Completeness and Super-Valuations

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Abstract

This paper uses the notion of Galois-connection to examine the relation between valuation-spaces and logics. Every valuation-space gives rise to a logic, and every logic gives rise to a valuation space, where the resulting pair of functions form a Galois-connection, and the composite functions are closure-operators. A valuation-space (resp., logic) is said to be complete precisely if it is Galois-closed. Two theorems are proven. A logic is complete if and only if it is reflexive and transitive. A valuation-space is complete if and only if it is closed under formation of super-valuations.

Keywords

Logic, valuation-space, Galois-connection, closure-operator, completeness, super-valuation.

1. Summary

Given a formal language $\mathcal{L}$, define an **argument** on $\mathcal{L}$ to be an ordered pair Γ/α, where Γ is a subset of formulas of $\mathcal{L}$, and α is a formula of $\mathcal{L}$. Define a **logic** on $\mathcal{L}$ to be *any* collection L of arguments on $\mathcal{L}$.

Given a formal language $\mathcal{L}$, define a **valuation** on $\mathcal{L}$ to be any function that assigns exactly one truth-value (T or F) to every formula in $\mathcal{L}$. Define a **valuation-space** on $\mathcal{L}$ to be *any* collection V of valuations on $\mathcal{L}$.

A valuation υ is said to **refute** an argument Γ/α if $\upsilon(\gamma)=T$ for every γ in Γ and $\upsilon(\alpha)=F$, and υ is said to **confirm** Γ/α if υ does not refute Γ/α.

Every valuation-space V gives rise to an associated logic $\mathbb{L}(V)$, which consists of all V-valid arguments. An argument Γ/α is **V-valid** iff every valuation in V confirms Γ/α. Dually, every logic L gives rise to an associated valuation-space $\mathbb{V}(L)$, which consists of all L-consistent valuations. A valuation υ is **L-consistent** iff υ confirms every argument in L.

The pair of maps $V \mapsto \mathbb{L}(V)$ and $L \mapsto \mathbb{V}(L)$ are what Birkhoff calls a **Galois-connection**. Accordingly, the two composite maps, $V \mapsto \mathbb{V}(\mathbb{L}(V))$ and $L \mapsto \mathbb{L}(\mathbb{V}(L))$, are both *closure-operators*. We say that a logic L is **complete** precisely if L is Galois-closed [i.e., $\mathbb{L}(\mathbb{V}(L)) = L$], and we say that a valuation-space V is **complete** precisely if V is Galois-closed [i.e., $\mathbb{V}(\mathbb{L}(V)) = V$].

In this connection, we have two theorems.

Theorem 1: A logic L is complete if and only if it satisfies the following conditions.

(c1) $\alpha \in \Gamma \rightarrow \Gamma \Vdash \alpha$ [reflexivity]
(c2) $\Gamma \Vdash \forall\Delta \;\&\; \Delta \Vdash \alpha \;.\!\!\rightarrow \Gamma \Vdash \alpha$ [transitivity]

Here, we employ the following shorthand definitions.

$\Gamma \Vdash \alpha \;=_{df}\; \Gamma/\alpha$ is L-valid
$\Gamma \Vdash \forall\Delta \;=_{df}\; \forall\delta\{\delta \in \Delta \rightarrow \Gamma \Vdash \delta\}$

Theorem 2: A valuation-space V is complete if and only if V is closed under the formation of super-valuations [i.e., V contains every super-valuation over V].

For any subset W of valuations, the **super-valuation** associated with W is the valuation σ_W defined so that, for every formula α, $\sigma_W(\alpha)=T$ if $\upsilon(\alpha)=T$ for every υ in W, and $\sigma_W(\alpha)=F$ otherwise. When $\upsilon = \sigma_W$ for some $W \subseteq V$, we say that υ is a **super-valuation** over V.

2. Logics

What is *a logic*? In what follows, we take a minimalist approach. Minimally, a logic L involves a formal language $\mathcal{L}$[1] and a specification of which arguments in $\mathcal{L}$ are valid according to L. We take an argument to consist of zero-or-more premises and exactly-one conclusion.[2] These ideas are formally rendered as follows.

(d1) Let $\mathcal{L}$ be a formal language. Then an ***argument in*** $\mathcal{L}$ is an ordered pair $\langle\Gamma,\alpha\rangle$, where Γ is a subset of formulas of $\mathcal{L}$, and α is a formula of $\mathcal{L}$.

Note: henceforth, we write '$\langle\Gamma,\alpha\rangle$' simply as '$\Gamma/\alpha$', which we read colloquially as "Γ, therefore α".

(d2) Let $\mathcal{L}$ be a formal language. Then a ***logic*** on $\mathcal{L}$ is, by definition, *any* collection of arguments on $\mathcal{L}$.

