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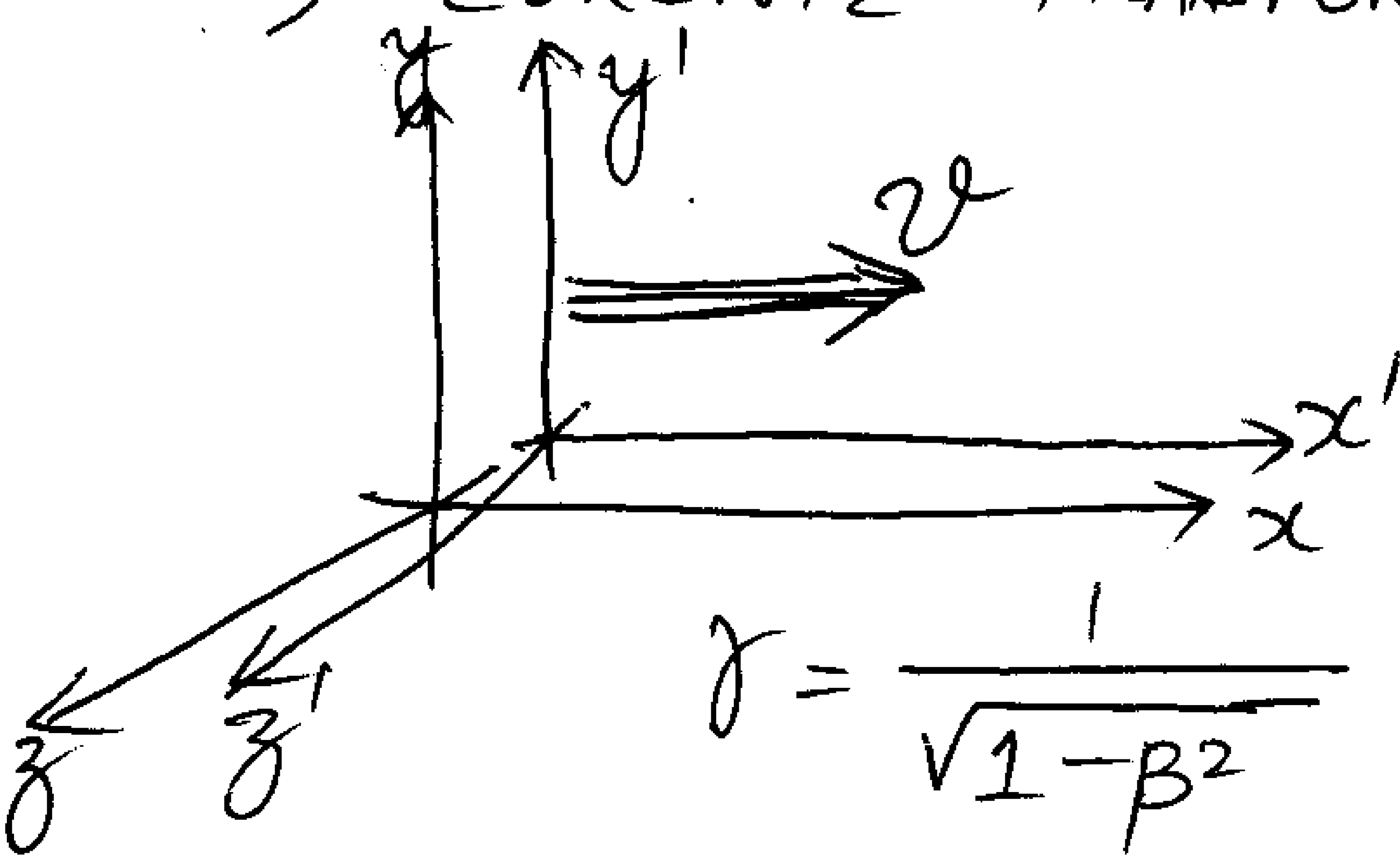
LECTURE II

SPECIAL RELATIVITY (Griffiths, Chapter 3)

<u>Postulates</u> (1) All inertial frames are equivalent
(2) c is a universal constant.

