10/11/05

LECTURE X

Complex Scalar Field

Suppose you had 2 scalar fields of the same mass:

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu \phi_1 \partial^\mu \phi_1 - m^2 \phi_1^2\right] + \frac{1}{2}\left[\partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_2^2\right]$$

Define $\phi = \dfrac{\phi_1 + i\phi_2}{\sqrt{2}}$ $\qquad \phi^* = \dfrac{\phi_1 - i\phi_2}{\sqrt{2}}$

$$\Longrightarrow \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

Observation: Nothing fixes the direction of ϕ_1 & ϕ_2
i.e. rotation in $\phi_1 - \phi_2$ space leaves $\mathcal{L}$ invariant.

$$\phi_1' = \phi_1 \cos\alpha + \phi_2 \sin\alpha$$
$$\phi_2' = -\phi_1 \sin\alpha + \phi_2 \cos\alpha$$

$$\Longrightarrow \phi' = e^{-i\alpha}\phi \qquad \phi'^* = e^{i\alpha}\phi^*$$

Invariance of $\mathcal{L}$ has important consequences:
(Noether's Theorem)

Consider an infinitesimal transformation

$$\phi' = (1 - i\alpha\phi) \Rightarrow \delta\phi = -i\alpha\phi \quad \& \quad \delta\phi^* = +i\alpha\phi^*$$

$$\delta\mathcal{L} = \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + (\delta(\partial^\mu \phi)) \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} + (\phi \longrightarrow \phi^*)$$

Demand $\delta\mathcal{L} = 0 \Longrightarrow \partial^\mu S_\mu = 0$ with

$$S_\mu = i(\phi \partial_\mu \phi^* - \phi^* \partial_\mu \phi); \qquad \partial^\mu S_\mu = \frac{\partial S^0}{\partial t} + \vec{\nabla} \cdot \vec{S}$$

(Conservation of "current")

10/11/05

LECTURE $\overline{X}$

E & M analogy: $\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$ (continuity Eqn)

$\int \rho\, d^3x = Q$ = Constant (charge); $\int \vec{\nabla}\cdot\vec{J}\, d^3x = \int_S \vec{J}\cdot\vec{n}\, ds$

$\Rightarrow$ change in "charge" density = flow of "current"

* $\int_V S^0 d^3x$ is a generic charge
* Interchange of ϕ & $\phi^* \Rightarrow$ change in sign of S^μ $\Rightarrow$ anti particles
* $e^{i\alpha} \Rightarrow$ "Global gauge transf^n" $\Rightarrow$ Invariance Law

__Example__: Conservation of probability from Schrödinger Eqn.

$$\frac{\partial \psi}{\partial t} = \frac{i\nabla^2}{2m}\psi \Longrightarrow \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \quad \text{iff}$$

$$\rho = \psi^*\psi \qquad J = \frac{-i}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

"probability" "Current"

__Dirac Equation__ Quest for a relativistically covariant eqn for massive particles. Motivation $\Rightarrow$ Conservation of probability $\Rightarrow$ linear in time derivative $\Rightarrow$ Lorentz invariance $\Rightarrow$ linear in space derivative

General form: $\frac{i\partial\psi}{\partial t} = H\psi$ with $H = -i\left[\alpha_j \partial^j + \beta m\right]$

Must obey $E^2 = p^2 + m^2 \Rightarrow -\frac{\partial^2\psi}{\partial t^2} = (-\nabla^2 + m^2)\psi$

10/11/05

LECTURE $\overline{\overline{X}}$

$\Rightarrow \alpha_i\alpha_j + \alpha_j\alpha_i = 0\ (i \neq j)$ & $\alpha_i\beta + \beta\alpha_i = 0$

If $m = 0$, then β is absent & $\alpha_i\alpha_j + \alpha_j\alpha_i \equiv \{\alpha_i, \alpha_j\} = 2\delta_{ij}$

One solution are the 2x2 pauli matrices

$$\Rightarrow i\frac{\partial \psi}{\partial t} = (\vec{\sigma} \cdot \vec{p})\psi$$

Here ψ ~~are~~ is a 1x2 spinor

For $m \neq 0$, no 2x2 soln $\Rightarrow$ must go to 4x4.

One set of solns:

$$\alpha_i \equiv \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

Introduce the γ matrices

$$\gamma^i = \beta\alpha^i \quad \& \ \beta^0 = \beta \Rightarrow \gamma^\mu \equiv (\gamma^0, \gamma^i)$$

Satisfies $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$i\frac{\partial \psi}{\partial t} = (-\vec{\alpha} \cdot \vec{\nabla} + \beta m)\psi$$

$$i\beta\frac{\partial \psi}{\partial t} = [(-i\beta\vec{\alpha} \cdot \vec{\nabla}) + m]\psi$$

$$i\gamma^0\frac{\partial \psi}{\partial t} + i\gamma^i\nabla^i\psi - m\psi = 0$$

$$\Rightarrow (i\gamma^\mu\partial_\mu - m)\psi = 0$$

ψ is a 4 element column matrix

$\psi^\dagger\psi$ is a number but not Lorentz invariant

Introduce $\bar{\psi} = \psi^\dagger\gamma^0 \Rightarrow \bar{\psi}\psi$ is a Lorentz scalar

10/11/05

LECTURE X

$\bar{\psi}$ is called the adjoint spinor. $\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

$\bar{\psi} \equiv (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$

$\bar{\psi}\gamma^\mu\psi$ is a Lorentz 4-vector.

Another important γ matrix is is $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$\bar{\psi}\gamma^5\psi$ is a pseudoscaler

$\bar{\psi}\gamma^\mu\gamma^5\psi$ is a pseudo-vector or axial-vector

Consider the Dirac Eqn in restframe $\Rightarrow \vec{p} = 0$

$\Rightarrow i\gamma^0 \frac{\partial\psi}{\partial t} = m\psi \qquad \because \gamma^0 \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$\Rightarrow \frac{\partial \psi_A}{\partial t} = -im\psi_A \quad \& \quad \frac{\partial \psi_B}{\partial t} = +im\psi_B$

ψ_A & ψ_B are 2 component spinors $\Rightarrow$ 4 total solutions

$e^{-imt}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{-imt}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, e^{+imt}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, e^{+imt}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

NOTE: $imt = imc^2t/\hbar$

Compare with Schrödinger Eqn Soln: $\psi = e^{-iEt/\hbar}\psi(0)$

ψ_A has $E = mc^2$ but ψ_B has $E = -mc^2$??!

$\Rightarrow$ Reinterpret as $E = +mc^2$ with $t \rightarrow -t$

$\Rightarrow$ Every Particle has antiparticle

2 fold degeneracy $\Rightarrow$ H commutes with new operator & ~~so~~ there exist simultaneous eigenfn's.

LECTURE X

Commuting operator is $\vec{\Sigma} \equiv \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$

$[H, \vec{L}] = -i(\vec{\alpha} \times \vec{p})$ $\quad [H, \vec{\Sigma}] = +2i(\vec{\alpha} \times \vec{p})$

$\Longrightarrow [H, \vec{J}] = 0$ with $\vec{J} = \vec{L} + \frac{1}{2}\vec{\Sigma}$

$\vec{L}$ not conserved but $\vec{J}$ is conserved $\Longrightarrow$ spin $\frac{1}{2}$!

$\bar{\psi}$ satisfies a different equation $\Longrightarrow$

Start with $\quad i\gamma^0 \frac{\partial \psi}{\partial t} + i\gamma^k \frac{\partial \psi}{\partial x^k} - m\psi = 0$

Conjugate Eqn: $\quad -i\frac{\partial \psi^\dagger}{\partial t}\gamma^0 - i\frac{\partial \psi^*}{\partial x^k}(-\gamma^k) - m\psi^\dagger = 0$

Since $\bar{\psi} = \psi^\dagger \gamma^0$: $\quad -i\frac{\partial \bar{\psi}}{\partial t} - i\,\bar{\psi}\gamma^0\gamma^k - m\bar{\psi}\gamma^0 = 0$

$(\because \gamma^0\gamma^k = -\gamma^k\gamma^0)$

Thus:

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \qquad (1)$$

$$i\partial_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \qquad (2)$$

$$\bar{\psi}(1) + (2)\psi \Longrightarrow \bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\gamma^\mu\psi = 0$$

$$\Longrightarrow \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

Introduce $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$ is a conserved current.