(d3) When an argument Γ/α is an element of a logic L, we also say that Γ/α is **L-valid**.

In other words, a logic is simply identified by what arguments it claims are valid. Bear in mind our minimalist stance; *any* partition of arguments into valid ones and invalid ones counts *minimally* as a logic.

3. Logic, Entailment, and Consequence

Given a logic L, minimally understood as *any* collection of arguments on a formal language $\mathcal{L}$, we can define two associated notions, as follows.[3]

(d4) $\Gamma\Vdash\alpha \quad =_{\text{df}} \quad \Gamma/\alpha$ is L-valid [entailment relation][4]

(d5) $Cn(\Gamma) \quad =_{\text{df}} \quad \{\alpha : \Gamma\Vdash\alpha\}$ [consequence operator][5,6]

We read '$\Gamma\Vdash\alpha$' as "Γ logically entails α (in L)", and we read '$Cn(\Gamma)$' as "the logical consequences of Γ (in L).

Notice that the different bits of terminology are inter-related in the obvious simple way; in particular, the following are all equivalent.

(1) $\Gamma/\alpha \in L$ argument Γ/α is valid (in L)
(2) $\Gamma\Vdash\alpha$ Γ logically entails α (in L)
(3) $\alpha \in Cn(\Gamma)$ α is one of the logical consequences of Γ (in L)

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[1] The precise nature of $\mathcal{L}$ is unimportant to this paper. For our purposes, $\mathcal{L}$ may simply be identified with the set of its formulas, which are treated as "points" in an abstract space.

[2] See footnote 18.

[3] This particular turnstile, '$\Vdash$' is used for "generic" logical-entailment, which is distinguished from proof-theoretic entailment (the single turnstile '$\vdash$') and model-theoretic entailment (the usual double turnstile '$\models$').

[4] Technically, since 'L' appears as a free variable in the *definiens*, it should also appear in the *definiendum*. For example, the turnstile should be subscripted by 'L'. Its absence is simply a matter of notational conciseness; the basic idea is that entailment is always relative to a logic L, which is usually tacitly understood.

[5] The consequence operator should be subscripted by 'L'. See Footnote 4.

[6] The term 'consequence operator' has a *narrow* meaning as well. See note 15. We adopt a *minimal* reading.

4. Cases and Valuations

It is common to explain argument-validity, *at least informally*, in terms of the notion of *possibility*, the following being a fairly typical definition.

(v1) An argument is valid if and only if it is *impossible* for the premises to the true without the conclusion also being true.

Furthermore, it is common to explicate the relevant notion of possibility in terms of *admissible-cases*. Applying this idea to (v1), we obtain the following.

(v2) An argument is valid if and only if *there is no admissible-case* in which the premises are true but the conclusion is not true.

The precise nature of cases (admissible and inadmissible) is outside the scope of this paper, which is intentionally abstract.[7] For our purposes, a case may simply be identified with the set of formulas it makes true, and accordingly may be identified with a *valuation*, which is defined as follows.

(d6) Let $\mathcal{L}$ be a formal language. Then a **valuation** on $\mathcal{L}$ is, by definition, a function that assigns to every formula of $\mathcal{L}$ exactly one truth-value (T or F).[8]

Notice that the valuation corresponding to a case is simply its characteristic function.[9]

Next, the notion of *admissible-case* can be formally rendered via the notion of *valuation-space*, which is defined as follows.

(d7) Let $\mathcal{L}$ be a formal language. Then a **valuation-space** on $\mathcal{L}$ is, by definition, *any* subset of valuations on $\mathcal{L}$.

In particular, every subset of valuations (i.e., valuation-space) represents a particular choice as to which cases count as admissible.

5. The Move from Valuation-Spaces to Logics

So far, we have discussed two demarcation techniques.

(1) A logic demarcates the valid and invalid arguments.
(2) A valuation-space demarcates the admissible and inadmissible cases.

Our goal is investigate how these are connected.

One obvious connection has already been suggested – if we know what the admissible cases are, we know what the valid arguments are. In particular, an argument is valid if and only if there is no case in which the premises are all true but the conclusion is false. These ideas are formally rendered as

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[7] For a description of various ways in which the notion of (*admissible*) *cases* might be formally rendered, the reader is referred to Beall & Restall, "Logical Pluralism", *Australian Journal of Philosophy*, 78 (2000) 475-493 (The special Year 2000 issue on Logic).

[8] We construe validity simply to be truth-preservation, so for the sake of simplicity, we define falsity to be (mere) lack of truth.

[9] Officially, given a subset S of a prior-defined universe $\mathbb{U}$, the characteristic function associated with S is the function χ_S that assigns 1 (alternatively, T) to every element of S, and assigns 0 (alternatively, F) to every element of $\mathbb{U}$ not in S.

follows, where $\mathcal{L}$ is a formal language, Γ/α is an argument in $\mathcal{L}$, υ is a valuation on $\mathcal{L}$, and V is a valuation-space on $\mathcal{L}$.

