompleteness theorems.

Let's see some examples.

$$\underline{\exists x\, Lxx, \forall x(\exists y\, Lxy \supset \forall y\, Lyx) \vdash \forall x\, \forall y\, Lxy}$$

$$\begin{array}{c}
\checkmark\ \exists x\, Lxx \\
\checkmark\ \forall x(\exists y\, Lxy \supset \forall y\, Lyx) \\
\checkmark\ \neg\, \forall x\, \forall y\, Lxy \\
\checkmark\ \exists x \neg\, \forall y\, Lxy \\
Laa \\
\checkmark\ \neg\, \forall y\, Lby \\
\checkmark\ \exists y \neg\, Lby \\
\neg\, Lbc \\
\checkmark\ \exists y\, Lay \supset \forall y\, Lya \\
\exists y\, Lby \supset \forall y\, Lyb \\
\checkmark\ \exists y\, Lcy \supset \forall y\, Lyc \\
\begin{array}{cc}
\begin{array}{c} \checkmark\ \neg\, \exists y\, Lay \\ \checkmark\ \forall y \neg\, Lay \\ \neg\, Laa \\ \times \end{array} &
\begin{array}{c} \checkmark\ \forall y\, Lya \\ Lba \\ Lca \\
\begin{array}{cc}
\begin{array}{c} \checkmark\ \neg\, \exists y\, Lcy \\ \checkmark\ \forall y \neg\, Lcy \\ \neg\, Lca \\ \times \end{array} &
\begin{array}{c} \checkmark\ \forall y\, Lyc \\ Lbc \\ \times \end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$

$$\underline{\forall x(Fx \supset Gx), \forall x(Fx \supset Hx) \nvdash \forall x(Gx \supset Hx)}$$

$$\begin{array}{c}
\checkmark\ \forall x(Fx \supset Gx) \\
\checkmark\ \forall x(Fx \supset Hx) \\
\checkmark\ \neg\, \forall x(Gx \supset Hx) \\
\checkmark\ \exists x \neg(Gx \supset Hx) \\
\checkmark\ \neg(Ga \supset Ha) \\
Ga \\
\neg\, Ha \\
\checkmark Fa \supset Ga \\
\checkmark Fa \supset Ha \\
\begin{array}{cc}
\begin{array}{c} \neg\, Fa \\ \begin{array}{cc} \neg\, Fa & Ga \end{array} \end{array} &
\begin{array}{c} Ha \\ \times \end{array}
\end{array}
\end{array}$$

Counter-models are read off open branches as you might expect. Let D be the set of constants on the branch. Interpret each constant as standing for itself, and include $\langle c_1, \ldots, c_n \rangle \in \nu(P)$ for predicate P just in case $Pc_1 \ldots c_n$ occurs on the branch.

In this case, close inspection will reveal that the "two" open branches are in fact identical. They describe a model where $D = \nu(G) = \{`a'\}$ and $\nu(a) = `a'$ and $\nu(F) = \nu(H) = \varnothing$. Nothing is F, so trivially, all Fs are G, and all Fs are H. But the one thing in the model is G but not H, so not all things are H.

If you don't like the idea of constants standing for themselves, you can use numbers, or anything else instead. Priest just makes up objects $\partial_a, \partial_b, \partial_c$, etc., as the default assignments to the constants, but tells us nothing more about them.

M.4 Identity Logic

To capture identity logic, we simply designate a dyadic predicate—P^0_2 will do—and force it to be interpreted as the identity relation. Hence $t_1 = t_2$ is taken as an alternative notation to Pt_1t_2 and $t_1 \neq t_2$ an alternative to $\neg\, Pt_1t_2$.

An interpretation $\langle D, \nu \rangle$ is *normal* iff $\nu(P^0_2) = \{\langle d, d \rangle : d \in D\}$, i.e., P^0_2 has as its extension in the interpretation the set of ordered pairs of a thing and itself.

In identity logic, we simply define validity as truth-preservation over normal interpretations. Clearly, identity logic is an extension of first-order logic generally.

Tableaux for identity logic use two new rules, one for reflexivity of identity, another for substitution of identicals:

$$\begin{array}{c} . \\ \downarrow \\ a = a \end{array} \qquad \begin{array}{c} a = b \\ A_x(a) \\ \downarrow \\ A_x(b) \end{array}$$

The first rule is applied for every constant a already on the branch. The second only needs to be applied when $A_x(a)$ is an atomic wff. Notice that $A_x(a)$ does not contain x in it, and does not need to have come from something of the form $A_x(x)$. This is really just a standard substitution rule, and must be applied to every atomic wff containing a for every identity of the form $a = b$. Notice that $a = b$ is itself an atomic wff, which gives us reflexivity and symmetry of identity, as in:

$$\underline{a = b, b = c \vdash a = c}$$

$$\begin{array}{c} a = b \\ b = c \\ a \neq c \\ a = c \\ \times \end{array}$$

Here $a = c$ comes from substituting c for b in the first line. I.e., $a = b$ is $(a = x)_x(b)$, and we have $b = c$ so we get $(a = x)_x(c)$, i.e., $a = c$.

Here are some issues that concern Priest:

1. We simply assume that the domain D is non-empty in any interpretation, and thus we have, e.g., $\vDash \exists x(Fx \vee \neg Fx)$. Especially with "$\exists x$" read as "there exists an x such that ..." this does not seem like a *logical truth*, as it seems logically possible that there be nothing.

2. Every constant is taken as having a denotation, which differs apparently from ordinary language. If we were to transcribe "Pegasus is a winged horse" as "$Hp \wedge Wp$", and took this to be true, we could validly infer "$\exists x(Hx \wedge Wx)$", which seems false, at least if "$\exists x$" is read as "there exists ...".

3. If, in response, the classical logician insists on forgoing non-referring names, how then to treat such apparent truths as these?

   (a) Sherlock Holmes lives on Baker Street.
   (b) Sherlock Holmes is a fictional character.
   (c) I am thinking about Sherlock Holmes.

4. In classical identity logic, codenoting constants can replace each other in any sentence validly. But this seems untrue in such cases as the expressions of belief or desire, or when modal operators are used (so-called "intensional" contexts):

   (1) Arthur knows Mark Twain is an author.
   (2) Mark Twain = Samuel Clemens
   (3) Therefore, Arthur knows Samuel Clemens is an author.

   (1) Necessarily, 8 > 7.
   (2) 8 = the number of planets
   (3) Therefore, necessarily, the number of planets > 7.

Unfortunately, Priest doesn't even mention the usual sorts of responses made to such issues by proponents of classical logic. Frege, for example, was one of the first to bring attention to the failures of substitutivity of identicals, and explained it in terms of his theory of *sense* and *reference*. The *reference* of a name is its denotation, the object it stands for. The sense is the method of presenting that reference, so "Mark Twain" and "Samuel Clemens" likely have different senses but the same reference.

Frege claimed that in certain contexts such as following "knows that ..." or "believes that ...", words "shift" from having their usual reference to having the usual sense as their reference instead. Thus, "Mark Twain" does not refer to the same thing in the two premises of the argument. In effect the argument actually has the form:

$$A_x(a'), a = b \therefore A_x(b')$$

... and this form is classically invalid.

Russell had a rather different response to such worries, which involved denying that most ordinary language proper names such as "Mark Twain", "Pegasus" or "Sherlock Holmes" ought to be transcribed into logical notation as simple constants. Instead, he claimed that they were truncated descriptions of some sort, and gave a quantificational analysis of descriptive phrases, where "the F is G" would be written:

$$\exists x(\forall y(Fy \equiv y = x) \wedge Gx)$$

In which case the form of the identity arguments is, again, not as clearcut as it seems. For example, the modal identity argument contains a scope ambiguity. Suppose we put the premises as:

(1) $\Box(8 > 7)$
(2) $\exists x(\forall y(Ny \equiv y = x) \wedge 8 = x)$

Then, what is the conclusion? Is it this?