$\Longrightarrow$ due to invariance of Dirac Eqn

(Equivalence of particle & antiparticle)

10/11/05

LECTURE X

Electromagnetic "current": $j^\mu = -e\bar{\psi}\gamma^\mu\psi$

For a free particle: $\psi = A e^{-ip_\mu x^\mu}$

$$\Rightarrow j^\mu = -ep^\mu$$

If we replace $E \to -E$ & $\vec{p} \to -\vec{p}$, then
$$j^\mu = +e(-p^\mu)$$

Thus, positrons have opposite charge

Electrons propagating forwards $\Longleftrightarrow$ positrons propagating backwards.

Next Steps: - Maxwell's Eqns in covariant form.
- Lagrangian for spin ½ fermions
- Lagrangian for free photons.
- Interaction between electrons & photons by demanding "local" gauge invariance

10/13/05

Lecture XI

Gauge Invariance in Classical E & M

$$\vec{\nabla}\cdot\vec{B} = 0 \implies \vec{B} = \vec{\nabla}\times\vec{A}$$

- $\vec{A}$ is the vector potential & ensures $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$
- $\vec{B}$ is unchanged if $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$

(Λ is an arbitrary scalar f^n)

However: $\vec{\nabla}\times\vec{E} = \dfrac{-\partial B}{\partial t} \implies \vec{\nabla}\times\left(\vec{E} + \dfrac{\partial \vec{A}}{\partial t}\right) = 0.$

This suggests $\vec{E} + \dfrac{\partial\vec{A}}{\partial t} = -\vec{\nabla}V$ & thus

if $\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$, then $V \rightarrow V - \partial\Lambda/\partial t$.

or $A_\mu \rightarrow A_\mu - \partial_\mu\Lambda \qquad (A_\mu \equiv (V, \vec{A}))$

Covariant formulation of Maxwell's Eqns:

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = \partial\vec{B}/\partial t$$
$$\vec{\nabla}\cdot\vec{E} = \rho \qquad \vec{\nabla}\times\vec{B} = \partial\vec{E}/\partial t + \vec{J}$$

$$\begin{array}{cc} & \nu\rightarrow \\ \mu\downarrow \; F^{\mu\nu} \equiv & \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \end{array}$$

$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$

Form guarantees $\vec{\nabla}\cdot\vec{B} = 0$

& $\vec{\nabla}\times\vec{E} - \partial\vec{B}/\partial t = 0$

Define $J_\mu \equiv (\rho, \vec{J})$

Then: $\partial_\mu F^{\mu\nu} = J^\nu$

It can be shown that $\partial_\nu J^\nu = 0 \Rightarrow$ Conserved current.

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu$ leads to E-L Eqn: $\partial_\mu F^{\mu\nu} = J^\nu$

10/13/05

LECTURE $\overline{XI}$

Phase Invariance in QM:

Consider an observable $\langle O \rangle \equiv \int \psi^* O \psi \, d^3x$

$\Rightarrow$ Unchanged by $\psi(x) \to e^{i\theta}\psi(x)$

But what if θ is a function of x?

Laws of physics should allow different phases.

Consider $-\dfrac{i\nabla^2\psi}{2m} = \dfrac{\partial \psi}{\partial t}$ If $\psi \to \psi' = e^{i\alpha(x,t)}\psi$

then $-\dfrac{i\nabla^2\psi'}{2m} \neq \dfrac{\partial \psi'}{\partial t}$ Unless,

$\vec{\nabla} \longrightarrow \vec{\nabla} - iq\vec{A} \equiv \vec{D}$ & $\dfrac{\partial}{\partial t} \to \dfrac{\partial}{\partial t} + iqV \equiv D^0$

Also $\vec{A} \longrightarrow \vec{A} + \vec{\nabla}(\alpha/q)$ & $V \to V - \dfrac{\partial}{\partial t}(\alpha/q)$

with $\alpha/q = \Lambda$

now $\Big\{ \underbrace{\dfrac{(-i\vec{\nabla} - q\vec{A})^2}{2m}}_{H} + qV \Big\}\psi = \dfrac{\partial \psi}{\partial t}$

This Hamiltonian leads to $\vec{F} = q\vec{E} + \vec{v} \times \vec{B}$

Thus, ~~the~~ $\vec{A}$ is the vector potential of E&M

Demanding ~~local~~ LOCAL phase invariance produces the $\vec{E}$ & $\vec{B}$ fields.

10/13/05

LECTURE XI

Covariant formulation: $\phi(x_\mu) \longrightarrow e^{iq\alpha(x_\mu)}\phi(x_\mu)$

$$\partial_\mu \phi \longrightarrow e^{iq\alpha}\left[\partial_\mu\phi + iq(\partial_\mu\alpha)\phi\right]$$

Introduce $D_\mu \equiv \partial_\mu + iqA_\mu$

Then $D_\mu\phi \longrightarrow e^{iq\alpha}\left[\partial_\mu\phi + iq(\partial_\mu\alpha)\phi + iqA_\mu\phi\right]$

Thus, $D_\mu \rightarrow e^{iq\alpha}D_\mu$ provided $A_\mu \longrightarrow A_\mu - \partial_\mu\alpha$

$\Rightarrow$ A_μ has the properties of A_μ of the Maxwell's Eqns.

Recipe for Quantum Electrodynamics

Dirac Eqn: $(i\gamma^\mu\partial_\mu - m)\psi = 0$ ①

$(i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi}) = 0$ ②

$$\mathcal{L}_{DIRAC} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0 \ \ \& \ \ \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi = 0 \Rightarrow ①$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}i\gamma^\mu \ \ \& \ \ \frac{\partial\mathcal{L}}{\partial\psi} = -\bar{\psi}m$$

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right] = i\partial_\mu\bar{\psi}\gamma^\mu = \frac{\partial\mathcal{L}}{\partial\psi} = -m\bar{\psi}$$

$$\Rightarrow i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \Rightarrow ②$$

10/13/05

LECTURE $\overline{XI}$

Start with $\mathcal{L}_{free} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$

Replace ∂_μ with $D_\mu \equiv \partial_\mu - iqA_\mu$

$$\Rightarrow \mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - qA_\mu\bar{\psi}\gamma^\mu\psi$$

$q\bar{\psi}\gamma^\mu\psi$ is the conserved electromagnetic current & A_μ is the vector potential of E&M.

We now have kinetic energy term for e^+ & e^-, mass term for e^+, e^-, & interaction term between photons & electrons

We need the kinetic energy term for photons:

$$\mathcal{L} = -\frac{1}{4}\underbrace{(\partial_\nu A_\mu - \partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu)}_{\frac{1}{2}(E^2+B^2)} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is invariant under gauge transf^n.

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - qA_\mu\bar{\psi}\gamma^\mu\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Note: $m^2 A^\mu A_\mu$ terms breaks invariance $\Rightarrow$ photons must be massless!

<u>RECAP</u>: - Formulated Dirac Eq^n $\Rightarrow$
- Eq^n of motion of massive spin 1/2 particles & antiparticles.
- Global phase transf^n reveals conserved current
- local phase invariance reveals new vector field & interaction term with conserved current
- finally add photon kinetic energy term

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LECTURE XI

Why is $\bar{\psi}\gamma^\mu\psi A_\mu = j^\mu A_\mu$ an interaction term?

Consider a scalar field and a static charge density.