$\Longrightarrow$ LORENTZ TRANSFORMATIONS:



$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

New co-ordinates: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$x^{0'} = \gamma(x^0 - \beta x^1)$$
$$x^{1'} = \gamma(x^1 - \beta x^0)$$
$$x^{2'} = x^2$$
$$x^{3'} = x^3$$

$$\Longrightarrow x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\ \nu} x^{\nu}$$

$(\mu = 0, 1, 2, 3)$

$$\Lambda \equiv \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Einstein Summation Convention.

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$$

(Sum over ν is implicit)

<u>INVARIANTS</u>: <u>e.g.</u> $r^2 = x^2 + y^2 + z^2$ is constant in Cartesian coordinates under rotations.

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2$$

is invariant under Lorentz transformations

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Introduce the metric g

$$g \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$g_{\mu\nu} x^\mu x^\nu \equiv x_\nu x^\nu$ is the invariant 4-scalar product

Define $g_{\mu\nu} x^\mu \equiv x_\nu \equiv$ covariant vector
$x^\nu \equiv$ contravariant vector.

$x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv$ position 4-vector.

Any 4 numbers $a^\mu \equiv (a^0, a^1, a^2, a^3)$ is a 4-vector if it transforms according to Lorentz transf$\underline{^n}$.

For every a^ν, there exist an $a_\nu = g_{\mu\nu} a^\mu$

$$a_\mu b^\mu = a^\mu b_\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$
$$= a'^0 b'^0 - a'^1 b'^1 - a'^2 b'^2 - a'^3 b'^3$$

where $a'^\mu = \Lambda^\mu{}_\nu a^\nu$

$a \cdot b = a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}$ is invariant.

$$a^2 = a \cdot a = (a^0)^2 - |\vec{a}|^2$$

$a^2 > 0 \Rightarrow a^\mu$ is time-like
$a^2 < 0 \Rightarrow a^\mu$ is space-like
$a^2 = 0 \Rightarrow a^\mu$ is light-like

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Now Some physics: Consider $d\tau$, a time interval in the rest frame.

Then, $c\,d\tau = ds$. Now change frames from $(c\,d\tau, 0, 0, 0)$ to $(c\,dt, x, y, z)$

Then: $d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$

$$\Rightarrow \frac{d\tau^2}{dt^2} = 1 - \frac{1}{c^2}\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right) = 1 - \frac{v^2}{c^2}$$

$$= 1 - \beta^2 = 1/\gamma^2$$

$$\Rightarrow d\tau = dt/\gamma \quad \text{or} \quad \boxed{dt = \gamma\, d\tau}$$

γ goes from 1 to ∞ $\Longrightarrow$ TIME DILATION

4-velocity: $\eta^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, v_x, v_y, v_z)$

$$\eta^2 = \gamma^2(c^2 - v^2) = c^2\gamma^2(1 - \beta^2) = c^2$$

$\Rightarrow$ η^2 is a constant as expected.

4-momentum: $p^\mu = m\eta^\mu = m\gamma(c, v_x, v_y, v_z)$

What is $p^0 = m\gamma c$? $\Longrightarrow$ γmc^2 has units of energy

$\Rightarrow$ $p^0 = E/c$ leads to conservation of energy.

$p^\mu \equiv (E/c, \vec{P})$ $\qquad P_x = \gamma m v_x \simeq m v_x$ for $v \ll c$

$$\gamma mc^2 \simeq c^2\left(m + \frac{1}{2}m\frac{v^2}{c^2} + \frac{3}{8}m\frac{v^4}{c^4} \cdots\right)$$

$$= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} \cdots\cdots$$

$$\Rightarrow \boxed{E = mc^2 + \text{KINETIC ENERGY}}$$

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What is $E_0 = mc^2 = E -$ KINETIC ENERGY?
Rest energy i.e. $E \neq 0$ in rest frame