(3) $\Box\exists x(\forall y(Ny \equiv y = x) \wedge x > 7)$

Or is it this?

(3′) $\exists x(\forall y(Ny \equiv y = x) \wedge \Box(x > 7))$

Neither is a simple application of substitution of identicals, but only (3′) (the *de re* reading) follows at all from (1) and (2). The *de dicto* reading, (3), does not, and it is the interpretation that seems problematic.

Whether or not these approaches can handle other claims involving non-referring names is less clear, especially those such as (b) and (c) of Priest's Holmes examples are arguably more problematic.

Perhaps we'll pick these issues up again if we have time. Let's look instead at a very different approach to such so-called "difficulties".

[N] Free Logic

Free logics are those that drop either or both the assumptions that the domain of quantification must be nonempty and that every closed term must have a denotation. You may have already studied a kind of "free description logic" in Hardegree's Intermediate logic course.

Similar to picking P^0_2 for $=$ in identity logic, we pick a monadic predicate—P^0_1 will do—and rewrite it as $\mathfrak{E}$. "$\mathfrak{E}a$" will then be interpreted to mean "a exists". Other common notations here are E!a or $\exists!a$.

A *free logic interpretation* is a triple $\langle D, E, \nu\rangle$, where:

- D is a non-empty set, the domain of discourse;
- E is a (possibly empty) subset of D, the existent objects; E is also called the "inner domain" or "domain of quantification";
- ν is a function assigning a member of D to each constant, and a subset of D^n for every n-place predicate, and $\nu(\mathfrak{E}) = E$.

ν is extended to cover complex wffs identically as in classical logic, except for the quantifiers, which are interpreted to range only over E:

$$\nu(\forall x\, A) = \begin{cases} 1, & \text{if for all } d \in E,\ \nu(A_x(k_d)) = 1. \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu(\exists x\, A) = \begin{cases} 1, & \text{if for some } d \in E,\ \nu(A_x(k_d)) = 1. \\ 0, & \text{otherwise.} \end{cases}$$

Validity is defined as truth preservation in all interpretations, as you would expect.

Tableaux can be done by modifying the classical rules for the quantifiers as follows:

$$\begin{array}{c} \forall x\, A \\ \begin{array}{cc} \neg\,\mathfrak{E}a & A_x(a) \end{array} \end{array} \qquad \begin{array}{c} \exists x\, A \\ \downarrow \\ \mathfrak{E}a \text{ (new } c) \\ A_x(c) \end{array}$$

The first rule must be applied to every constant on the branch. We still must apply it at least once.

Some examples:

$$\underline{\forall x(Fx \supset Gx), \exists x\, Fx \vdash \exists x\, Gx}$$

$$\begin{array}{c} \checkmark\ \forall x(Fx \supset Gx) \\ \checkmark\ \exists x\, Fx \\ \checkmark\ \neg\,\exists x\, Gx \\ \mathfrak{E}a \\ Fa \\ \checkmark\ \forall x\, \neg\, Gx \\ \begin{array}{cc} \begin{array}{c} \neg\,\mathfrak{E}a \\ \times \end{array} & \begin{array}{c} \neg\, Ga \\ \begin{array}{cc} \begin{array}{c} \neg\,\mathfrak{E}a \\ \times \end{array} & \begin{array}{c} \checkmark Fa \supset Ga \\ \begin{array}{cc} \begin{array}{c} \neg\, Fa \\ \times \end{array} & \begin{array}{c} Ga \\ \times \end{array} \end{array} \end{array} \end{array} \end{array} \end{array} \end{array}$$

$$\underline{\nvDash \exists x(Fx \vee \neg\, Fx)}$$

$$\begin{array}{c} \checkmark\ \neg\,\exists x(Fx \vee \neg\, Fx) \\ \checkmark\ \forall x\, \neg(Fx \vee \neg\, Fx) \\ \begin{array}{cc} \neg\,\mathfrak{E}a & \begin{array}{c} \checkmark\ \neg(Fa \vee \neg\, Fa) \\ \neg\, Fa \\ \neg\,\neg\, Fa \\ \times \end{array} \end{array} \end{array}$$

And to "solve" the Pegasus problem:

$$\underline{Wp \wedge Hp \nvDash \exists x(Wx \wedge Hx)}$$

$$\begin{array}{c} \checkmark Wp \wedge Hp \\ \checkmark\ \neg\,\exists x(Wx \wedge Hx) \\ Wp \\ Hp \\ \checkmark\ \forall x\, \neg(Wx \wedge Hx) \\ \begin{array}{cc} \neg\,\mathfrak{E}p & \begin{array}{c} \checkmark\ \neg(Wp \wedge Hp) \\ \begin{array}{cc} \begin{array}{c} \neg\, Wp \\ \times \end{array} & \begin{array}{c} \neg\, Hp \\ \times \end{array} \end{array} \end{array} \end{array} \end{array}$$

Identity logic can be added on top of free logic in just the same way as in classical logic.

Priest also discusses some variant systems, such as those that deny that nonexistent objects can have any "positive" properties, i.e., forcing $\nu(P_n) \subseteq E^n$ rather than $\nu(P_n) \subseteq D^n$, for every predicate letter. To get a system in which statements involving nonexistent objects are *neither true nor false*, however, we'd have to use a gappy base system instead (K_3 or similar). The treatment of identity becomes complicated if we deny "positive" properties to nonexistent objects, as even "$a = a$" would be false.

Another option Priest considers is a system with two sets of quantifiers, inner and outer. The outer quantifiers $\exists x$ and $\forall x$ would range over all of D, but $\exists^E x$ and $\forall^E x$ would range only over existent quantifiers. Then $\exists x$ could not be read "there exists..." but rather "for some x" or similar. This would allow one to interpret, e.g., "there is someone Tut worshipped—viz., Ra," without being held to say that Ra exists. (With outer quantifiers, the inner one could be defined, i.e., $\exists^E x A$ for $\exists x(\mathfrak{E}x \wedge A)$; technically the system simply is classical logic but with a certain predicate receiving a fixed interpretation.)

Since a classical interpretation may be defined as a free interpretation where $E = D$, classical logic is a proper ex-

tension of free logic. However, if $\Delta \vDash A$ in classical logic, and Σ is the set of premises of the form $\mathfrak{E}c$ for constants c in the original argument, then $\Sigma \cup \Delta \vDash A$ in free logic.

Homework

Priest, §13.10 (pp. 305–6), problem 2, any five of (b)–(i). (I did (a) above.)

[O] Quantified Modal Logic

O.1 *Introduction and the Barcan Formulæ*

One of the complications that arises when formulating quantified versions logical systems involving possible worlds semantics—modal logic most obviously—has to do with whether or not one regards the same objects as existing, or as being quantified over (which may or may not be the same), at all worlds, of whether one regards the domains as varying from world to world. On the first approach one gets *Constant Domain* (e.g.) *Modal Logic*; on the other you get *Variable Domain Modal Logic*.

We precede a propositional modal logic's name with either C or V to get the name of the corresponding quantified version, e.g., CK for "constant domain K" or VS5 for "variable domain S5", and so on.

Historically, this debate originated over whether or not one should accept as valid these schemata:

$$\forall x \Box A \supset \Box \forall x A \qquad \text{(Barcan Formula)}$$
$$\Box \forall x A \supset \forall x \Box A \qquad \text{(Converse Barcan Formula)}$$

The first seems false if the domain can increase going from one world to its accessible worlds. Everything actual may be necessarily A, but there may be other things at other worlds which are not necessarily A. If the domain can shrink going from the actual world to its accessible worlds, the second seems false.