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right] - \phi\rho$$

$$\Longrightarrow \partial_\mu\partial^\mu\phi - m^2\phi = \rho$$

Consider a point source $\rho \equiv g\,\delta(\vec{x})$

$$\text{Then } (\nabla^2 + m^2)\phi = g\,\delta(\vec{x})$$

Introduce $\tilde{\phi}(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\phi(\vec{x})$

$$\Longrightarrow \tilde{\phi}(\vec{k}) = \frac{g}{(2\pi)^{3/2}}\frac{1}{k^2+m^2}$$

$\frac{1}{k^2+m^2}$ is the propagator which becomes $\frac{1}{k^2}$ for $m=0$

$$\phi(x) = \frac{g}{4\pi}\frac{e^{-mr}}{r} \Longrightarrow \text{infinite range } 1/r \text{ potential for } m=0$$

In 1934, Yukawa proposed that ϕ is a meson field produced by source nucleons.

RECAP: In a Quantum field Theory:

- Interaction term $\phi\rho$ or $j^\mu A_\mu$ denotes the emission or absorption of force quanta
- This gives rise to the potential term
- This leads to a force or interaction

Thus: Local phase Invariance $\Longrightarrow$ Force.

10/18/05

LECTURE XII

Having motivated $\mathcal{L}_{QED}$, we now derive Feynman rules, which allow us to carry out scattering calculations.

Example from non-relativistic perturbation theory:

Let $H_0 \phi_n = E_n \phi_n$ with $\int \phi_m^* \phi_n d^3x = \delta_{mn}$

Add a time dependent potential:

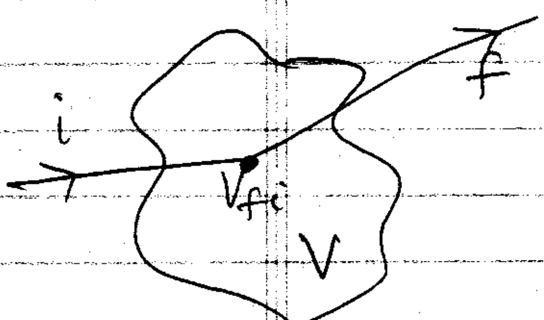
$$H_0 + V(\vec{x},t) = i\,\partial\psi/\partial t$$

$$\psi = \sum_n a_n(t)\,\phi_n(\vec{x})\,e^{-iE_n t}$$

One obtains: $$\frac{da_f}{dt} = -i \sum_n a_n(t) \int \phi_f^* V_{fn} \phi_n d^3x \, e^{-i(E_f - E_n)t}.$$

This is a recursive relation.

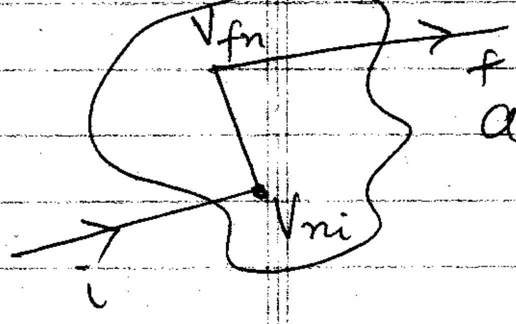
a_f for a given final state is related M_{fi} in Fermi's Golden Rule.



FIRST ORDER

$$a_f \equiv T_{fi} = i \int d^4x \, \phi_f^* V \phi_i$$

SECOND ORDER



$$a_f = \sum_{n \neq i} \frac{V_{fn} V_{ni}\, \delta(E_f - E_i)}{E_i - E_n}$$

$\overrightarrow{\text{XII}}$ LECTURE

Need free particle solns to Dirac Hamiltonian

$$\psi = a\, u(p_\mu)\, e^{i x\cdot p}$$

$$p \equiv (E, \vec{p}) \qquad x\cdot p = x_\mu p^\mu$$

$$\partial_\mu \psi = -i p_\mu a e^{i x\cdot p} u$$

$$\Longrightarrow (\gamma^\mu p_\mu - m) u = 0 \quad (\text{Momentum-Space Dirac Eqn})$$

Let $u \equiv \begin{pmatrix} u_A \\ u_B \end{pmatrix} \Rightarrow \gamma^\mu p_\mu - m = \begin{pmatrix} E-m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E-m \end{pmatrix}$

$$\Longrightarrow u_A = \frac{\vec{p}\cdot\vec{\sigma}}{E-m} u_B \qquad u_B = \frac{\vec{p}\cdot\vec{\sigma}}{E+m} u_A$$

One set of solutions:

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ p_z \\ \frac{p_x + i p_y}{E+m} \end{pmatrix} \quad u^2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} \quad u^3 = \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + i p_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \text{ \& } u^4$$

u^i are not eigenstates of Σ_z unless $\vec{p}$ is along z-axis.
$\Rightarrow$ Then u^1 is e^- with spin "up" & u^2 is e^- with spin "down".

Rewrite negative energy states:

$$u^{3,4}(-E, -\vec{p}) \equiv v^{2,1}(E, \vec{p})$$

$$(\gamma^\mu p_\mu - m) u = 0$$

$$(\gamma^\mu p_\mu + m) v = 0$$

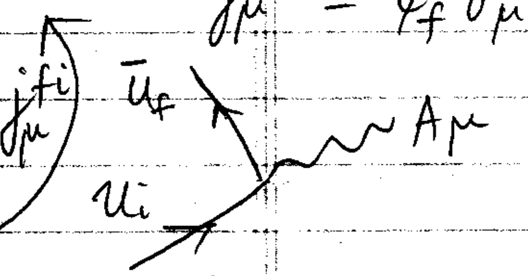
10/18/05

LECTURE XII

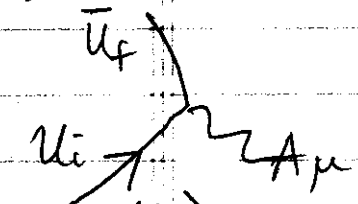
Now we introduce $V = -e\gamma_\mu A^\mu$

$$T_{fi} = -i\int \psi_f^\dagger V \psi_i d^4x = ie\int \bar{\psi}_f A^\mu \psi_i d^4x = i\int j_\mu^{fi} A^\mu d^4x$$

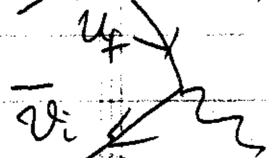
$$j_\mu^{fi} = \bar{\psi}_f \gamma_\mu \psi_i = -e\bar{u}_f \gamma_\mu u_i e^{-i(p_f - p_i)\cdot x}$$



electron emitting a photon.



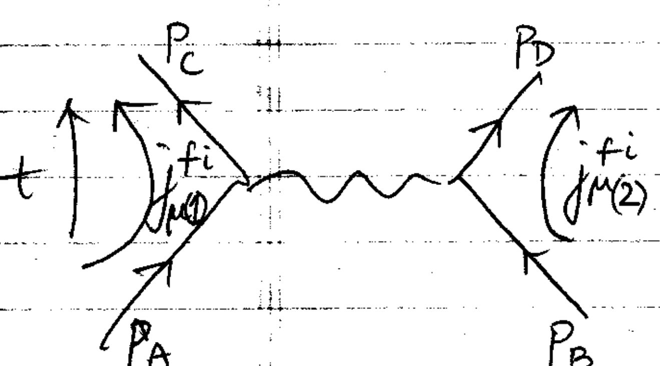
electron absorbing a photon



positron absorbing a photon.

These processes do not satisfy energy-momentum conservation. Now, let A_μ be produced by a second charged fermion.