Relativistic kinetic energy $= \gamma mc^2 - mc^2 = (\gamma - 1)mc^2$

Since p^μ is a four-vector, one can look at it in 2 different frames: $p^\mu = (E/c, \vec{P})$ & $(mc, 0)$
where the second 4-vector is in the rest frame

$$\Rightarrow p^2 = \left(\frac{E}{c}\right)^2 - |P|^2 = m^2c^2 \Rightarrow \boxed{E^2 = m^2c^4 + |P|^2c^2}$$

If $m = 0$ then $E = pc \Rightarrow$ massless particle travels at the speed of light! It has energy!
Einstein: $E = h\nu$ (photo-electric effect)

Collisions: $a + b \longrightarrow c + d$

$$\Rightarrow p_a^\mu + p_b^\mu = p_c^\mu + p_d^\mu$$ (Energy-Momentum conservation)

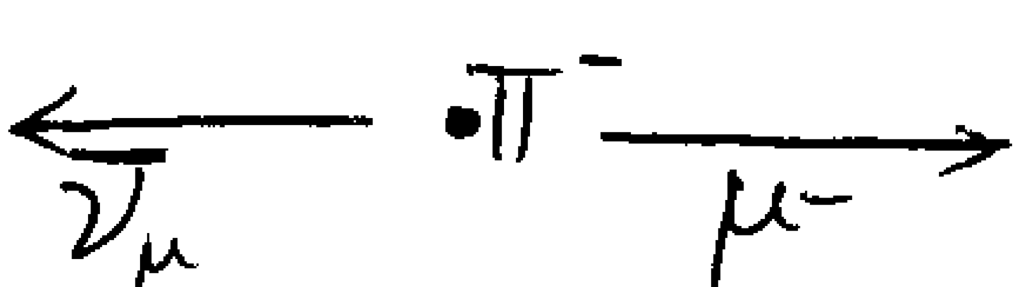
Each of the 4-components is separately conserved.

Decays: $a \rightarrow b + c \Rightarrow p_a^\mu = p_b^\mu + p_c^\mu$

xxx

PROBLEM: $\pi^- \rightarrow \mu^- + \bar{\nu}_\mu$. In rest frame of pion, what is the speed of the muon?

$$E_\pi = E_\mu + E_\nu$$
$$|P_\mu| = |P_\nu|$$



$E_\pi = m_\pi c^2 = E_\mu + E_\nu \quad \Longrightarrow \quad m_\pi c^2 - E_\mu = c|P_\nu| = c|P_\mu|$

$E_\nu = |P_\nu| c$ (Since $m_\nu = 0$) $\quad m_\pi c^2 - \gamma mc^2 = \gamma m v c$

$\Longrightarrow$ Hard to find v!

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Pion Decay: Alternate solution:

Use $E^2 = m^2c^4 + p^2c^2$

$$E_\pi = m_\pi c^2 \qquad E_\mu^2 = m_\mu^2 c^4 + p_\mu^2 c^2 \qquad E_\nu = |p_\nu| c$$

$$m_\pi c^2 = c\sqrt{m_\mu^2 c^2 + |p_\mu|^2} + |p_\mu| c$$

$$(m_\pi c - |p_\mu|)^2 = m_\mu^2 c^2 + |p_\mu|^2$$

$$\Rightarrow |p_\mu| = \left[\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right] c$$

$$\Rightarrow E_\mu = \sqrt{m_\mu^2 c^4 + |p_\mu|^2 c^2} = \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi}\right) c^2$$

$$E = \gamma m c^2 \quad p = \gamma m v \Rightarrow \frac{p}{E} = \frac{v}{c^2}$$

$$\Rightarrow \beta_\mu = \frac{|p_\mu| c}{E_\mu} = \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right)$$

<u>Best</u> <u>Solution</u>: Use the fact that $m_\nu = 0$

$$p_\nu = p_\pi - p_\mu$$

$$p_\nu^2 = p_\pi^2 + p_\mu^2 - 2 p_\mu \cdot p_\pi$$

$$\Rightarrow 0 = m_\pi^2 c^2 + m_\mu^2 c^2 - 2 m_\pi E_\mu$$

$$\Rightarrow E_\mu = \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi}\right) c^2$$