(d8) υ **verifies** α $=_{\mathrm{df}}$ $\upsilon(\alpha)=\mathrm{T}$

(d9) υ **verifies** Γ $=_{\mathrm{df}}$ υ verifies *every* element of Γ

(d10) υ **falsifies** α $=_{\mathrm{df}}$ $\upsilon(\alpha)=\mathrm{F}$

(d11) υ **refutes** Γ/α $=_{\mathrm{df}}$ υ verifies Γ but falsifies α

(d12) υ **confirms** Γ/α $=_{\mathrm{df}}$ υ does not refute Γ/α

(d13) V **refutes** Γ/α $=_{\mathrm{df}}$ *some* element of V refutes Γ/α

(d14) Γ/α is V-**valid** $=_{\mathrm{df}}$ V does *not* refute Γ/α

(d15) $\mathbb{L}(V)$ $=_{\mathrm{df}}$ $\{\Gamma/\alpha : \Gamma/\alpha \text{ is } V\text{-valid}\}$

$\mathbb{L}(V)$, which consists of all V-valid arguments, is called the **logic associated with** valuation-space V.

6. The Move from Logics to Valuation-Spaces

Given a valuation-space V, there is a corresponding logic $\mathbb{L}(V)$. The obvious question is whether we can go the other direction. Given a logic L, is there a corresponding valuation-space $\mathbb{V}(L)$? The proposed answer is formally given as follows.

(d16) Let L be a logic on formal language $\mathcal{L}$, and let υ be a valuation on $\mathcal{L}$. Then υ is said to be L-**consistent** precisely if υ confirms every argument in L [alternatively, υ refutes no argument in L].

(d17) $\mathbb{V}(L)$ $=_{\mathrm{df}}$ $\{\upsilon : \upsilon \text{ is } L\text{-consistent}\}$

7. Moving Back and Forth between Logics and Valuation-Spaces – Galois-Connections

We have now set up a two-way path between valuation-spaces and logics. Given a valuation-space V, there is a corresponding logic $\mathbb{L}(V)$, which consists of all V-valid arguments. Conversely, given a logic L, there is a corresponding valuation-space $\mathbb{V}(L)$, which consists of all L-consistent valuations.

It is natural to consider what happens when we make "two-way trips". For example, we take a logic L, and form the associated valuation-space $\mathbb{V}(L)$, then take $\mathbb{V}(L)$ and form the associated logic $\mathbb{L}(\mathbb{V}(L))$. Conversely, we take a valuation-space V, and form the associated logic $\mathbb{L}(V)$, then take $\mathbb{L}(V)$ and form the associated valuation-space $\mathbb{V}(\mathbb{L}(V))$. The following depicts these processes.

$$V \curvearrowright \mathbb{L}(V) \curvearrowright \mathbb{V}(\mathbb{L}(V))$$

$$L \curvearrowright \mathbb{V}(L) \curvearrowright \mathbb{L}(\mathbb{V}(L))$$

As it turns out, the maps $V \mapsto \mathbb{L}(V)$ and $L \mapsto \mathbb{V}(L)$ form what Birkhoff[10] calls a *Galois-connection*, which may be defined as follows.

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[10] Garrett Birkhoff, "Lattice Theory", *American Mathematical Society Colloquium Publications*, Vol. 25, 1940.

(d18) Let $\langle P_1, \leqslant\rangle$ and $\langle P_2, \leqslant\rangle$ be partially-ordered sets, which is to say the two structures satisfy the following conditions.

(1) $x \leqslant x$ [reflexivity]
(2) $x \leqslant y \;\&\; y \leqslant z \;.\!\rightarrow x \leqslant z$ [transitivity]
(3) $x \leqslant y \;\&\; y \leqslant x \;.\!\rightarrow x = y$ [anti-symmetry]

Then a ***Galois-connection*** between these two partially-ordered sets is, by definition, a pair of functions $\langle f_1, f_2\rangle$ satisfying the following restrictions.

(1) f_1 maps P_1 into P_2
f_2 maps P_2 into P_1

(2) $x \leqslant y \;\rightarrow\; f_1(x) \geqslant f_1(y)$
$x \leqslant y \;\rightarrow\; f_2(x) \geqslant f_2(y)$

(3) $x \leqslant f_1(f_2(x))$
$x \leqslant f_2(f_1(x))$

8. A General Method for Manufacturing Galois-Connections

Oftentimes, the partially-ordered set under consideration is *a* collection of subsets of a given set S, ordered by set-inclusion, and sometimes, the collection of sets is *the* collection of *all* subsets of S – the power set of S [$\wp(S)$].