We will look briefly at both options.

O.2 *Constant Domain Modal Logic*

For the syntax we simply add the operators $\Box$ and $\Diamond$ to the syntax of first-order logic (free or otherwise, with identity or not—usually with identity).

A CK-interpretation is a quadruple $\langle D, W, R, \nu \rangle$, where:

- D is the domain of quantification, the same at all worlds;
- W is a non-empty set of worlds;
- R is a binary accessibility relation on W (technically $R \subseteq W^2$).
- ν is a function that:
  1. Assigns to each constant c a member $d \in D$; we write this $\nu(c) = d$.
  2. Assigns to each predicate P_n an extension which is a set of ordered n-tuples from D at each world; we write this as $\nu_w(P)$. (Technically $\nu_w(P_n) \subseteq D^n$ and may be a different subset at different worlds as predicates vary their extension world by world).

When discussing metatheory, we again extend the set of constants to include one, k_d for every $d \in D$. We extend ν to assign truth-values 1 and 0 to closed formulæ at worlds. For atomics:

$$\nu_w(P_n c_1 \ldots c_n) = \begin{cases} 1, & \text{if } \langle \nu_w(c_1), \ldots, \nu_w(c_n) \rangle \in \nu_w(P_n) \\ 0, & \text{otherwise.} \end{cases}$$

We extend ν according to the rules for propositional K for $\wedge, \vee, \neg, \supset, \equiv, \Box$ and $\Diamond$, and for the quantifiers, we have just what you would expect:

$$\nu_w(\forall x A) = \begin{cases} 1, & \text{if for all } d \in D, \nu_w(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}$$
$$\nu_w(\exists x A) = \begin{cases} 1, & \text{if for some } d \in D, \nu_w(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Tableaux are done exactly as you would expect, just like classical first-order logic, except with world numbers, and adding the K rules for $\Box$ and $\Diamond$. $\neg\Box$ and $\neg\Diamond$ change to $\Diamond\neg$ and $\Box\neg$ respectively, $\Diamond A$ gives you A at a new accessible world, $\Box A$ gives you A at every accessible world, etc. For each extension of K, the corresponding extension of CK can be formulated with the appropriate restriction on R, and by adopting the corresponding rule for tableaux.

Here is an instance of the Barcan Formula (the point generalizes to other instances):

$$\underline{\vdash_{\text{CK}} \forall x \Box Fx \supset \Box \forall x Fx}$$

$$
\begin{array}{c}
\checkmark \neg(\forall x \Box Fx \supset \Box \forall x Fx), 0 \\
\checkmark \forall x \Box Fx, 0 \\
\checkmark \neg \Box \forall x Fx, 0 \\
\checkmark \Diamond \neg \forall x Fx, 0 \\
0\, r\, 1 \\
\checkmark \neg \forall x Fx, 1 \\
\checkmark \exists x \neg Fx, 1 \\
\checkmark \neg Fa, 1 \\
\checkmark \Box Fa, 0 \\
Fa, 1 \\
\times
\end{array}
$$

Here is an invalid example:

$$
\begin{array}{c}
\Box \exists x Fx \nvdash_{\text{CK}} \exists x \Box Fx \\
\hline
\checkmark \Box \exists x Fx, 0 \\
\checkmark \neg \exists x \Box Fx, 0 \\
\checkmark \forall x \neg \Box Fx, 0 \\
\checkmark \neg \Box Fa, 0 \\
\checkmark \Diamond \neg Fa, 0 \\
0\, r\, 1 \\
\neg Fa, 1 \\
\checkmark \exists x Fx, 1 \\
Fb, 1 \\
\checkmark \neg \Box Fb, 0 \\
\checkmark \Diamond \neg Fb, 0 \\
0\, r\, 2 \\
\neg Fb, 2 \\
\checkmark \exists x Fx, 2 \\
Fc, 2 \\
\checkmark \neg \Box Fc, 0 \\
\checkmark \Diamond \neg Fc, 0 \\
0\, r\, 3 \\
\neg Fc, 3 \\
\checkmark \exists x Fx, 3 \\
Fd, 3 \\
\vdots
\end{array}
$$

The tableaux goes on forever, but it is not hard to see how to construct a counter-model, even a finite one (in this case). Let $W = \{w_1, w_2\}, D = \{d_1, d_2\}, v_{w_1}(F) = \{d_1\}, v_{w_2}(F) = \{d_2\}, R = \{\langle w_1, w_2\rangle, \langle w_2, w_1\rangle\}$. At every world, something is F, so this is necessary, but no particular object is F at all worlds.

O.3 *Variable Domain Modal Logic*

For variable domain systems, we need to use a free logic base, so that the extension of $\mathfrak{E}$ (existence) can vary from world to world. In fact, that's pretty much the only change we need to make.

A *VK interpretation* is a quadruple $\langle D, W, R, v\rangle$, i.e., a domain, set of worlds, an accessibility and an evaluation function, with a new role assigned to v, i.e., a value, written $v(w)$ or D_w for every $w \in W$, a subset of D which is the domain of quantification at that world, which is also identified with $v_w(\mathfrak{E})$.

v is extended to cover complex wffs just as in the constant domain case, except:

$$
v_w(\forall x A) = \begin{cases} 1, & \text{if for all } d \in D_w,\ v_w(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}
$$

$$
v_w(\exists x A) = \begin{cases} 1, & \text{if for some } d \in D_w,\ v_w(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}
$$

Validity is defined as one would expect as truth preservation at all worlds in all interpretations.

Tableaux are just like constant domain tableaux, except with the free logic rules for the quantifiers instead of the classical logic rules.

Here we show the invalidity of an instance of the Barcan formula:

$$
\begin{array}{c}
\nvdash_{\text{VK}} \forall x \Box Fx \supset \Box \forall x Fx \\
\hline
\checkmark \neg(\forall x \Box Fx \supset \Box \forall x Fx), 0 \\
\checkmark \forall x \Box Fx, 0 \\
\checkmark \neg \Box \forall x Fx, 0 \\
\checkmark \Diamond \neg \forall x Fx, 0 \\
0\, r\, 1 \\
\checkmark \neg \forall x Fx, 1 \\
\checkmark \exists x \neg Fx, 1 \\
\mathfrak{E}a, 1 \\
\neg Fa, 1 \\
\begin{array}{cc}
\neg \mathfrak{E}a, 0 & \checkmark \Box Fa, 0 \\
 & Fa, 1 \\
 & \times
\end{array}
\end{array}
$$

The counter-model this open branch describes is a little funny. The world w_0 is empty; nothing exists there. So *trivially*, everything there is necessarily F. But there is a world, w_1 containing an entity a, which is not F, so it's not necessarily true that everything is F.

Extensions of VK are developed precisely as you would expect, by placing constraints on the accessibility relation R. Thus you get VD, VT, VB, VK4, VS4 and VS5, etc.