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$$P_\mu = P_\pi - P_\nu$$

$$P_\mu^2 = P_\pi^2 + P_\nu^2 - 2P_\pi \cdot P_\nu$$

$$m_\mu^2 c^2 = m_\pi^2 c^2 + 0 - 2E_\pi |p_\nu|$$

$$= m_\pi^2 c^2 - 2E_\pi |p_\mu|$$

$$\Longrightarrow \quad |p_\mu| = \left(\frac{m_\pi^2 - m_\mu^2}{2m_\pi}\right) c$$

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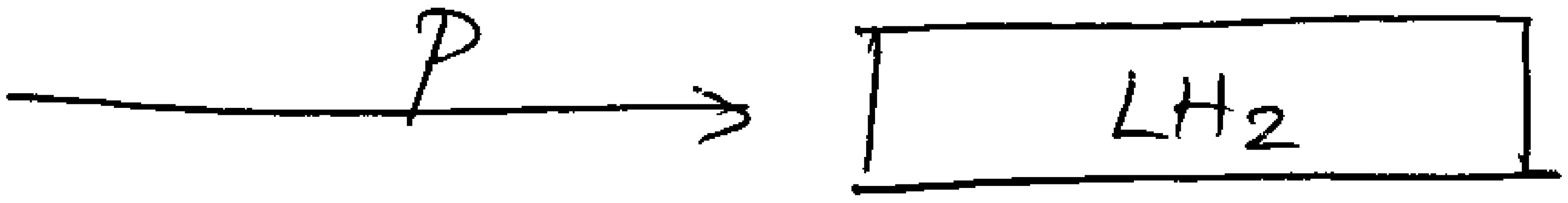
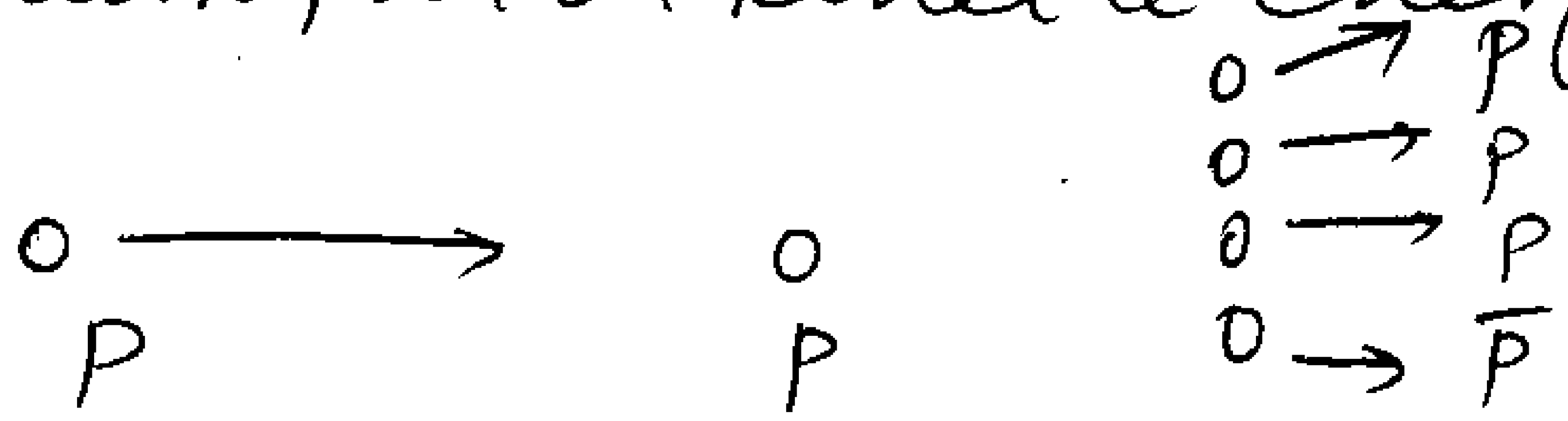
Special Relativity Problem