Consider two such partially-ordered sets – $\mathcal{P}_1 = \langle \wp(S_1), \subseteq\rangle$ and $\mathcal{P}_2 = \langle \wp(S_2), \subseteq\rangle$. It is then easy to produce a Galois-connection between $\mathcal{P}_1$ and $\mathcal{P}_2$ as follows. Take any relation R from S_1 to S_2, and define the functions f_1 [from $\wp(S_1)$ to $\wp(S_2)$], and f_2 from $\wp(S_2)$ to $\wp(S_1)$ – as follows.

(1) $f_1(X) \;=\; \{\, y : \forall x[x \in X \rightarrow xRy] \,\}$ (where X is any subset of S_1)
(2) $f_2(Y) \;=\; \{\, x : \forall y[y \in Y \rightarrow xRy] \,\}$ (where Y is any subset of S_2)

It is then routine to show that the pair $\langle f_1, f_2\rangle$ is a Galois-connection between $\mathcal{P}_1$ and $\mathcal{P}_2$.

As a fairly intuitive example, let S_1 be a set of individuals (e.g., natural numbers), and S_2 be a set of traits (e.g., being even, odd, prime, etc.), and let R be the instantiation relation [$iRt =_{\text{df}}$ individual i *has* trait t]. Then, for any subset X of individuals, $f_1(X)$ consists of all those traits instantiated by all individuals in X; and, for any subset Y of traits, $f_2(Y)$ consists of all individuals who instantiate every trait in Y.

An historically interesting example occurs when S_1 and S_2 are both the set of rational numbers, and the relation R is the numerical less-than-or-equal relation $\leqslant$. Then, for any subset X of rational numbers, $f_1(X)$ consists of all upper-bounds of X, and $f_2(X)$ consists of all lower-bounds of X.[11]

Finally, let us consider the example that is pertinent to this paper. In this case, S_1 is the set of *all* valuations over $\mathcal{L}$, and S_2 is the set of *all* arguments on $\mathcal{L}$, and R is the confirmation-relation. Then, for any subset of valuations (i.e., valuation-space) V, $f_1(V)$ consists of all arguments confirmed by every valuation in V – in other words, $f_1(V) = \mathbb{L}(V)$. Similarly, for any subset of arguments (i.e., logic) L, $f_2(L)$ consists of all valuations that confirm every argument in L; in other words, $f_2(L) = \mathbb{V}(L)$.

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[11] Why this example is interesting will be discussed shortly. See Footnote 12.

9. Closure Operators

We now have two partially-ordered sets, given as follows.

($\mathcal{P}_1$) the set of all valuation-spaces on $\mathcal{L}$, ordered by set inclusion;
($\mathcal{P}_2$) the set of all logics on $\mathcal{L}$, ordered by set-inclusion.

One can readily show that the pair of maps $\langle V \mapsto \mathbb{L}(V), L \mapsto \mathbb{V}(L)\rangle$, defined earlier, is a Galois-connection between $\mathcal{P}_1$ and $\mathcal{P}_2$. We can accordingly apply various theorems about Galois-connections to this pair of functions, including the following.

(t1) the map $V \mapsto \mathbb{V}(\mathbb{L}(V))$ is a closure operator on $\mathcal{P}_1$;
(t2) the map $L \mapsto \mathbb{L}(\mathbb{V}(L))$ is a closure operator on $\mathcal{P}_2$.

The following is the relevant definition.

(d19) Let $\langle P, \leqslant\rangle$ be a poset, and let cl be a function on P. Then cl is said to be a ***closure operator*** on $\langle P, \leqslant\rangle$ if and only it satisfies the following conditions for all x,y in P.

(c1) $x \leqslant cl(x)$
(c2) $cl(cl(x)) \leqslant cl(x)$
(c3) $x \leqslant y \ \rightarrow\ cl(x) \leqslant cl(y)$

10. Galois-Closure and Completeness

Once one has a closure operator, one has an affiliated notion of being *closed*, defined quite simply as follows.

(d20) Let cl be a closure operator on poset $\langle P, \leqslant\rangle$. Let x be an element of P.
Then x is said to be ***closed*** precisely if: $cl(x) = x$.

In the context of Galois-connections, the notion of *closure* becomes the notion of *Galois-closure*. And in the special case of logics and valuation spaces, the notion of *Galois-closure* becomes the notion of *completeness*, defined as follows.[12]

(d21) Let $\mathcal{L}$ be a formal language, and let L be a logic on $\mathcal{L}$.
Then L is said to be ***complete*** precisely if $\mathbb{L}(\mathbb{V}(L)) = L$.