In S5, one can just leave off accessibility lines, as with its propositional cousin.

$$\underline{\forall x \Box \mathfrak{E}x, \exists x \Diamond \Diamond Fx \vdash_{VS5} \Box \Diamond \exists x Fx}$$

$\checkmark\ \forall x \Box \mathfrak{E}x, 0$
$\checkmark\ \exists x \Diamond \Diamond Fx, 0$
$\checkmark\ \neg \Box \Diamond \exists x Fx, 0$
$\checkmark\ \Diamond \neg \Diamond \exists x Fx, 0$
$\mathfrak{E}a, 0$
$\checkmark\ \Diamond \Diamond Fa, 0$
$\checkmark\ \Diamond Fa, 1$
$Fa, 2$

$\neg \mathfrak{E}a, 0$	$\checkmark\ \Box \mathfrak{E}a, 0$
$\times$	$\checkmark\ \neg \Diamond \exists x Fx, 3$
	$\checkmark\ \Box \neg \exists x Fx, 3$
	$\checkmark\ \neg \exists x Fx, 2$
	$\checkmark\ \forall x \neg Fx, 2$

$\neg \mathfrak{E}a, 2$	$\neg Fa, 2$
$\mathfrak{E}a, 2$	$\times$
$\times$	

Besides the usual kinds of extensions obtained by placing restrictions on accessibility by itself, there are also extensions that place restrictions on the domains at various worlds. E.g., one possibility would be to accept, e.g.:

$$\text{if } wRw' \text{ then } D_w \subseteq D_{w'}$$

So the domain is always increasing over accessibility. This might make sense, e.g., for a tense logic where it is taken to mean that nothing goes out of existence (though its *state* may change over time), or something like that.

One may make radical suggestions such as that no two words ever overlap, i.e., there is no such thing as *trans-world identity*. Pursuing this route usually means adopting something like Lewis's Counterpart Theory in its place, or else one ends up with counterintuitive results, such as that all my properties are necessary. In counterpart theory, what determines what properties are possible or necessary of me is what properties my "counterparts" (entities relevantly similar to me) have at other worlds.

All the issues about positive and negative properties that we saw with free logic also arise here, as well the choice between inner and outer quantifiers. Modal logics with outer quantifiers (quantifying over all things at all worlds) often go long with views that make variable domain modal logic unnecessary, at least as a true rival to constant domain modal logic.

Homework

Priest, §15.12 (p. 347), problem 2 (any one of (a)–(c)), and problem 3 (any two of (a)–(h)).

O.4 *Necessary Identity in Modal Logic*

Syntactically and semantically we could add identity to quantified modal logic just as we added it to classical logic. Simply set aisde a predicate = for identity and define validity as truth preservation over those models such that $\nu_w(=) = \{\langle d, d\rangle : d \in D\}$ for all $w \in W$.

One potential limitation of this way of doing things is that we have defined interpretations so that the interpretation of constants is constant across worlds, i.e., we have $\nu(c) = d$ not $\nu_w(c) = d$. This makes all constants what Kripke calls *rigid designators*.

A *designator* is an expression that refers to (or purports to refer to) an individual.

A *rigid designator* is an expression that refers to the same individual at all worlds (or, at all worlds where it refers at all).

A *nonrigid designator* is an expression that refers to different individuals at different individual worlds.

Kripke argued that proper names, such as "Socrates" or "Mitt Romney" are rigid designators, while descriptions, e.g., "the inventor of bifocals" or "the winner of the 2012 election" are non-rigid.

Because our constants are *rigid*, an identity statement between them is necessarily true if true at all. But it seems we do sometimes want contingent identities. To allow for this, we would need to add yet another kind of term to the language.

A descriptor is the Greek letter α written with a numerical superscript: $\alpha^0, \alpha^1, \alpha^2, \ldots$.

I use $\alpha, \beta, \gamma, \ldots$ for arbitrary descriptors.

Syntactically, these are just like constants, and may be used wherever a constant may be used.

Semantically, the difference is that the evaluation function ν in an interpretation will assign it a member of D only relative to a world, so we may have $\nu_w(\alpha) = d$ and $\nu_{w'}(\alpha) = d'$ where $d \neq d'$.

When we state the truth conditions for quantification we may assume there is a constant k_d for every $d \in W$. Thus, every descriptor's semantic value will be the same as some constant's, at every world, and we need not revise the truth conditions for the quantifiers.

The tableaux quantifier rules apply to constants, but there are new tableaux for identity and descriptors as follows. Here t and u may be either constants or descriptors, a, b and c must be constants, and α must be a descriptor:

$$\begin{array}{cccc}
\begin{array}{c} . \\ \downarrow \\ t = t, n \end{array} &
\begin{array}{c} t = u, n \\ A_x(t), n \\ \downarrow \\ A_x(u), n \end{array} &
\begin{array}{c} a = b, n \\ \downarrow \\ a = b, m \end{array} &
\begin{array}{c} . \\ \downarrow \\ c = \alpha, n \\ \text{(new } c\text{)} \end{array}
\end{array}$$

The second rule is applied to atomic formulæ only. The last is applied to every descriptor on the branch, and separately at every world where one does not already have something of this form.

These may be added to either VK or CK or their extensions. The results are written VK(NI) or CK(NI). The "NI" is for "necessary identity", since, for constants (and variables as we shall see), identity is necessary if true.

$$\begin{array}{c}
\vdash_{\mathrm{CK(NI)}} \forall x\, \forall y(x = y \supset \Box x = y) \\ \hline
\checkmark \neg \forall x\, \forall y(x = y \supset \Box x = y), 0 \\
\checkmark \exists x \neg \forall y(x = y \supset \Box x = y), 0 \\
\checkmark \neg \forall y(a = y \supset \Box a = y), 0 \\
\checkmark \exists x \neg (a = y \supset \Box a = y), 0 \\
\checkmark \neg (a = b \supset \Box a = b), 0 \\
a = b, 0 \\
\checkmark \neg \Box a = b, 0 \\
\checkmark \Diamond a \neq b, 0 \\
0 r 1 \\
a \neq b, 1 \\
a = b, 1 \\
\times
\end{array}$$

$$\begin{array}{c}
\nvdash_{\mathrm{CK(NI)}} \forall x(x = \alpha \supset \Box x = \alpha) \\ \hline
\checkmark \neg \forall x(x = \alpha \supset \Box x = \alpha), 0 \\
\checkmark \exists x \neg (x = \alpha \supset \Box x = \alpha), 0 \\
\checkmark \neg (a = \alpha \supset \Box a = \alpha), 0 \\
a = \alpha, 0 \\
\checkmark \neg \Box a = \alpha, 0 \\
\checkmark \Diamond a \neq \alpha, 0 \\
0 r 1 \\
a \neq \alpha, 1 \\
b = \alpha, 1 \\
\alpha = \alpha, 0 \\
\alpha = \alpha, 1 \\
a = a, 0 \\
a = a, 1 \\
b = b, 0 \\
b = b, 1 \\
\alpha = a, 0 \\
\alpha = b, 1
\end{array}$$

To deal with the "number of planets" problem, we would treat it as a descriptor rather than a constant.

O.5 *Contingent Identity*

Still, one might object to all identities being necessary even for constants/rigid designators. This is perhaps more salient with interpretations of $\Box$ and $\Diamond$ other than metaphysical necessity and possibility. If we take $\Box$ as epistemic necessity, for example, it seems that $a = b$ is possible even when $\neg \Box a = b$. To capture systems that allow for contingent identity, even between constants, we need to complicate the semantics as follows.

Let an interpretation be a structure $\langle D, H, W, R, \nu, \rangle$, where W and R are the usual set of worlds and accessibility relation, D is a domain, H is a set of *avatars* of D (these may be, e.g., modal or temporal "parts" of members of D), so that the members d of D determine a function $|d|_w$ such that $|d|_w \in H$. For each constant $\nu(c) \in D$, for each descriptor α, $\nu_w(\alpha) \in D$, but for each predicate $\nu_w(P_n) \subseteq H^n$, and for atomic statements, we specify:

$$\nu_w(P_n t_1 \dots t_n) = \begin{cases} 1, & \text{if } \langle |\nu_w(t_1)|_w, \dots, |\nu_w(t_1)|_w \rangle \in \nu_w(P_n); \\ 0, & \text{otherwise.} \end{cases}$$

(When t is a constant, $\nu_w(t) = \nu(c)$.)