Antiproton Production: $p + p \longrightarrow p + p + p + \bar{p}$



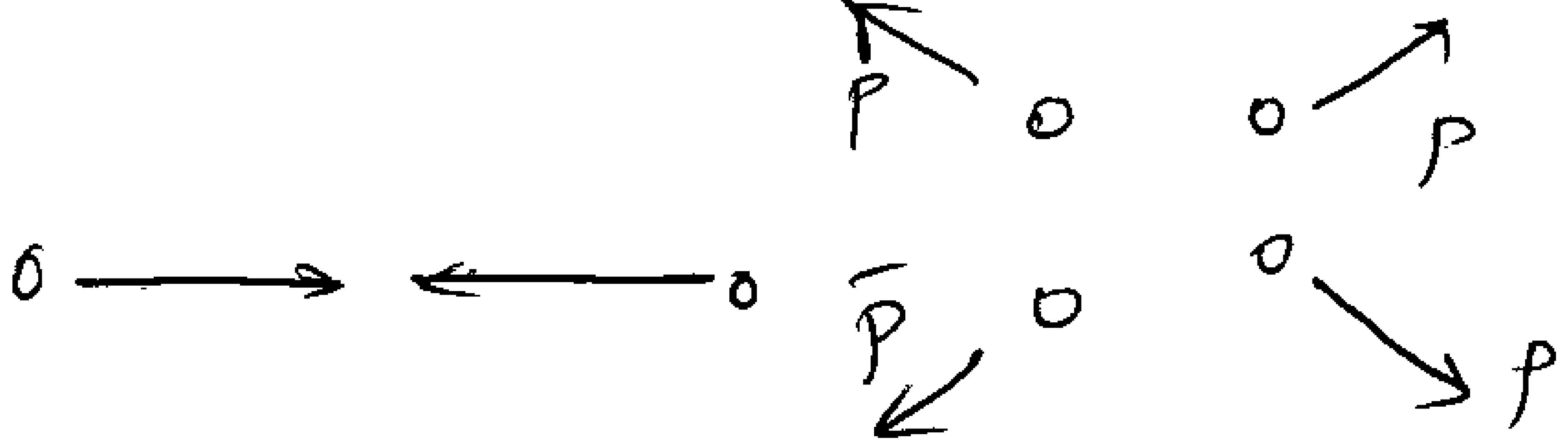
What is the minimum proton kinetic Energy?

Lab frame:



Not enough information in Lab frame

COM frame



Minimum KE: all 4 particles at rest in COM.

LAB 4-momenta: (Before Collision) $\quad P_1 \equiv \left(\frac{E_1}{c}, 0, 0, P_{1z}\right)$

$P_2 \equiv (m_p c, 0, 0, 0)$

COM 4-momenta (After collision) $\quad \vec{P}_{1c} = \vec{P}_{2c} = \vec{P}_{3c} = \vec{P}_{4c} = 0$

$P'_{1c} \equiv (m_p c, 0, 0, 0), \; P'_{2c}, \; P'_{3c}, \; P'_{4c}$

$$\underbrace{P_1 + P_2}_{P_T} = \underbrace{P_1' + P_2' + P_3' + P_4'}_{P_T'}$$

$$\boxed{P_T^2 = P_T'^2 = P_{TC}^2 = P_{TC}'^2}$$

$$\underbrace{P_{1c} + P_{2c}}_{P_{TC}} = \underbrace{P'_{1c} + P'_{2c} + P'_{3c} + P'_{4c}}_{P'_{TC}}$$

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$$P_T = \left(\frac{E_1 + m_p c^2}{c}, 0, 0, P_{1z} \right)$$

$$P'_{Tc} = \left(4 m_p c, 0, 0, 0 \right)$$

$$\left(\frac{E_1 + m_p c^2}{c^2} \right)^2 - P_{1z}^2 = 16 m_p^2 c^2$$

$$E_1^2 + 2 E_1 m_p c^2 + m_p^2 c^4 - P_{1z}^2 c^2 = 16 m_p^2 c^4$$

$$2 E_1 m_p c^2 = 14 m_p^2 c^4$$

$$\Longrightarrow \quad E_1 = 7 m_p c^2$$

$$\boxed{KE = (\gamma - 1) m_p c^2 = 6 m_p c^2}$$