$\Longrightarrow \partial_\mu F^{\mu\nu} = j^\mu_{(2)}$ & since A_μ can go to $A_\mu + \partial_\mu \Lambda$, choose $\partial_\mu A^\mu = 0 \Longrightarrow \partial_\mu \partial^\mu A^\nu = j^\nu_{(2)}$



$$j^\mu_{(2)} = \bar{u}(P_D)\gamma^\mu u(P_B) e^{-i(P_D - P_B)\cdot x}$$

$$P_D - P_B \equiv q$$

Since $\partial_\mu \partial^\mu e^{iq\cdot x} = -q^2 e^{iq\cdot x}$,

$$A_\mu = -\frac{1}{q^2} j^\mu_{(2)}$$

10/18/05

LECTURE XII

Thus, $T_{fi} = -i \int j_\mu^{(1)} \left(-\frac{1}{q^2}\right) j^\mu_{(2)} \, d^4x$.

Use Dirac δ-function.

$$\delta(x) = \begin{matrix} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{matrix} \quad \& \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

$$\Longrightarrow \int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$$

If $\phi(x) = \frac{1}{\sqrt{2\pi}} \int dp\, e^{-ipx} \tilde{\phi}(p)$ & $\tilde{\phi}(p) = \frac{1}{\sqrt{2\pi}} \int dx\, e^{ipx} \phi(x)$

If $\tilde{\phi}(p) = \delta(p)$ then $\phi(x) = \frac{1}{\sqrt{2\pi}}$

Thus $\delta(p) = \frac{1}{2\pi} \int dx\, e^{ipx}$

$$\Longrightarrow \int d^4x\, e^{i(p_A - p_C + p_B - p_D)\cdot x} = (2\pi)^4 \delta^4(p_A - p_C + p_B - p_D)$$

$$\Longrightarrow T_{fi} = \left[-i \left(-e \bar{u}_C \gamma_\mu u_A\right) \left(\frac{-1}{q^2}\right) \left(-e \bar{u}_D \gamma^\mu u_B\right) \right] \times (2\pi)^4 \delta^4(p_A + p_B - p_C - p_D)$$

The quantity in [ ] is M_{fi}

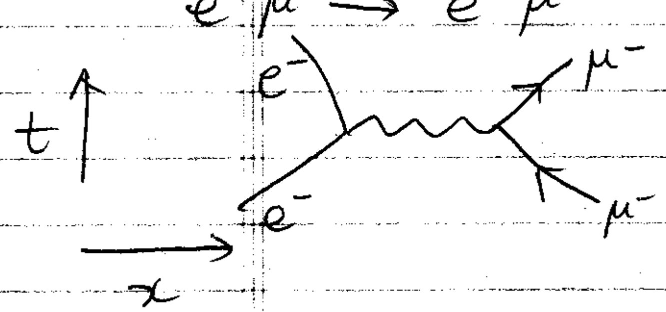
10/18/05

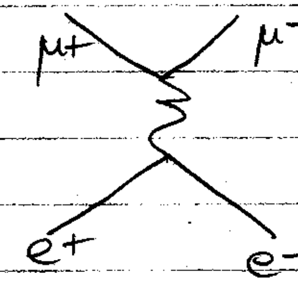
LECTURE XII

FEYNMAN RULES.

I Draw all possible diagrams taking initial state to final state.

$e^- \mu^- \longrightarrow e^- \mu^-$ $\qquad\qquad$ $e^+ e^- \longrightarrow \mu^+ \mu^-$





II Label all incoming & outgoing 4-momenta:

- $P_1, P_2 \cdots P_n$ and corresponding spins $S_1, S_2 \cdots S_n$.
- Internal 4-momenta $q_1 \cdots q_n$.
- Assign arrows: external fermions go "forward"
- external anti-fermions go "backward"
- internal fermions go in direction of 'flow'
- external photons go "forward"
- Internal photon lines are arbitrary

III External lines contribute following factors:

Fermions { Incoming: u ; Outgoing: $\bar{u}$ }

Anti-fermions { Incoming: $\bar{v}$; Outgoing: v }

Photons { Incoming: ϵ^μ ; Outgoing: $\epsilon^{\mu *}$ }

$A_\mu = a e^{ip\cdot x} \epsilon_\mu^{(s)}$

$\epsilon_\mu^{(s)} p^\mu = 0$

$s = 1, 2$

10/18/05

LECTURE XII

IV Vertex factors: $i g_e \gamma^{\mu}$

$$g_e = e\sqrt{\frac{4\pi}{\hbar c}} = \sqrt{4\pi\alpha}$$

V Propagator: photons: $-\frac{i g_{\mu\nu}}{q^2}$

electrons: $\frac{i(\gamma^{\mu} q_{\mu} + m)}{q^2 - m^2}$

VI For each vertex; add a factor

$$(2\pi)^4 \delta^4(K_1 + K_2 + K_3)$$

K_1, K_2 & K_3 are 4-vectors; Sign convention: outward = -ve

(NOTE: This is actual direction; opposite for antifermions!)

VII For each internal momentum q_i, write

$\frac{d^4 q_i}{(2\pi)^4}$ & integrate over d^4x.

VIII Cancel the δ-function:

The result will include a factor

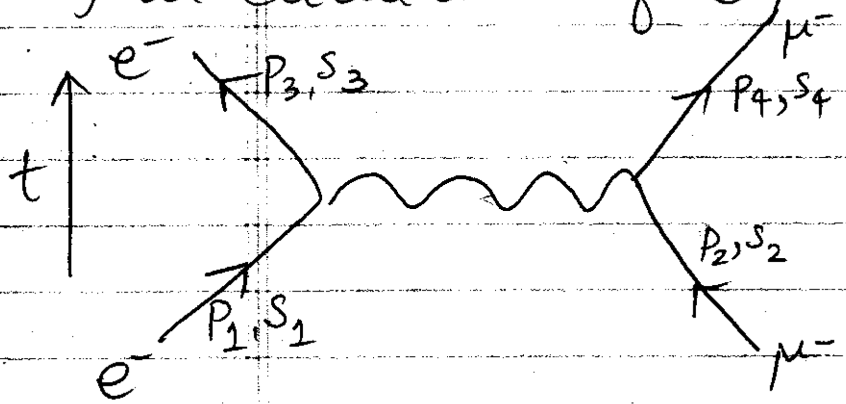
$$(2\pi)^4 \delta(P_1 + P_2 \ldots - P_n)$$

reflecting overall energy-momentum conservation.

$\Rightarrow$ Remove this factor & what remains is $-iM$.

LECTURE XIII

Full calculation of $e^-\mu^- \longrightarrow e^-\mu^-$



Proceed backward along each fermion line.

$$\int \left[\bar{u}^{s_3}(p_3)(ig_e\gamma^\mu)u^{s_1}(p_1)\right] \frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}^{s_4}(p_4)(ig_e\gamma^\nu)u^{s_2}(p_2)\right] \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta(p_1-p_3-q)(2\pi)^4\delta^4(p_2+q-p_4)$$

$$\swarrow$$

$$(2\pi)^4\delta^4(p_2+p_1-p_3-p_4)\frac{ig_e^2}{(p_1-p_3)^2}\left[\bar{u}^{s_3}(p_3)\gamma^\mu u^{s_1}(p_1)\right]\left[\bar{u}^{s_4}\gamma_\mu u^{s_2}(p_2)\right]$$

$$\Longrightarrow M = \frac{-g_e^2}{(p_1-p_3)^2}\left[u(3)\gamma^\mu u(1)\right]\left[u(4)\gamma_\mu u(2)\right]$$

M is just a number (1x1 matrix)

Golden Rule for Scattering: $1+2 \longrightarrow 3+4 \cdots n$

$$d\sigma = |M|^2 \frac{S}{4\sqrt{(p_1\cdot p_2)^2-(m_1m_2)^2}}\left[\frac{d^3\vec{p}_3}{(2\pi)^3 2E_3}\cdots\frac{d^3\vec{p}_n}{(2\pi)^3 2E_n}\right]$$

$S = 1/j!$ for each group of j identical particles

10/20/05

LECTURE $\overline{XIII}$

- Simplify form of $d\sigma$ for the special case of $1+2 \to 3+4$
- Integrating over $|P_3|$ & $\vec{P_4}$ gives the probability of observing particle 3 into a solid angle $d\Omega$.