Tableaux are like the previous systems except dropping the rule taking you from $a = b, n$ to $a = b, m$. Obviously, then the previous tableaux showing $\nvdash_{\mathrm{CK(NI)}} \forall x\, \forall y(x = y \supset \Box x = y)$ will no longer work. Priest calls these systems CK(CI) or VK(CI) and similarly for extensions.

There is still a difference between descriptors and constants as evinced by these two tableaux:

$$\begin{array}{c|c}
\begin{array}{c}
\checkmark \Box Pc, 0 \\
\checkmark \neg \exists x \Box Px, 0 \\
\checkmark \forall x \neg \Box Px, 0 \\
\neg \Box Pc, 0 \\
\times \\
\\ \\ \\ \\ \\ \\ \\
\end{array} &
\begin{array}{c}
\checkmark \Box P\alpha, 0 \\
\checkmark \neg \exists x \Box Px, 0 \\
\checkmark \forall x \neg \Box Px, 0 \\
c = \alpha, 0 \\
\neg \Box Pc \\
\Diamond \neg Pc \\
0 r 1 \\
\neg Pc, 1 \\
b = \alpha, 1 \\
\neg \Box Pb, 0 \\
\vdots
\end{array}
\end{array}$$

O.6 *Quantified Non-normal Modal Logics*

Generating quantified non-normal modal logics is, for the most part, simply a matter of modifying non-normal modal logics such as N, L, etc., to yield, e.g., CN, VN, CL, VL, etc., in precisely the same way we modified K to get CK and VK, etc. Indeed for N-based systems that's pretty much all there is to it. We simply add a set $N \subseteq W$ of normal worlds, and at worlds in $W - N$, we make $v_w(\Box A) = 0$ and $v_w(\Diamond A) = 1$ for all closed wffs A.

Matters are a tad different for L-based systems, as we do not want the truth-values $v_w(\Box A)$ and $v_w(\Diamond A)$ to be completely lawless. In particular, we want $v_w(\Box A_x(a)) = v_w(\Box A_x(b))$ when $v(a) = v(b)$ to preserve substitutivity of identicals in all contexts. To allow for this, rather than having interpretations directly make modal wffs true or false, we define the notion of a matrix. Let A be a closed wff; then let the matrix $A_{x_1,\ldots,x_n}(x_1,\ldots,x_n)$ be formed by substituting variables $x_1, x_2, \ldots, x_n$ for all the constants $c_1, c_2, \ldots, c_n$ in A. A matrix $A_{x_1,\ldots,x_n}(x_1,\ldots,x_n)$ can then be thought of as an n-place predicate that an interpretation can "do as it pleases with" at non-normal worlds. I.e., the interpretation will directly assign some extension to such a matrix, where $v(A_{x_1,\ldots,x_n}(x_1,\ldots,x_n)) \subseteq D^n$. then $v(A_{x_1,\ldots,x_n}(c_1,\ldots,c_n)) = 1$ just in case $\langle v(c_1),\ldots,v(c_2)\rangle \in v(A_{x_1,\ldots,x_n}(x_1,\ldots,x_n))$.

For example, for the wff $\Box(Rab \supset Fc)$, the corresponding matrix is $\Box(Rx_1x_2 \supset Fx_3)$. At a non-normal world, the semantic value of $\Box(Rab \supset Fc)$ is not determined by the extensions of R or F at any world. Rather, it's as if we have a new predicate $\Box(Rx_1x_2 \supset Fx_3)$, whose extension is directly set, here being predicated of a, b, c. If $c = d$, then if $\Box(Rab \supset Fc)$ holds there, so will $\Box(Rab \supset Fd)$. For further details, consult chap. 18 of the book.

O.7 *Quantified Conditional Logic*

If we wish we may add a *ceteris paribus* conditional $>$ to the kinds of quantified modal logics we have been studying. We may simplify the modal core by assuming all worlds are accessible to each other à la S5, but we will now need to add, once again, a whole set of accessibility relations $\{R_A : A \text{ is a wff}\}$, one for each wff, used when considering the truth conditions of conditionals of the form $A > \ldots$. Again, using $f_A(w)$ for the set of worlds $\{w' : wR_Aw'\}$ and $[B]$ for the set of worlds where $v_w(B) = 1$, we let $v_w(A > B)$ be 1 just in case $f_A(w) \subseteq [B]$. This basically amounts to simply combining the approaches of the earlier chapter on propositional conditional logics with the quantified modal logics we've been considering.

There is one additional wrinkle. We let $f_{Fa}(w)$ be the "closest" worlds to w where Fa is true. Now suppose $a = b$, where these are constants, thus rigid designators. It should hold that these are exactly the closest worlds where Fb is true. Thus we adopt the additional provision on CC or VC interpretations:

$$\text{if } v(a) = v(b) \text{ then } R_{A_x(a)} = R_{A_x(b)}.$$

Crudely, we could put this as saying that substitution of identicals even holds in the subscripts to R. (This would even be applied in tableaux rules if the object-language identity predicate were used in tableaux.)

VC and CC could be extended to generate, e.g., VC^+, CC^+, VS, CS, VC_1, CC_1, and so on, by placing restrictions similar to those discussed in our earlier chapter on conditional logic.

[P] Quantified Non-Classical Logics

P.1 *Quantified Intuitionist Logic*

There aren't two forms of quantified intuitionist logic: constant domain and variable domain. Recall that intuitionists think of the subject matter of mathematics as *mental constructions*, and the possible worlds in intuitionist models are like states of mind. Certain constructions only exist in certain states of mind, and so something closer to variable domain logic is appropriate, though variation is not entirely lawless, as, once constructed, a mental construction stays constructed in later (accessible) states of mind.

Moreover, because mathematics is concerned centrally with equations, and we do not want merely possible equations to spread to all possible states of mind, interpretations allowing for contingent identity must be adopted.

Quantified statements are given different truth-conditions, or better, proof-conditions. Just like $\sqsupset$ is like a necessary $\supset$, $\forall x$ in intuitionism is more like a $\Box\,\forall x$; but $\exists x$ will only be true if the instance can be constructed in the world in question. Informally:

- $\exists x\, A$ is true (proven) if there is a constructed object a for which it is proven that $A_x(a)$;
- $\forall x\, A$ is true (proven) if there is a construction that can be applied to every object, a, we may construct now or later, to prove that $A_x(a)$.

Thus, an interpretation is a structure $\langle D, H, W, R, \nu\rangle$ where D is a domain, H a set of avatars, W a set of worlds, R a *reflexive and transitive* accessible relation, and ν is an evaluation function, which obeys the hereditary condition, which means, for example:

- If wRw', then for each predicate P_n, $\nu_w(P_n) \subseteq \nu_{w'}(P_n)$;
- As a special case of the above, if wRw', then $\nu_w(\mathfrak{E}) \subseteq \nu_{w'}(\mathfrak{E})$, so $D_w \subseteq D_{w'}$.
- As another special case of the above, if wRw', and $|d|_w = |e|_w$ then $|d|_{w'} = |e|_{w'}$.
- For every constant a (in the orig. language, before k_d constants are added for each $d \in D$), and every $w \in W$, $\nu(a) \in D_w$. (One cannot speak of objects one has not constructed.)