(d22) Let $\mathcal{L}$ be a formal language, and let V be a valuation-space on $\mathcal{L}$.
Then V is said to be ***complete*** precisely if $\mathbb{V}(\mathbb{L}(V)) = V$.

Two questions naturally arise.

(q1) Under what conditions is a logic L complete?
(q2) Under what conditions is a valuation-space V complete?

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[12] In the case of the upper-bounds and lower-bounds of sets of rational numbers, the Galois-closed sets of rational numbers are the Dedekind cuts.

11. Completeness Theorem for Logics

In the present section, we consider question (q1), concerning what conditions are both necessary and sufficient in order for a logic L to be complete. The theorem is stated as follows.

THEOREM 1

A logic L is complete if and only if its associated entailment relation $\Vdash$ satisfies the following conditions.

(c1) $\alpha \in \Gamma \;\rightarrow\; \Gamma \Vdash \alpha$ [reflexivity]
(c2) $\Gamma \Vdash \forall\Delta \;\;\&\;\; \Delta \Vdash \alpha \;.\!\rightarrow\; \Gamma \Vdash \alpha$ [transitivity]

Note, '$\Gamma \Vdash \forall\Delta$' means that Γ entails every (element of) Δ.

Whereas (c1) is a natural generalization of *simple reflexivity* [$\alpha \Vdash \alpha$][13], (c2) is a natural generalization of *simple* transitivity [$\alpha \Vdash \beta \;\;\&\;\; \beta \Vdash \gamma \;.\!\rightarrow\; \alpha \Vdash \gamma$].

By way of proving Theorem 1, we state and prove two lemmas.

Lemma 1: Suppose logic L is transitive. Let Γ be any subset of formulas of $\mathcal{L}$. Define υ_Γ as follows:
$\upsilon_\Gamma(\alpha)=\mathrm{T}$ if $\Gamma \Vdash \alpha$ [$\alpha \in Cn(\Gamma)$]
$\upsilon_\Gamma(\alpha)=\mathrm{F}$ otherwise
Then υ_Γ is L-consistent, and hence $\upsilon_\Gamma \in \mathbb{V}(L)$.

Proof. Suppose otherwise – that υ_Γ is *not* L-consistent. Then there is an L-valid argument that υ_Γ refutes, call it Δ/β. Since υ_Γ refutes Δ/β, υ_Γ verifies Δ but falsifies β. Given the way υ_Γ is defined, this means that $\Gamma \Vdash \forall\Delta$. But Δ/β is L-valid, so $\Delta \Vdash \beta$. By hypothesis, L is transitive, so it follows that $\Gamma \Vdash \beta$, so by the definition of υ_Γ, $\upsilon_\Gamma(\beta)=\mathrm{T}$, which contradicts our earlier assumption that υ_Γ falsifies β.

Lemma 2: Suppose logic L is complete. Let Γ/α be an argument. Then Γ/α is L-valid if and only if $\mathbb{V}(L)$ does not refute Γ/α.

Proof. This is proven by noticing the following. (1) $\mathbb{L}(\mathbb{V}(L)) = L$, provided L is complete. (2) $\Gamma/\alpha \in L$ iff Γ/α is L-valid (by definition). (3) $\Gamma/\alpha \in \mathbb{L}(\mathbb{V}(L))$ iff $\mathbb{V}(L)$ does not refute Γ/α (by definition).

Proof of Theorem 1: A logic L is complete if and only if it is reflexive and transitive.

($\leftarrow$) Suppose L is reflexive and transitive. We wish to show that L is complete, which is to say that $\mathbb{L}(\mathbb{V}(L)) = L$. We already know that $L \subseteq \mathbb{L}(\mathbb{V}(L))$, given that the maps form a Galois-connection.[14] So we only need to show the converse – that $\mathbb{L}(\mathbb{V}(L)) \subseteq L$. Arguing contrapositively, suppose that argument Γ/α is not an element of L, which is to say that Γ/α is not L-valid, to show that Γ/α is not an element of $\mathbb{L}(\mathbb{V}(L))$, which is to say that Γ/α is refuted by $\mathbb{V}(L)$. Consider the valuation υ_Γ, which by Lemma 1 is in $\mathbb{V}(L)$. By its definition, υ_Γ verifies all the logical consequences of Γ. Since L is reflexive, every element of Γ is a logical consequence

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[13] It is customary to write '$\alpha \Vdash \beta$' in place of the official '$\{\alpha\} \Vdash \beta$', to write '$\alpha,\beta \Vdash \gamma$' in place of the official '$\{\alpha,\beta\} \Vdash \gamma$', etc.