(1) $$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{S\,|M|^2}{(E_1+E_2)^2} \frac{|P_f|}{|P_i|}$$

in the center of mass frame.

(2) $$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi m_B)^2} |M|^2$$

in the lab frame with $m_B \gg m_A$

- Calculation of M_{fi} can be simplified by summing over initial and final state polarizations.
- If one starts with unpolarized particles & do not observe polarization in the final state; then one must <u>Average</u> over intial spins & <u>Sum</u> over final spins

$\Rightarrow$ Specific Algorithm for summing spins.

Start with $M = -\dfrac{g_e^2}{(P_1-P_3)^2} [\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)]$

$$|M|^2 = MM^* = \frac{g_e^4}{(P_1-P_3)^4} [\bar{u}(3)\gamma^\mu u(1)][\bar{u}(4)\gamma_\mu u(2)][\bar{u}(3)\gamma^\nu u(2)]^*[\bar{u}(4)\gamma_\nu u(2)]^*$$

Need to compute $[\bar{u}(a)\Gamma_1 u(b)][\bar{u}(a)\Gamma_2 u(b)]^*$

10/20/05

LECTURE XIII

$$[\bar{u}(a)\Gamma_2 u(b)]^* = [u(a)^\dagger \gamma^0 \Gamma_2 u(b)]^\dagger = u(b)^\dagger \Gamma_2^\dagger \gamma^{0\dagger} u(a)$$
$$= u(b)^\dagger \gamma^0 \gamma^0 \Gamma_2^\dagger \gamma^0 u(a) = \bar{u}(b)\bar{\Gamma}_2 u(a)$$

where $\bar{\Gamma}_2 = \gamma^0 \Gamma_2^\dagger \gamma^0$.

Let $G \equiv [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(b)\bar{\Gamma}_2 u(a)]$

Then $\sum_{b\,\text{spins}} G = \bar{u}(a)\,\Gamma_1 \left\{ \sum_{S_b} u^{S_b}(P_b)\,\bar{u}^{S_b}(P_b) \right\} \bar{\Gamma}_2\, u(a)$

We use $\sum_{S_1 S_2} u^{S_1}(p)\,\bar{u}^{S_2}(p) = \gamma^\mu p_\mu + m$

Then $\sum_{a\,\text{spins}} \sum_{b\,\text{spins}} G = Tr\left[\Gamma_1(\gamma^\mu P_{a\mu}+m)\,\bar{\Gamma}_2(\gamma^\mu P_{b\mu}+m)\right]$

For μ-e scattering, $\Gamma_2 = \gamma^\nu \Longrightarrow \bar{\Gamma}_2 = \gamma^0\gamma^{\nu\dagger}\gamma^0 = \gamma^\nu$

Thus, $\langle |M|^2 \rangle = \sum\sum_{\text{spins}} |M|^2/4$

$$= \frac{g_e^4}{4(P_1-P_3)^4}\, Tr\left[\gamma^\mu(\not{P}_1+m)\gamma^\nu(\not{P}_3+m)\right] Tr\left[\gamma_\mu(\not{P}_2+M)\gamma_\nu(\not{P}_4+M)\right]$$

$\not{P} \equiv \gamma^\mu P_\mu$, $m \equiv m_e$, $M \equiv m_\mu$

Let us evaluate the first Tr: $Tr[\gamma^\mu(\not{P}_1+m)\gamma^\nu(\not{P}_3+m)]$

$$= Tr(\gamma^\mu\not{P}_1\gamma^\nu\not{P}_3) + m[Tr(\gamma^\mu\not{P}_1\gamma^\nu)] + Tr(\gamma^\mu\gamma^\nu\not{P}_3) + m^2 Tr(\gamma^\mu\gamma^\nu)$$

Use following Trace identities

$$Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$$

$$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

Trace of odd number of γ matrices is 0

10/20/05

LECTURE XIII

$$\mathrm{Tr}(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3) = p_{1\lambda} p_{3\sigma} \mathrm{Tr}(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma)$$

$$= p_{1\lambda} p_{3\sigma} \cdot 4 (g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu})$$

$$= 4 \left[(p_1^\mu p_3^\nu - g^{\mu\nu}(p_1 \cdot p_3) + p_3^\mu p_1^\nu) \right]$$

$\Longrightarrow$

$$\mathrm{Tr}[\gamma^\mu (\not{p}_1 + m) \gamma^\nu (\not{p}_3 + m)] = 4 [p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}(m^2 - p_1 \cdot p_3)]$$

Similarly

$$\mathrm{Tr}[\gamma_\mu (\not{p}_2 + M) \gamma_\nu (\not{p}_4 + M)] = 4 [p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g_{\mu\nu}(M^2 - p_2 \cdot p_4)]$$

$$\langle |M|^2 \rangle = \frac{8 g_e^4}{(p_1 - p_3)^4} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_3 \cdot p_2) - M^2 (p_1 \cdot p_3) - m^2 (p_2 \cdot p_4) + 2 m^2 M^2 \right]$$

Choose to work in LAB frame.

$p_1 = (E, \vec{p}_1)$ $\quad p_2 = (M, 0)$: Assume negligible recoil

$\Longrightarrow$ $|p_1| = |p_3| = |p|$, $\quad p_3 \equiv (E, \vec{p}_3)$, $\quad p_4 \equiv (M, 0)$

$$(p_1 - p_3)^2 = -(\vec{p}_1 - \vec{p}_3) \cdot (\vec{p}_1 - \vec{p}_3) = -2|p|^2 - 2|p|^2 \cos\theta = -4|p|^2 \sin^2\theta/2$$

$$p_1 \cdot p_3 = E^2 - \vec{p}_1 \cdot \vec{p}_3 = 2|p|^2 \sin^2\theta/2 + m^2$$

$$(p_1 \cdot p_2) = (p_3 \cdot p_4) = (p_3 \cdot p_2) = (p_1 \cdot p_4) = ME$$

$$p_2 \cdot p_4 = M^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi M)^2} \langle |M|^2 \rangle$$

$$\frac{8 g_e^4}{(4|p|^2 \sin^2\theta/2)^2} \left[M^2 E^2 + M^2 E^2 - M^2 (2|p|^2 \sin^2\theta/2 + m^2) - m^2 M^2 + 2 m^2 M^2 \right]$$

$$= \langle |M|^2 \rangle$$

10/20/05

LECTURE XIII

$$\langle |M|^2 \rangle = \frac{g_e^4 M^2}{|P|^4 \sin^4\theta/2}\left[m^2 + |P|^2\cos^2\theta/2\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar^2 c^2)}{4|P|^4\sin^4\theta/2}\left[m^2 + |P|^2\cos^2\theta/2\right]$$

If $|P| << m \Rightarrow$ Rutherford scattering!

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 m^2(\hbar^2 c^2)}{4|P|^4\sin^4\theta/2}$$

If $|P| >> m$, then;

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar^2 c^2)}{4|P|^2\sin^4\theta/2}\cos^2\theta/2$$

Use $q^2 = 4E^2\sin^2\theta/2$,

Then $$\frac{d\sigma}{d\Omega} = \frac{\alpha^2(\hbar^2 c^2)E^2}{q^4}\cos^2\theta/2$$

Mott Scattering.