Our truth conditions for quantifiers can be stated:

$$\nu_w(\exists x\, A) = \begin{cases} 1, & \text{if for some } d \in D_w, \nu_w(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_w(\forall x\, A) = \begin{cases} 1, & \text{if for all } w' \text{ such that } wRw', \\ & \qquad \text{for all } d \in D_{w'}, \nu_{w'}(A_x(k_d)) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Priest gives two methods for doing tableaux here. The first is closer to the generic method for variable domain systems, but is more complicated (using $\mathfrak{E}$ a lot, etc.) A simpler method is also sketched, which I focus on here. The rules for propositional intuitionist logic (including the *hereditary*, *reflexivity* and *transitivity* rules) are supplemented with these new rules:

The first two yield new constants:

$$\begin{array}{c} \exists x\, A, +n \\ \downarrow \\ A_x(c), +n \text{ (new } c) \end{array} \qquad \begin{array}{c} \forall x\, A, -n \\ \downarrow \\ n\ r\ m \text{ (new } m) \\ A_x(c), -m \end{array}$$

The next are applied to (all) the old constants on the branch; in the first case, however, the constant must "belong" to world n, i.e., there must be a line of the form $m\ r\ n$, where m is the world where the constant a was "introduced":

$$\begin{array}{c} \exists x\, A, -n \\ \downarrow \\ A_x(a), -n \end{array} \qquad \begin{array}{c} \forall x\, A, +n \\ n\ r\ m \\ \downarrow \\ A_x(a), +m \end{array}$$

Finally, we have identity rules. The rule on the left must be applied at least once at world 0, and so if there is no constant there, a new constant must be picked.

$$\begin{array}{c} . \\ \downarrow \\ a = a, +n \end{array} \qquad \begin{array}{c} a = b, +n \\ A_x(a), +n \\ \downarrow \\ A_x(b), +n \end{array}$$

Example:

$$\begin{array}{c} \underline{\vdash \forall x\, \forall y(x = y \supset (Fx \supset Fy))} \\ \\ \forall x\, \forall y(x = y \supset (Fx \supset Fy)), -0 \\ 0\ r\ 0, 0\ r\ 1, 1\ r\ 1 \\ \forall y(a = y \supset (Fa \supset Fy)), -1 \\ 1\ r\ 2, 2\ r\ 2, 0\ r\ 2 \\ a = b \supset (Fa \supset Fb), -2 \\ 2\ r\ 3, 3\ r\ 3, 1\ r\ 3, 0\ r\ 3 \\ a = b, +3 \\ Fa \supset Fb, -3 \\ 3\ r\ 4, 4\ r\ 4, 2\ r\ 4, 1\ r\ 4, 0\ r\ 4 \\ Fa, +4 \\ Fb, -4 \\ a = b, +4 \\ Fb, +4 \\ \times \end{array}$$

P.2 Quantified Many-Valued Logic

We defined a simple (propositional many-valued logic as a structure $\langle \mathcal{V}, \mathcal{D}, \{f_c : c \in \mathcal{C}\}\rangle$, where $\mathcal{V}$ is a set of truth-values, $\mathcal{D}$ the set of *designated* values, and $\{f_c : c \in \mathcal{C}\}$ the truth functions between those values represented by the connectives $\mathcal{C}$. To get a quantified version, we add a domain D, and a set $\{f_q : q \in \mathcal{Q}\}$, where $\mathcal{Q}$ is a set of quantifiers. The functions in this set are mappings from non-empty subsets of values from $\mathcal{V}$ to values in $\mathcal{V}$. Thus, for classical logic, we have:

$$\begin{array}{cc} \forall : & \exists : \\ \{1\} \longrightarrow 1 & \{1\} \longrightarrow 1 \\ \{0,1\} \longrightarrow 0 & \{0,1\} \longrightarrow 1 \\ \{0\} \longrightarrow 0 & \{0\} \longrightarrow 0 \end{array}$$

Consider wffs of the form $\forall x\, A$ and $\exists x\, A$. If every instance of these, i.e., wffs of the form $A_x(k_d)$, are true, then we are in the top row, since only 1 is represented among the instances. If some are true, and some are false, then we are in the middle, since both 0 and 1 are represented. If every instance is false, we are in the bottom. We can see how this chart then tells us whether or not $\forall x\, A$ and $\exists x\, A$ are true or false.

In a finitely-valued many-valued logic in which there is a natural *ordering* of the truth-values—say the order $0, i, 1$ for systems like K_3 or $Ł_3$—it is natural to take the value of $\forall x\, A$ to be the *minimum* of the instances, and the value

of $\exists x\, A$ to be the maximum. For these systems we then have:

$$\begin{array}{rcl@{\qquad\qquad}rcl} & \forall : & & & \exists : & \\ \{1\} & \longrightarrow & 1 & \{1\} & \longrightarrow & 1 \\ \{i,1\} & \longrightarrow & i & \{i,1\} & \longrightarrow & 1 \\ \{0,i,1\} & \longrightarrow & 0 & \{0,i,1\} & \longrightarrow & 1 \\ \{0,1\} & \longrightarrow & 0 & \{0,1\} & \longrightarrow & 1 \\ \{i\} & \longrightarrow & i & \{i\} & \longrightarrow & i \\ \{0,i\} & \longrightarrow & 0 & \{0,i\} & \longrightarrow & i \\ \{0\} & \longrightarrow & 0 & \{0\} & \longrightarrow & 0 \end{array}$$

A similar chart with b in for i would be appropriate for quantified versions of LP or RM_3.

If we wanted free-logic version, we could have two domains, an outer domain D and an inner domain E, and define quantification as having only to do with thos einstances $A_x(k_d)$ where $d \in E$.

Of course, other systems may not have a natural ordering for their truth-values, and may use quantifiers distinct from $\exists$ and $\forall$ with rules all their own.

Identity can be added in the normal way. Matters are complicated however if one wishes to allow statements of the form $a = b$ to have truth-values other than 0 or 1, e.g., value i in gappy systems. One then needs to respond to an argument from Gareth Evans. Objects are distinct when they have different properties. Clearly, a is determinately a. Hence, if b is only indeterminately a, not determinately a, it cannot be a, because otherwise b would have a property a lacks, being determinately a.

We can formalize this by adding a "indeterminately" operator:

$$v(\nabla A) = \begin{cases} 1, & \text{if } v(A) = i; \\ 0, & \text{otherwise.} \end{cases}$$

Evans's argument is then:

$$\neg \nabla a = a, \nabla b = a \mathrel{\therefore} a \neq b$$

Exactly how one could react to this argument would depend on the conditions under which $a = b$ is to have value i. Suppose, e.g., one held that $v(a = b) = 1$ whenever $v(a) = v(b)$ but, for nonexistent entities, $v(a = b) = i$ when $v(a) \neq v(b)$ but $v(a), v(b) \notin E$. In that case, one might argue that Evans's argument is invalid, because the premises may be true while the conclusion has value i, which is not designated. Notice that this can be, even though, the "G. E. Moore Shift" of this argument, $\neg \nabla a = a, a = b \mathrel{\therefore} \neg \nabla b = a$ is a valid instance of the substitution of identicals.

P.3 *Quantified FDE*

As in the propositional case, we could describe FDE using different kinds of semantics: many-valued, relational and Routley star semantics. I'll skip the many-valued version.

In the relational semantics, we define interpretations as couples $\{D, v\}$ where D is the domain and v is a function (yes, a function—we'll get the relation ρ in a bit). v assigns each constant a member of D; $v(c) \in D$, but for each predicate v assigns it a couple $\langle \mathcal{E}, \mathcal{A} \rangle$, where $\mathcal{E}$ is called the *extension* of the predicate, and $\mathcal{A}$ is the *anti-extension* of the predicate. We write these as $v^{\mathcal{E}}(P_n)$ and $v^{\mathcal{A}}(P_n)$, and both are subsets of D^n.

The extension and anti-extension may overlap, creating gluts, or may fail to be exhaustive, creating gaps.