[14] That $L \subseteq \mathbb{L}(\mathbb{V}(L))$ simply amounts to the following. Every argument in L is confirmed by every valuation that confirms every argument in L.

of Γ [$\Gamma \subseteq Cn(\Gamma)$]. Thus, υ_Γ verifies Γ. On the other hand, by hypothesis, Γ/α is not L-valid, so $\alpha \notin Cn(\Gamma)$, so υ_Γ falsifies α. Since υ_Γ verifies Γ, but falsifies α, υ_Γ refutes Γ/α. It follows that $\mathbb{V}(L)$ refutes Γ/α, so Γ/α is not an element of $\mathbb{L}(\mathbb{V}(L))$.

(→) Suppose L is complete. We wish to show that L is (1) reflexive, and (2) transitive.
(1) Suppose $\alpha \in \Gamma$, to show $\Gamma \Vdash \alpha$. Since L is complete, by Lemma 2, $\Gamma \Vdash \alpha$ iff every valuation in $\mathbb{V}(L)$ confirms Γ/α. The latter is true if and only if every valuation that verifies every element of Γ also verifies α. The latter is obviously true, since α is an element of Γ.
(2) Suppose (a) $\Gamma \Vdash \forall\Delta$, and (b) $\Delta \Vdash \alpha$, to show that $\Gamma \Vdash \alpha$. L is complete, so by Lemma 2, given (a), every valuation in $\mathbb{V}(L)$ that verifies Γ also verifies Δ. Similarly, given (b), every valuation that verifies Δ also verifies α. Therefore, every valuation in $\mathbb{V}(L)$ that verifies Γ also verifies α. So $\Gamma/\alpha \in \mathbb{L}(\mathbb{V}(L))$. But L is complete, so $\mathbb{L}(\mathbb{V}(L)) \subseteq \mathbb{L}$, so $\Gamma/\alpha \in \mathbb{L}$, which is to say that $\Gamma \Vdash \alpha$.

12. Alternative Formulation of Completeness Theorem for Logics

In the present section, we prove an alternative form of the completeness theorem.

THEOREM 1′

A logic L is complete if and only if its associated consequence operator *Cn* is a closure operator – which is to say the following conditions are satisfied.[15]

(c1*)	$\Gamma \subseteq Cn(\Gamma)$	[reflexivity]
(c2*)	$Cn(Cn(\Gamma)) \subseteq Cn(\Gamma)$	[quasi-transitivity]
(c3*)	$\Gamma_1 \subseteq \Gamma_2 \rightarrow Cn(\Gamma_1) \subseteq Cn(\Gamma_2)$	[monotonicity]

Proof. In light of the proof of Theorem 1, it is sufficient to show that conditions (c1*)-(c3*) are jointly equivalent to conditions (c1)-(c2) of Theorem 1. We divide the proof into 4 lemmas, which together yield the desired equivalence.

(1) (c1) and (c1*) are equivalent. This is obvious given the relationship between $\Vdash$ and *Cn*.

(2) (c2) entails (c2*). Suppose L is transitive (c2), to show that L is quasi-transitive. Suppose $\alpha \in Cn(Cn(\Gamma))$, to show that $\alpha \in Cn(\Gamma)$. Then $Cn(\Gamma) \Vdash \alpha$. But we also have that $\Gamma \Vdash \gamma$ for every γ in $Cn(\Gamma)$, so by (c2), $\Gamma \Vdash \alpha$, and hence $\alpha \in Cn(\Gamma)$.

(3) (c1)+(c2) entails (c3*). Suppose $\Gamma_1 \subseteq \Gamma_2$, and suppose $\alpha \in Cn(\Gamma_1)$, to show $\alpha \in Cn(\Gamma_2)$. Since $\Gamma_1 \subseteq \Gamma_2$, every element of Γ_1 is an element of Γ_2, so by (c1) every element of Γ_1 is a consequence of Γ_2; in other words, $\Gamma_2 \Vdash \forall\Gamma_1$. Also, since $\alpha \in Cn(\Gamma_1)$, $\Gamma_1 \Vdash \alpha$, so by (c2), $\Gamma_2 \Vdash \alpha$, and hence $\alpha \in Cn(\Gamma_2)$.

(4) (c2*)+(c3*) entails (c2). Suppose (a) $\Gamma \Vdash \forall\Delta$, and (b) $\Delta \Vdash \alpha$, to show $\Gamma \Vdash \alpha$. Given (a), $\Delta \subseteq Cn(\Gamma)$, so by (c3*) $Cn(\Delta) \subseteq Cn(Cn(\Gamma))$. By (c2*), $Cn(Cn(\Gamma)) \subseteq Cn(\Gamma)$, so $Cn(\Delta) \subseteq Cn(\Gamma)$. Given (b), $\alpha \in Cn(\Delta)$, so $\alpha \in Cn(\Gamma)$, which is to say that $\Gamma \Vdash \alpha$.