10/25/05

LECTURE XIV

Continue discussion of $e^- \mu^- \to e^- \mu^-$.
- now assume $m=0$ but M can recoil $\Rightarrow$ results can carry over to $ep \to ep$

$$\langle |M|^2 \rangle = \frac{8 g_e^4}{(P_1-P_3)^4}\left[(P_1\cdot P_2)(P_3\cdot P_4)+(P_2\cdot P_3)(P_1\cdot P_4)-(P_1\cdot P_3)M^2\right]$$

$$P_1 \equiv (E,\vec{P}) \quad P_2 \equiv (M,0) \quad P_3 \equiv (E',\vec{P}') \quad P_4 = P_1+P_2-P_3$$

$$P_1^2 = 0 \quad P_3^2 = 0 \quad E = |\vec{P}| \quad E' = |\vec{P}'|$$

$$(P_1-P_3)\cdot(P_1-P_3) = q^2 = -2(P_1\cdot P_3) = -2EE'(1+\cos\theta) = -4EE'\sin^2\theta/2$$

$$(P_3\cdot P_4) = (P_3\cdot P_1)+(P_3\cdot P_2) \qquad (P_1\cdot P_4) = (P_1\cdot P_2)-(P_1\cdot P_3)$$

$$\Rightarrow \frac{8 g_e^4}{q^4}\left[-\frac{q^2}{2}\left\{(P_1\cdot P_2)-(P_2\cdot P_3)-M^2\right\}+2(P_1\cdot P_2)(P_2\cdot P_3)\right]$$

$$P_1\cdot P_2 = ME \qquad P_2\cdot P_3 = ME'$$

$$\Rightarrow \frac{8 g_e^4}{q^4}\left[-\frac{q^2}{2}\left\{ME-ME'-M^2\right\}+2M^2EE'\right]$$

$$= \frac{8 g_e^4}{q^4}\left[2M^2EE'\cos^2\theta/2 - \frac{q^2}{2M}\left\{M^2(E-E')\right\}\right]$$

$$q = P_1-P_3 = P_4-P_2 \Rightarrow P_4^2 = (q+P_2)^2 = q^2+2P_2\cdot q+P_2^2$$

$$\Rightarrow q^2 = -2P_2\cdot q = -2M(E-E')$$

$$\Rightarrow E-E' = -q^2/2M = 4EE'/2M \sin^2\theta/2$$

$$\Rightarrow \langle |M|^2 \rangle = \frac{8 g_e^4}{q^4}\, 2M^2EE'\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right]$$

LECTURE XIV

For fixed target scattering with recoil:

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi M}\right)^2 \frac{E'^2}{E^2} \langle |M|^2 \rangle$$

$$= \frac{\alpha^2}{4E^2 \sin^4\theta/2} \frac{E'}{E} \left[\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

The first term is the Mott term.
The second term is a new term due to the target magnetic moment. Let us see why?

Suppose the electron had charge but no spin.
Then $(\partial_\mu \partial^\mu + m^2)\phi = -V\phi$ where V contains A_μ

$$V \propto A^\mu j_\mu^{fi} \equiv ie\,(\phi_f^* \partial_\mu \phi_i - \partial_\mu \phi_f^* \phi_i) A^\mu$$

$$j_\mu^{fi} \propto (p_i + p_f)_\mu e^{i(p_f - p_i)\cdot x}$$

For spin 1/2, $j_\mu^{fi} \propto \bar{u}_f \gamma^\mu u_i = \frac{1}{2m} \bar{u}_f \left[(p_f+p_i)^\mu + i\sigma^{\mu\nu}(p_f-p_i)_\nu\right] u_i$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. The nonrelativistic limit of $\sigma^{\mu\nu}(p_f-p_i)_\nu = \vec{\sigma}\cdot\vec{B}$.

Thus: magnetic moment $= -e/2m$
$\Rightarrow$ predict ~~for~~ magnetic moment of a fundamental fermion.

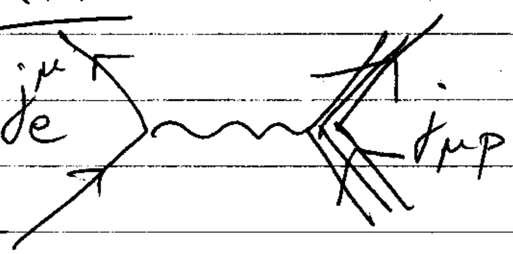
10/25/05

LECTURE XIV

$e^- p \to e^- p$

$j^\mu_e = -e\bar{u}(p_3)\gamma^\mu u(p_1)e^{iq\cdot x}$

$j^\mu_{p} = e\bar{u}(p_4)\,\Gamma_\mu\, u(p_2)e^{iq\cdot x}$



$$\Gamma_\mu = F_1(q^2)\gamma^\mu + \frac{K}{2M}F_2(q^2)\,i\sigma^{\mu\nu}q_\nu$$

F_1 & F_2 are functions of q^2, the only independent Lorentz scalar available at the proton vertex.

As $q \to 0$, proton becomes a dirac particle

$\Longrightarrow F_1(0) = F_2(0) = 1$

Proton magnetic moment $= 1 + K/2M \Rightarrow K = +1.79$

For the neutron, $F_1(0) = 0$ & $F_2(0) = -1.91$

$$\frac{d\sigma}{d\Omega} \propto \cos^2\theta/2\left[\left(F_1^2 - \frac{K^2q^2}{4M^2}\right)\right] + \sin^2\theta/2\left[\frac{-q^2}{2M}(F_1 + KF_2)^2\right]$$

$F_1 \cdot F_2$ interference terms are hard to measure.

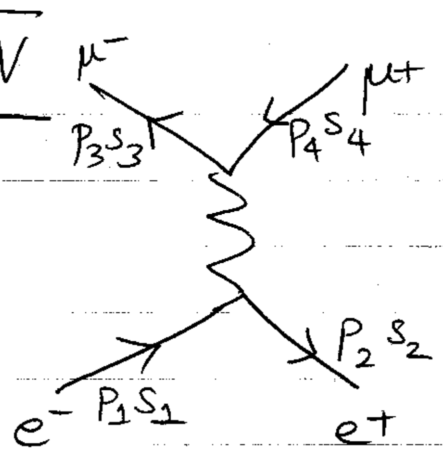
Instead define $G_E = F_1 + K\tau F_2 \quad G_M = F_1 + KF_2 \quad \tau = q^2/4M^2$

$$\frac{d\sigma}{d\Omega} \propto \cos^2\theta/2\left[\frac{G_E^2 + \tau G_M^2}{1+\tau}\right] + \sin^2\theta/2\left[2\tau G_M^2\right]$$

One can measure G_E & G_M by varying Q^2 & θ

LECTURE XIV

$e^+e^- \longrightarrow \mu^+\mu^-$



$$M = (2\pi)^4 \int \left[\bar{u}^{s_3}(P_3)(ig_e\gamma^\mu)v^{s_4}(P_4)\right] \frac{-ig_{\mu\nu}}{q^2}\left[\bar{v}^{s_2}(P_2)(ig_e\gamma^\nu)u^{s_1}(P_1)\right]$$
$$d^4q\, \delta(P_1+P_2-q)\,\delta(q-P_3-P_4)$$

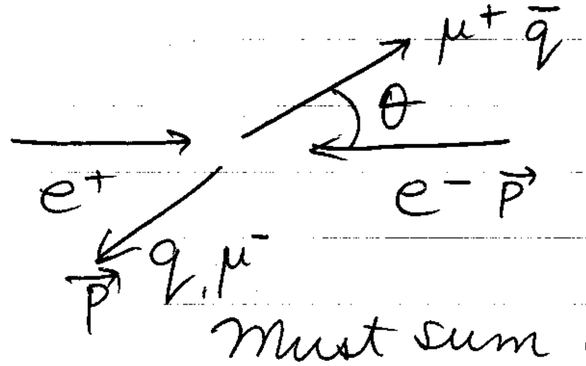
$$= 2\pi^4 \delta^4(P_1+P_2-P_3-P_4)\frac{ig_e^2}{(P_1+P_2)^2}\left[\bar{u}(3)\gamma^\mu v(4)\right]\left[\bar{v}(2)\gamma_\mu u(1)\right]$$

Must evaluate $\mathrm{Tr}\left[\gamma^\mu(\not{P}_4-M)\gamma^\nu(\not{P}_3+M)\right]\mathrm{Tr}\left[\gamma^\mu(\not{P}_1+m)\gamma^\nu(\not{P}_2-m)\right]$

For $E \gg M$: $\langle |M|^2 \rangle = g_e^4\{1+\cos^2\theta\}$

$$\left(\frac{d\sigma}{d\Omega}\right)_{COM} = \frac{\alpha^2}{16E^2}(1+\cos^2\theta) \quad \text{with}$$

$P_1 = (E, \vec{P})$, $P_2 = (E, -\vec{P})$, $P_3 = (E, \vec{P}')$, $P_4 = (E, -\vec{P}')$



For $e^+e^- \longrightarrow q\bar{q}$

$\Longrightarrow ig_e\gamma^\mu = iQg_e\gamma^\mu$

Must sum over all species of quarks possible

$$R = \frac{\sigma(e^+e^- \longrightarrow q\bar{q})}{\sigma(e^+e^- \longrightarrow \mu^+\mu^-)} = \sum Q_i^2$$

10/24/05

LECTURE XIV

If total beam energy is less than 3 GeV, then there are 3 kinds of quarks, u, d, s.

$$\sum Q_i^2 = \left(\frac{2}{3}\right)^2 + 2\left(\frac{1}{3}\right)^2 = \frac{2}{3}$$ But observed R is 2!!