The relation ρ is defined in terms of v, beginning with the atomic case:

$$P_n c_1 \ldots c_n \; \rho \; 1 \text{ iff } \langle v(c_1), \ldots, v(c_n) \rangle \in v^{\mathcal{E}}(P_n)$$
$$P_n c_1 \ldots c_n \; \rho \; 0 \text{ iff } \langle v(c_1), \ldots, v(c_n) \rangle \in v^{\mathcal{A}}(P_n)$$

As usual for FDE, the same atomic formula may relate to 0, or to 1, or to both, or to neither.

The relation ρ is then extended to cover molecular formulæ using the rules from our previous FDE unit, and for quantified formulæ, we stipulate:

$$\forall x\, A \; \rho \; 1 \text{ iff for all } d \in D, A_x(k_d) \; \rho \; 1.$$
$$\forall x\, A \; \rho \; 0 \text{ iff for some } d \in D, A_x(k_d) \; \rho \; 0.$$
$$\exists x\, A \; \rho \; 1 \text{ iff for some } d \in D, A_x(k_d) \; \rho \; 1.$$
$$\exists x\, A \; \rho \; 0 \text{ iff for all } d \in D, A_x(k_d) \; \rho \; 0.$$

These conditions are just the same as in the case of classical logic, except that the same formula may relate to both truth and falsity (or neither), because the instances too may relate to both truth and falsity (or neither).

One can define quantified K_3 or LP interpretations as subsets of the above obeying exclusion or exhaustion, where those might be stated as follows:

Exclusion: For each P_n, $v^{\mathcal{E}}(P_n) \cap v^{\mathcal{A}}(P_n) = \varnothing$.

Exhaustion: For each P_n, $v^{\mathcal{E}}(P_n) \cup v^{\mathcal{A}}(P_n) = D^n$.

Quantified FDE tableaux (or quantified K_3 or LP tableaux, with the exclusion or exhaustion rules) can be done by supplementing the (relational semantics version) tableau method for FDE with the following quantifier rules (and here I use the convention of c for a new constant, a for any old constant, adding a new one only if need be):

$$
\begin{array}{cccc}
\forall x\,A,+ & \forall x\,A,- & \neg\forall x\,A,+ & \neg\forall x\,A,- \\
\downarrow & \downarrow & \downarrow & \downarrow \\
A_x(a),+ & A_x(c),- & \exists x\neg A,+ & \exists x\neg A,-
\end{array}
$$

$$
\begin{array}{cccc}
\exists x\,A,+ & \exists x\,A,- & \neg\exists x\,A,+ & \neg\exists x\,A,- \\
\downarrow & \downarrow & \downarrow & \downarrow \\
A_x(c),+ & A_x(a),- & \forall x\neg A,+ & \forall x\neg A,-
\end{array}
$$

Example:

$$\underline{\forall x\,Fx \wedge \forall x\,Gx \vdash \forall x(Fx \wedge Gx)}$$

$$
\begin{array}{c}
\checkmark\ \forall x\,Fx \wedge \forall x\,Gx,+ \\
\checkmark\ \forall x(Fx \wedge Gx),- \\
\checkmark Fc \wedge Gc,- \\
\checkmark\ \forall x\,Fx,+ \\
\checkmark\ \forall x\,Gx,+ \\
\checkmark Fc,+ \\
\checkmark Gc,+ \\
\begin{array}{cc}
Fc,- & Gc,- \\
\times & \times
\end{array}
\end{array}
$$

If we wish to have a free logic version instead, we make interpretations have the form $\langle D, E, \nu\rangle$ instead, where $E \subseteq D$, and require that $\nu^{\mathcal{E}}(\mathfrak{E}) = E$. Notice, however, that $\nu^{\mathcal{A}}(\mathfrak{E})$ may be anything, and so may overlap E, and thus objects may both exist and not exist (or neither). The tableau method is changed slightly to adopt something more like the free logic rules; details on p. 481 of the book.

Equivalent to the relational semantics is the Routley star semantics, where an interpretation takes the form $\langle D, W, *, \nu\rangle$. There, rather than dealing with an extension and anti-extension for each predicate, one has its extension at w and its extension at w^* play a similar role. (The relative complement of D^n and the extension at w^* is basically the anti-extension.) The tableaux rules are reduced; one has, in effect, rules like the four on the left above, except with a world number (which for now will always be 0 or $0^\#$), and instead of negation rules such as those on the right, we have the regular star semantics negation rule which takes $\neg A,+0$ to $A,-0^\#$ and so on. As an example, we show that an argument relying on disjunctive syllogism is still invalid.

$$\underline{\forall x(Fx \vee Gx), \exists x\neg Fx \nvdash \exists x\,Gx}$$

$$
\begin{array}{c}
\checkmark\ \forall x(Fx \vee Gx),+0 \\
\checkmark\ \exists x\neg Fx,+0 \\
\checkmark\ \exists x\,Gx,-0 \\
\checkmark\ \neg Fa,+0 \\
Fa,-0^\# \\
Ga,-0 \\
\checkmark Fa \vee Ga,+0 \\
\begin{array}{cc}
Fa,+0 & Ga,+0 \\
 & \times
\end{array}
\end{array}
$$

Identity can be added to quantified FDE with no special fuss, just as you would expect. However, if one takes identity to be necessary between constants in the star semantics, or true at the star world of w when true at w, the equivalence of the relational semantics and star semantics breaks down, as one then validates the law of excluded middle for such identities:

$$\underline{\vdash a = b \vee a \neq b}$$

$$
\begin{array}{l}
a = b \vee a \neq b,-0 \\
\quad a = b,-0 \\
\quad a \neq b,-0 \\
\quad a = b,+0^\# \text{ (necessary identity} \Rightarrow\text{)} \\
\quad a = b,+0 \\
\qquad \times
\end{array}
$$

This would not go through in the relational semantics.

P.4 ... And with Strict Implication

To obtain quantified versions of those systems founded on FDE but with a strict implication $\to$ (or $\sqsupset$ as the case may be), such as K_4, K_*, N_4, N_*, I_4, W, etc., for the most part, one simply combines approaches we have already seen. Indeed, in most cases it is not worth our time to go over the details; they are in the book. Tableaux rules are simply those of chap. 9 modified in just the way FDE is modified to obtain these systems by adding rules for the conditional.

I here mention only a couple hiccups involving the non-normal world-based systems N_4 and N_*. In these systems, the truth-values of strict implications are allowed near complete freedom at non-normal worlds. However, for reasons exactly parallel to those involved with quantified versions of non-normal modal logic, we cannot simply let interpretations assign truth-values directly to conditionals, but we must, in order to preserve substitutivity of identicals, make use of matrices instead. Thus, an N_* interpretation will treat, e.g., $Fx \to Gy$ as though it were a primitive dyadic predicate, and assign it whatever extension it likes at each world, completely independent of what it does with F and G at those worlds. An N_4 interpretation will do the same, assigning it both an extension and an anti-extension, however it pleases, as each non-normal world.

If identity is added, and one wishes to preserve N_* and N_4 as relevant logics, one must also allow the truth-values

of identity statements to be non-standard at non-normal worlds as well, or else one will have such irrelevantly valid conditionals as:

$$\forall x Fx \to a = a$$

The solution is simply to require that $v_w(=) = \{\langle d, d\rangle : d \in D\}$ only if w is normal. One then only applies the reflexivity of identity rule at world 0, and the substitution rule only when the identity holds at world 0.

P.5 *Quantified Relevant Logic*

Once again, one may obtain quantified versions of B and its extensions simply by combining approaches we have already seen; tableaux systems may be obtained just by adding quantifier rules to the tableaux rules for relevant logic, etc. If identity is added, then for reasons akin to those for N_* and N_4, we require $v_w(=) = \{\langle d, d\rangle : d \in D\}$ only at normal worlds.