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[15] Note carefully that it is common to define a consequence operator to be an operator *Cn* that satisfies (c1*)-(c3*). We take a minimal approach to consequence operators, allowing one to study more general forms of (logical) consequence. Nevertheless, given the completeness theorem, consequence operators in the narrow sense are still very important.

13. Valuation-Consistency

We now turn our attention to the dual question concerning the conditions under which a valuation-space is complete. Toward this end, we introduce some further terminology.

(d23) Let V be a valuation-space, and let υ be a valuation, on a formal language $\mathcal{L}$. Then υ is said to be V-*consistent* precisely if υ confirms every V-valid argument.

Lemma 3: υ is V-consistent iff $\upsilon \in \mathbb{V}(\mathbb{L}(V))$.

Proof. By definition, $\mathbb{L}(V)$ consists of all V-valid arguments. Also, by definition, $\mathbb{V}(L)$ consists of all valuations that confirm every argument in L. So, $\mathbb{V}(\mathbb{L}(V))$ consists all valuations that confirm every argument in $\mathbb{L}(V)$, which is to say every V-valid argument.

Lemma 4: V is complete iff V contains every V-consistent valuation.

Proof. By definition, V is complete if and only if $V = \mathbb{V}(\mathbb{L}(V))$. Given that the associated maps are a Galois-connection, the latter is true if and only if $\mathbb{V}(\mathbb{L}(V)) \subseteq V$.[16] Given Lemma 3, the latter is true if and only if V contains every V-consistent valuation.

14. Examples of V-Consistent Valuations

Example 1:

First, we consider a "degenerate" example. Define valuation υ_1 as follows.

$\upsilon_1(\alpha) = \mathrm{T}$, for every sentence α in S

This is a kooky valuation, to be sure, but notice that υ_1 is V-consistent for any valuation-space V. This is because υ_1 refutes no argument whatsoever, because υ_1 does not falsify any formula.

Example 2:

Next, we consider a non-degenerate example. In particular, consider classical sentential logic (CSL). A valuation is officially regarded as CSL-admissible precisely if it satisfies the usual truth-functional conditions. Next, consider the valuation υ_2 defined as follows.

$\upsilon_2(\alpha) = \mathrm{T}$	if every CSL-valuation verifies α
$\upsilon_2(\alpha)=\mathrm{F}$	otherwise

First, υ_2 is not CSL-admissible, because $\upsilon_2(\mathrm{P}) = \upsilon_2(\sim\mathrm{P}) = \mathrm{F}$, which violates one of the truth-functional restraints on CSL-admissible valuations. Although υ_2 is not CSL-admissible, it is nevertheless CSL-*consistent*. To see this, we argue as follows.

Proof. Suppose υ_2 is not CSL-consistent. Then υ_2 refutes at least one CSL-valid argument, call it Γ/α. Then υ_2 verifies every element of Γ, but falsifies α. Let γ be an arbitrary element of Γ. Then υ_2 verifies γ, so by the definition of υ_2, every CSL-valuation verifies γ; in other words, γ is CSL-valid [i.e., a tautology]. Thus, every element of Γ is CSL-valid. But since the argument Γ/α is CSL-valid, α must

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[16] The converse, $V \subseteq \mathbb{V}(\mathbb{L}(V))$, amounts simply to the claim that every valuation in V confirms every argument confirmed by every valuation in V.

also be CSL-valid. Accordingly, every CSL-valuation verifies α, so by the definition of υ_2, $\upsilon_2(\alpha)$=T. This contradicts our earlier assumption that υ_2 falsifies α.

15. Super-Valuations

Are there other valuations that are CSL-consistent but not CSL-admissible? As it turns out, there are infinitely-many such valuations. This is in fact a special case of a more general theorem. In this connection, we offer the following definitions.

(d24) Let W be any set of valuations on $\mathscr{L}$. Define σ_w as follows.

$$\begin{array}{llll} \sigma_W(\alpha) & = & T & \text{if} \quad \forall\upsilon\{\upsilon\in W \rightarrow \upsilon(\alpha)=T\} \\ & = & F & \text{otherwise} \end{array}$$

The function σ_W is called the **super-valuation** associated with, or determined by, W.[17]

(d25) Let V be a valuation-space over formal language $\mathscr{L}$, and let υ be a valuation on $\mathscr{L}$. Then υ is said to be **a super-valuation over** V precisely if $\upsilon = \sigma_W$ for some $W\subseteq V$.

Example 1:

If W=$\varnothing$, then we obtain Example 1 of Section 14; $\sigma_\varnothing(\alpha)$=T for every α in $\mathscr{L}$.

Example 2:

If W=V, then we obtain Example 2 of Section 14.

Example 3:

if W={υ}, then $\sigma_W = \upsilon$.

This means that, technically speaking, every valuation in V is automatically a super-valuation over V.