Direct evidence that there are 3 colors!

Above $E_{beam} = 3.15\,GeV$, $R = 10/3 \Rightarrow$ new charmed quark of charge $2/3$.

Above $E_{beam} = 5\,GeV$, $R = 11/3 \Rightarrow$ b quark.
$R = 11/3$ until $\sim 70\,GeV \Rightarrow$ no new light quarks.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}(1+\cos^2\theta).$$ Simple explanation for angular distribution

Consider the operators: $P_L \equiv \frac{1}{2}(1-\gamma^5)$ $\quad P_R \equiv \frac{1}{2}(1+\gamma^5)$

$P_L + P_R = I \qquad P_L P_R = 0 \qquad P_L P_L = P_L \qquad P_R P_R = P_R$

$$(1-\gamma^5)(1+\gamma^5) = 1 - \gamma^{5\,2} = 0$$
$$\frac{1}{4}(1-\gamma^5)^2 = \frac{1}{4}(1 - 2\gamma^5 + \gamma^{5\,2}) = \frac{1}{2}(1-\gamma^5)$$

Recall: if $|V\rangle = \sum_n |n\rangle\langle n|V\rangle$, define $|i\rangle\langle i| \equiv P_i$

Then $\sum P_i = I \qquad P_i P_j = \delta_{ij} P_j$

$\Rightarrow$ P_L & P_R form a complete set of projection operators

10/26/05

LECTURE XV

$e^+e^- \to \mu^+\mu^-$ $\qquad \dfrac{d\sigma}{d\Omega} = \dfrac{\alpha^2}{16E^2}(1+\cos^2\theta)$

(For $E \gg m, M$)

- Simple explanation for angular distribution

Consider the operators $P_L \equiv \frac{1}{2}(1-\gamma^5)$ $\quad P_R \equiv \frac{1}{2}(1+\gamma^5)$

$P_L + P_R = I \qquad P_L P_R = 0 \qquad P_L P_L = P_L \qquad P_R P_R = P_R$

$P_L P_R = (1-\gamma^5)(1+\gamma^5) = 1-(\gamma^5)^2 = 0$

$P_L P_L = \frac{1}{4}(1-\gamma^5)^2 = \frac{1}{4}(1-2\gamma^5+(\gamma^5)^2) = \frac{1}{2}(1-\gamma^5) = P_L$

Recall: if $|V\rangle = \sum_n |n\rangle\langle n|V\rangle$, define $|i\rangle\langle i| \equiv P_i$

with $\sum_i P_i = I$ & $P_i P_j = \delta_{ij} P_j$

$\Rightarrow$ P_L & P_R form a complete set of projection operators

What do they project? $\gamma^5 u(p) = \begin{bmatrix} \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & 0 \\ 0 & \frac{\vec{p}\cdot\vec{\sigma}}{E-m} \end{bmatrix} u$

Consider the $m=0$ case:

$\gamma^5 u = (\vec{p}\cdot\vec{\Sigma})u = \pm u \Rightarrow u$ is an eigenfn with eigenvalues $\pm 1 = h$

If $(\vec{p}\cdot\vec{\Sigma})u = +u \Longrightarrow \frac{1}{2}(1-\gamma^5)u = 0$

If $(\vec{p}\cdot\vec{\Sigma})u = -u \Longrightarrow \frac{1}{2}(1-\gamma^5)u = u$

$\Rightarrow P_L \equiv \frac{1}{2}(1-\gamma^5)$ project $h=-1$ state of $m=0$ fermion.

h is the <u>helicity</u> or handed-ness

LECTURE XV

P_L & P_R are projection operators or chirality

chirality $\Longleftrightarrow$ helicity iff $m=0$

$\vec{\Sigma}\cdot\vec{p}$ commutes with H i.e. conserved h but NOT LORENTZ INVARIANT

- P_L & P_R are Lorentz invariant but NOT CONSERVED
- u_L & u_R do not satisfy the Dirac Eqn.

A freely propagating u_R will develop an u_L component (unless $m=0$)

$u = u_L + u_R \qquad \bar{u} = \bar{u}_L + \bar{u}_R \qquad (u_L = P_L u \text{ etc})$

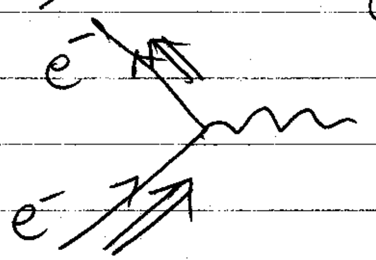
$\bar{u}_L = u_L^\dagger \gamma^0 = u^\dagger \frac{1}{2}(1-\gamma^5)\gamma^0 = \bar{u}\,\frac{1}{2}(1+\gamma^5)$

$(\gamma^{5\dagger} = \gamma^5, \quad \gamma^5\gamma^0 = -\gamma^0\gamma^5)$

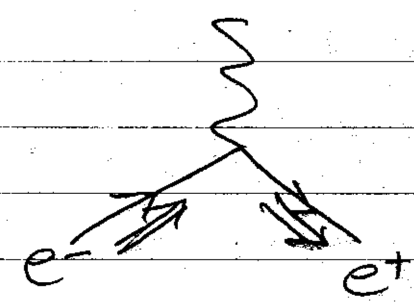
$$\bar{u}_L\gamma^\mu u_R = \frac{1}{4}\,\bar{u}(1+\gamma^5)\gamma^\mu(1+\gamma^5)u = \frac{1}{4}\,\bar{u}\gamma^\mu(1-\gamma^5)(1+\gamma^5)u = 0$$

$$\Rightarrow \bar{u}\gamma^\mu u = (\bar{u}_L + \bar{u}_R)\gamma^\mu(u_L + u_R) = \bar{u}_L\gamma^\mu u_L + \bar{u}_R\gamma^\mu u_R$$

$\Rightarrow$ helicity is conserved at each vertex.



$e^-_L\mu^- \to e^-_L\mu^-$



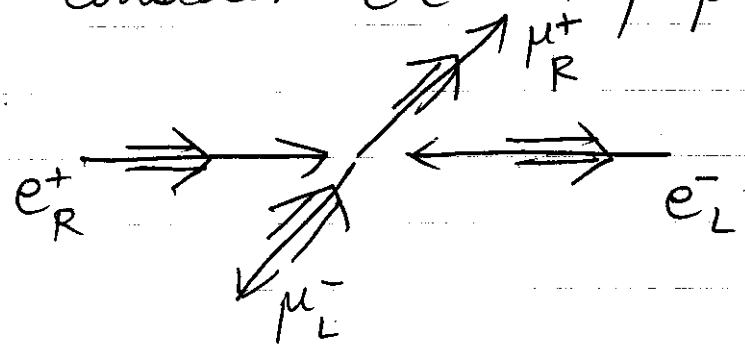
$e^-_L e^+_R \to \mu^+\mu^-$

$e^-_L e^+_L \not\to \mu^+\mu^-$

10/26/05

LECTURE XV

Consider $e^+e^- \longrightarrow \mu^+\mu^-$



If γ^* has spin +1 along a certain axis, what is the probability of finding ± 1 along an axis at angle θ ?