One limitation to only doing this, however, is that it becomes difficult to express restricted quantification—how does one say "all *F*s are Gs" in quantified relevant logic? If one defines $A \supset B$ as $\neg A \vee B$, and analyzes "all *F*s are Gs" as "$\forall x(Fx \supset Gx)$" then the simple syllogism below fails:

$$\forall x(Fx \supset Gx), Fa \nvDash Ga$$

Modus ponens for $\supset$ is invalid in relevant logic just as it is for FDE. One might then attempt to use the relevant conditional $\to$ instead. But "$\forall x(Fx \to Gx)$" seems to require a closer connection between the *F*s and Gs than "all *F*s are Gs". Consider:

$$\forall x Gx \nvDash \forall x(Fx \to Gx)$$

However, if everything is G, it seems to be true that "all *F*s are Gs", even when there is no relevant connection between being *F* and being G.

One solution to this problem sometimes adopted is to add yet another, non-relevant, but still *ponens*-able conditional $\mapsto$, defined using content inclusion $\sqsubseteq$, where:

$$v_w(A \mapsto B) = \begin{cases} 1, & \text{if for all } w_1, w_2 \in W \text{ such that } Rww_1w_2 \text{ and } w \sqsubseteq w_2, \text{ if } v_{w_1}(A) = 1, \text{ then } v_{w_2}(B) = 1. \\ 0, & \text{otherwise.} \end{cases}$$

We then have (no, this is not obvious—just take my word for it):

$$\forall x(Fx \mapsto Gx), Fa \vDash Ga$$
$$\text{and:}\quad \forall x Gx \vDash \forall x(Fx \mapsto Gx)$$
$$\text{but:}\quad \forall x \neg Fx \nvDash \forall x(Fx \mapsto Gx)$$

(This last invalidity is important, as if this were not true, the system would no longer be paraconsistent.) So $\mapsto$ is in between $\supset$ and $\to$ in strength.

P.6 *Quantified Fuzzy Logic*

Quantified fuzzy logic counts as a simple but infinitely-valued many-valued logic. To characterize the quantified version of continuum-valued Ł, we need only specify $\{f_q : q \in \mathcal{Q}\}$, i.e., specify the functions from sets of values to values for the quantifiers. Here we put:

$$v(\exists x A) = \text{the least upper bound of the set of values } v(A_x(k_d)) \text{ for } d \in D.$$
$$v(\forall x A) = \text{the greatest lower bound of the set of values } v(A_x(k_d)) \text{ for } d \in D.$$

Typically, the least upper bound is the maximum of the set, and the greater lower bound is the minimum. Here, the sets may be infinite, however, and hence have no minimum or maximum. In such cases, the least upper bound or greater lower bound is the limit of ascent or descent, respectively. For example, if the series $Fa_1, Fa_2, Fa_3, Fa_4, \ldots$ have values 0.1, 0.01, 0.001, 0.0001, etc., then $v(\forall x Fx)$ will have value 0, which is the limit of the descent.

As in the propositional case, one can define a different notion of validity $\vDash_\varepsilon$ by setting the designated values as everything $\geq \varepsilon$, for every real ε in the interval $[0, 1]$. We also write $\vDash_{Ł_\aleph}$ for $\vDash_1$. If we leave off the subscript, or put only Ł, then $\Delta \vDash_Ł A$ means that for *every* ε in the interval $[0, 1]$, $\Delta \vDash_\varepsilon A$. For a finite list of premises $B_1, \ldots, B_n$, we again have:

$$B_1, \ldots, B_n \vDash_Ł A \text{ iff } \vDash_{Ł_\aleph} (B_1 \wedge \ldots \wedge B_n) \to A$$

Hence, one can get a grip on Ł by studying conditionals in $Ł_\aleph$. In addition to the propositional axiom or theorem schema considered in chap. 11, the quantified form of $Ł_\aleph$ validates:

$$\vDash \forall x A \to A_x(a)$$
$$\vDash A_x(a) \to \exists x A$$
$$\vDash \forall x \neg A \leftrightarrow \neg \exists x A$$
$$\vDash \neg \forall x A \leftrightarrow \exists x \neg A$$
$$\vDash \forall x(A \vee B) \leftrightarrow (A \vee \forall x B) \text{ if } x \text{ not free in } A$$
$$\vDash \forall x(A \to B) \leftrightarrow (A \to \forall x B) \text{ if } x \text{ not free in } A$$
$$\vDash (\exists x A \to \exists x B) \leftrightarrow \exists x(A \to B)$$
$$\text{if } \vDash A_x(a) \text{ then } \vDash \forall x A, \text{ if } a \text{ is not in } A$$

One may use these results to construct deductions if one wishes.

Identity in fuzzy logic raises certain interesting questions, including, once again, Evans's argument. In favor of "fuzzy identity", consider this example of Priest's: suppose he has two identical cars, a and b. Each day, he swaps one part from a to b and vice versa. After doing this long enough, eventually b is where a was, and vice versa. On the second day, it seems clear that despite having one or two parts swapped, the car in the first spot is still the same car as a. In the middle, however, it seems the car there is neither fully identical to a nor fully identical to b but not fully non-identical to them either. There is no clear place to draw the boundary.

Similarly, if one considers mountains or forests, etc., certain spatial points in them are "on the boundary" as to whether they count as "the same forest" or "the same mountain" any more. Priest calls such things "vague objects" and argues they give rise to "vague identity".

In order to stave off a *sorites* like paradox in the cars case, one will need to say that in Ł (though not $Ł_{\aleph}$), that the transitivity of identicals is not valid (but only valid-esque or valid-ish or something like that). But what sorts of semantics for identity would give that result? If there is some way of measuring or quantifying the would-be identical entities x and y in terms of "sameness", so that one could speak of the amount of difference between them with a numeric value $\delta(x, y)$ Priest suggests something like:

$$v(a = b) = 1 - \delta(v(a), v(b))$$

Time-slice 0 of the car may be 0.98-identical to time-slice 1 of the car, and 0.47-identical to time-slice 223 of the car, etc.

Now let us reconsider Evans's argument. Suppose:

$$v(\nabla A) = \begin{cases} 0, & \text{if } v(A) = 1 \text{ or } v(A) = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Consider now the argument:

$$\neg \nabla a = a, \nabla b = a \therefore a \neq b$$

Let a be the car on day zero, and b be the "same" car on day one. We can assume that $v(a = a) = 1$ and $v(b = a) = v(a = b) = 0.98$. Then the premises are both true, $v(\neg \nabla a = a) = v(\nabla a = b) = 1$, but $v(a \neq b) = 0.02$, so the argument can be dismissed as invalid.

Or so an advocate of fuzzy identity might claim. These are, of course, difficult metaphysical issues which we could discuss *ad nauseum*.

Homework

Choose one of the options (i)–(vii) below to complete as homework. (You will have the opportunity to choose additional options from this list on the final exam.)

(i) Priest, §18.10 (p. 397), problem 2. (Quantified non-normal modal logic tableaux.)

(ii) Priest, §19.9 (p. 419), problem 2. (Quantified conditional logic tableaux.)

(iii) Priest, §20.13 (p. 454), problem 6. (Quantified intuitionist tableaux.)

(iv) Priest, §22.12 (p. 502), problem 2 (any 5 of (a)–(j); also, you don't need to convert counter-models to many-valued countermodels). (Quantified FDE tableaux—relational style.)

(v) Priest, §22.12 (p. 502), problem 5 items (g)–(j). (Quantified FDE tableaux—Routley star style.)

(vi) Priest, §23.15 (p. 533), do either problem 2 or 3 (any 4 of (a)–(j)). (Quantified N_4 or N_* tableaux.)

(vii) Priest §25.11 (p. 582), do any three items from either problem 2 (a)–(g) or problem 3 (a)–(c). (Quantified fuzzy logic.)