16. Completeness Theorem for Valuation-Spaces

We are now in position to state and prove the corresponding completeness theorem for valuation-spaces. We do it in parts, starting with the following lemma.

Lemma 5:
Let W be any subset of V, and let σ_W be the super-valuation determined by W. Then σ_W is V-consistent. In other words, every super-valuation over V is V-consistent.

Proof. Let W be a subset of V, and let σ_W be the super-valuation determined by W. We wish to show that σ_W is V-consistent. Suppose to the contrary that σ_W is not V-consistent. Then σ_W refutes at least one V-valid argument, call it Γ/α. Since σ_W refutes Γ/α, σ_W verifies Γ but falsifies α. Given the definition of σ_W, every valuation in W verifies Γ, but at least one valuation in W falsifies α. It follows that at least one valuation in W verifies Γ but falsifies α, call it υ_0. Since $W\subseteq V$, $\upsilon_0 \in V$, so some valuation in V refutes Γ/α, so Γ/α is not V-valid, and we are in contradiction.

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[17] The notion of *super-valuation* is originally due to van Fraassen. “Singular Terms, Truth-Value Gaps, and Free Logic”, *Journal of Philosophy*, 63 (1966), 481-495. Following his original definition, it is customary to define a super-valuation so that σ_W falsifies α iff every W falsifies α. Our official view is that falsity is mere lack-of-truth. There are accordingly no "truth-value gaps" in the usual sense; on the other hand, a formula can be such that both it and its ordinary negation are false.

Lemma 6:
Suppose υ is V-consistent. Then there is a subset W of V, and associated super-valuation σ_W, such that $\upsilon = \sigma_W$. In other words, every V-consistent valuation is a super-valuation over V.

Proof. Suppose υ_0 is V-consistent. We wish to show that υ_0 is a super-valuation σ_W over V (for some $W \subseteq V$).

First, define $\sqsubseteq$ as follows:

$$\upsilon_1 \sqsubseteq \upsilon_2 \quad =_{df} \quad \forall\alpha\{\ \upsilon_1 \text{ verifies } \alpha \ \rightarrow\ \upsilon_2 \text{ verifies } \alpha\ \}$$

Next, let $W \;=\; \{\upsilon : \upsilon_0 \sqsubseteq \upsilon\}$

Claim: $\upsilon_0 = \sigma_W$

Suppose to the contrary that $\upsilon_0 \neq \sigma_W$. Then there is a formula α such that $\upsilon_0(\alpha) \neq \sigma_W(\alpha)$. There are two cases to consider, both of which we show contradict earlier claims.

Case 1: $\upsilon_0(\alpha) = T$, but $\sigma_W(\alpha) = F$

In this case, since $\sigma_W(\alpha) = F$, given the definition of σ_W, some valuation in W falsifies α, call it υ_2. Since $\upsilon_2 \in W$, given the definition of W, we have $\upsilon_0 \sqsubseteq \upsilon_2$. This means that υ_2 verifies every formula υ_0 verifies. But, in the case we are considering, υ_0 verifies α. So υ_2 also verifies α. This contradicts the earlier claim that υ_2 falsifies α.

Case 2: $\upsilon_0(\alpha) = F$, but $\sigma_W(\alpha) = T$

Let $\Gamma_0 \;=\; \{\alpha : \upsilon_0 \text{ verifies } \alpha\}$

Now, consider the argument Γ_0/α. By the definition of Γ_0, υ_0 verifies every element of Γ_0; in case 2, υ_0 falsifies α; so, υ_0 refutes Γ_0/α. But, by hypothesis, υ_0 is V-consistent, so V must also refute Γ_0/α. This means that there is a valuation in V that refutes Γ_0/α – call it υ_3. So, υ_3 verifies Γ_0 and falsifies α. Since Γ_0 is the set of *all* formulas verified by υ_0, and υ_3 verifies Γ_0, υ_3 verifies every formula verified by υ_0. This means that $\upsilon_0 \sqsubseteq \upsilon_3$, and so $\upsilon_3 \in W$, given the definition of W. But in case 2, $\sigma_W(\alpha) = T$, so every element of W verifies α, so υ_3 verifies α. This contradicts our earlier claim that υ_3 falsifies α.

With these lemmas in hand, we can now state the major theorem.

THEOREM 2

A valuation-space V is complete if and only if V contains all its super-valuations.[18]

Proof. According to Lemmas 5 and 6, a valuation is V-consistent if and only if it is a super-valuation over V. According to Lemma 4, V is complete if and only V contains every V-consistent valuation.

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[18] For those readers who wonder how this result generalizes to multiple-conclusion logics, the result is quite simple – every valuation space is (Galois) complete. Consult Dunn & Hardegree, *Algebraic Methods in Philosophical Logic*, Oxford University Press, 2001.