$\Rightarrow$ Matrix elements for finite rotations.

- The generator of rotations by angle θ about Y-axis is $\exp(i\theta J_y/\hbar)$

Define $d^j_{mm'} \equiv \langle j m' | \exp(i\theta J_y/\hbar) | j m \rangle$

For $j = 1/2$, $d^j_{mm'} \Rightarrow$

$$
\begin{array}{c}
m' \downarrow \quad \xrightarrow{\;m\;} \\
\begin{array}{cc}
\begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} & \begin{array}{c} +1/2 \\ -1/2 \end{array} \\
\begin{array}{cc} \quad +1/2 & \quad -1/2 \end{array} &
\end{array}
\end{array}
$$

For $j = 1$,

$$
\begin{array}{c}
m' \downarrow \quad \xrightarrow{\;m\;} \\
\begin{array}{cc}
\begin{pmatrix} \frac{1}{2}(1+\cos\theta) & \frac{\sin\theta}{\sqrt{2}} & \frac{1}{2}(1-\cos\theta) \\ -\frac{\sin\theta}{\sqrt{2}} & \cos\theta & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1}{2}(1-\cos\theta) & -\frac{\sin\theta}{\sqrt{2}} & \frac{1}{2}(1+\cos\theta) \end{pmatrix} & \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \\
\begin{array}{ccc} \quad 1 & \qquad 0 & \qquad -1 \end{array} &
\end{array}
\end{array}
$$

4 amplitudes:

$+1 \to +1$, $+1 \to -1$,
$-1 \to +1$, $-1 \to -1$.

$$\sum_4 d^2(\theta) = 2 \times \frac{1}{4}(1+\cos\theta)^2 + 2 \times \frac{1}{4}(1-\cos\theta)^2$$

$$= \boxed{1 + \cos^2\theta}$$

LECTURE XV

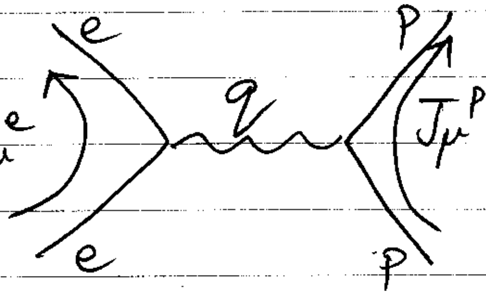
<u>ELECTROWEAK INTERACTIONS</u>: We will take a novel approach: We will first introduce the electroweak Lagrangian via local gauge invariance & then study quantitative aspects of weak interactions

Weak interactions $\Longrightarrow$ "long" lifetime

$\pi^- \to \mu^- \bar{\nu}_\mu : \tau_\pi \cong 2 \times 10^{-8} s$ $\quad \mu^- \to e^- \bar{\nu}_e \nu_\mu : \tau_\mu \cong 2.2 \times 10^{-6} s$

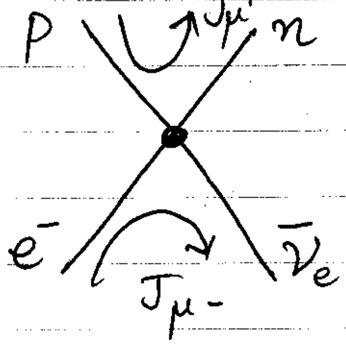
Fermi proposed a "current-current" Lagrangian to describe weak interactions.

Recall:



$$A_\gamma = e(\bar{u}_p \gamma^\mu u_p)\left(-\frac{1}{q^2}\right) e(\bar{u}_e \gamma_\mu u_e)$$
$$= -\frac{e^2}{q^2} (J^\mu)_p (j_\mu)_e$$

Consider $n \to p + e^- + \bar{\nu}_e$: 4-Fermi interaction.



$$A_W \equiv G\underbrace{(\bar{u}_n \gamma^\mu u_p)}_{J^{\mu+}}\underbrace{(\bar{u}_{\nu_e} \gamma_\mu u_e)}_{J_\mu^-}$$

This theory successfully dealt with nuclear & muon decay.

$J^{\mu+}, J_\mu^-$ are 4-vectors & A_W is a Lorentz scalar.

LECTURE XV

1956 : Weak interactions violate parity
: A_W has pseudo-scalar terms.

This can be accomplished by introducing an axial-vector current $\Longrightarrow$

$$A_W \sim C_V\, \bar{u}_{\nu_e} \gamma^\mu u_e + C_A\, \bar{u}_{\nu_e} \gamma^\mu \gamma^5 u_e$$

Change nomenclature $\Rightarrow$ $\bar{u}_{\nu_e} \equiv \bar{\nu}_e$

$$J_\mu^- = C_V\, \bar{\nu}_e \gamma_\mu e + C_A\, \bar{\nu}_e \gamma_\mu \gamma^5 e$$

Experiments showed that $C_V/C_A = -1 \Rightarrow$ Maximal $\Longrightarrow$ V−A universal interaction.

In 1950s, theorists began to see patterns that could lead to electroweak unification:

For example: 

$J_\mu^- = \bar{\nu}_e \gamma_\mu \left(\frac{1-\gamma^5}{2}\right) e$

$$\frac{\gamma_\mu}{2} - \frac{\gamma_\mu \gamma^5}{2} = \frac{\gamma_\mu}{2} + \frac{\gamma^5 \gamma_\mu}{2} \Longrightarrow \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu = \gamma_\mu\left(\frac{1-\gamma^5}{2}\right)$$

Thus, $\left(\frac{1-\gamma^5}{2}\right)^2 = \frac{1-2\gamma^5+(\gamma^5)^2}{4} = \frac{1-\gamma^5}{2} \quad \because (\gamma^5)^2 = 1$

Therefore: $\left(\frac{1+\gamma^5}{2}\right)\gamma_\mu\left(\frac{1-\gamma^5}{2}\right) = \gamma_\mu\left(\frac{1-\gamma^5}{2}\right)^2 = \gamma_\mu\left(\frac{1-\gamma^5}{2}\right)$

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LECTURE XV

$$(j^{\mu -})_e = \bar{\nu}_e \gamma_\mu \left(\frac{1-\gamma^5}{2}\right) e = \bar{\nu}_e \left(\frac{1+\gamma^5}{2}\right) \gamma_\mu \left(\frac{1-\gamma^5}{2}\right) e$$

$$u_L = \left(\frac{1-\gamma^5}{2}\right) u \qquad u_R = \left(\frac{1+\gamma^5}{2}\right) u$$

$$\bar{u}_L = u_L^{+} \gamma^0 = \bar{u} \left(\frac{1+\gamma^5}{2}\right)$$

$\Rightarrow (j_\mu^-)_e = \bar{\nu}_{e_L} \gamma_\mu e_L \Longrightarrow$ CHARGED CURRENT CONSISTS ONLY OF LEFT-CHIRAL STATES

$$j_\mu^{EM} = q \bar{e} \gamma_\mu e = q (\bar{e}_L + \bar{e}_R) \gamma_\mu (e_L + e_R)$$

But $\bar{e}_L \gamma_\mu e_R = \bar{e}_R \gamma_\mu e_L = 0$

$$\Longrightarrow \quad j_\mu^{EM} = q \bar{e}_L \gamma_\mu e_L + q \bar{e}_R \gamma_\mu e_R$$

i.e. the electromagnetic current consists of left- & right-handed states of equal strength.

But the weak-interaction current only couples to left-chiral states and does not couple at all to the right-chiral states $\Longrightarrow$ maximal parity-violation

NOTE:

- e_L & e_R do not represent different particles
- u_L & u_R do not satisfy the Dirac Eqn.