2   *Relations*

1.   Ordered-Pairs

After the concepts of set and membership, the next most important concept of set theory is the concept of *ordered-pair*. We have already dealt with the notion of *unordered-pair*, or doubleton. A doubleton is *unordered* insofar as the following is a theorem.

(t1) $\{a,b\} = \{b,a\}$

For, $x \in \{a,b\}$ iff $x=a$ or $x=b$, and $x \in \{b,a\}$ iff $x=b$ or $x=a$. But '$x=a$ or $x=b$' and '$x=b$ or $x=a$' are logically equivalent (by SL).

Thus, we cannot magically order the elements of a set simply by ordering the names of its elements. We need a further set-theoretic device; in particular, we need a method of aggregating things in which the order of aggregation makes a difference.

For the moment at least, we propose the following informal definition of the two-place function-sign 'op(,)'.

(d1) $op(a,b)$: the ordered-pair whose first component is a and whose second component is b

This simply presents an additional primitive symbol. In order to flesh out the idea, we propose the following principle.

The Fundamental Principle of Ordered-Pairs

(POP) $op(a,b) = op(p,q) \leftrightarrow. a=p \& b=q$

In other words, ordered-pairs are individuated according to their respective components: ordered-pairs p and q are identical if and only if the first component of p is identical to the first component of q and the second component of p is identical to the second component of q. This can also be written as follows; see later section for official definitions of 'OP', '1st', and '2nd'.

(POP*) $OP[p] \& OP[q] \& p=q .\rightarrow. 1\text{st}(p)=1\text{st}(q) \& 2\text{nd}(p)=2\text{nd}(q)$

This is completely analogous to the individuation of words: words w_1 and w_2 are the same word if and only if their respective first, second, third, etc., letters are the same.

Before continuing, it is important to note that (POP) divides naturally into two halves.

(P1) $op(a,b) = op(p,q) \rightarrow. a=p \& b=q;$
(P2) $a=p \& b=q .\rightarrow op(a,b) = op(p,q).$

(P2) is a theorem of identity logic, and accordingly need not be separately postulated. On the other hand, (P1) is *not* a logical truth; to see this, simply interpret '$op(,)$' as 'the quotient of _ and _'.

2.   Reducing Ordered-Pairs to Unordered-Pairs

In the development of the concept of ordered-pair, there are essentially two approaches. According to the first approach, one posits 'op(,)' as an additional primitive expression of set theory, on a par with epsilon. In this case, one further postulates principle (POP) – or equivalently, (P1) – as an additional non-logical axiom of set theory.

According to the second approach, one defines $op(a,b)$ to be a special sort of (unordered!) set, but in a way that the order information is "coded in". The trick, then, is to come up with a definition that enables one to deduce (POP) from the definition and the prior axioms of set theory.

Each approach has its shortcoming. The problem with the first approach is that it is not as conceptually economical as the second approach. The problem with the second approach is that, no matter how we define $op(a,b)$, we end up ascribing adventitious properties to it (see below).

If one regards set theory as essentially reductionistic, or foundational, in nature (the idea being to construct as much of mathematics as possible from as few primitive concepts as possible), then one will likely opt for reducing ordered-pairs to unordered sets. On the other hand, if one regards set theory to be just another branch of mathematics, not essentially different from analysis, algebra, geometry, etc., then there seems to be no harm in treating the concept of ordered-pair as yet another primitive.

Our own approach is largely foundational in character. We ultimately want to show how mathematics reduces to set theory. Accordingly, in keeping with our reductionistic orientation, we opt for defining ordered-pairs in terms of already existing concepts.

That leaves us with the task of producing an adequate definition. To say that a definition of $op(a,b)$ is *adequate* is just to say that one can deduce the non-logical half of principle (POP) from the definition.

For example, neither of the following definitions is adequate.

(d1) $op_1(a,b)$ $=_{df}$ $\{a,b\}$
(d2) $op_2(a,b)$ $=_{df}$ $\{a,\{b\}\}$

The inadequacy of (d1) has already been seen; demonstrating the inadequacy of (d2) is left as an exercise.

By contrast, both of the following definitions are adequate in the sense explained (exercise).

(d3) $op_3(a,b)$ $=_{df}$ $\{\{a,\varnothing\}, \{b,\{\varnothing\}\}\}$
(d4) $op_4(a,b)$ $=_{df}$ $\{\{a\},\{a,b\}\}$

At this point, we have two viable definitions, (d3) and (d4), so we need to make a choice. So long as Principle (POP) can be deduced, our choice is largely arbitrary. In light of the relative simplicity of (d4), we make that our choice.

Furthermore, we introduce new notation, in keeping with mathematical custom, for ordered-pairs, given as follows.

(D1) (a,b) $=_{df}$ $\{\{a\},\{a,b\}\}$

In other words, (a,b) is the ordered-pair whose first component is a and whose second component is b.

Before continuing, we note that the following notation is also common in the literature.

$$\langle a,b \rangle$$

However, we propose to employ corner-bracket notation for a closely related concept, that of *sequence*, which is defined in terms of functions, which are defined in terms of ordered-pairs, and which will be discussed in Chapter 3. Although there is a "practical" identity between (a,b) and $\langle a,b \rangle$, they are technically speaking distinct.

As mentioned earlier, the definitional approach to ordered-pairs has a number of odd consequences. We list a few.

(o1) $a \notin (a,b)$; $b \notin (a,b)$;
(o2) $\{a\} \in (a,b)$; $\{b\} \notin (a,b)$;
(o3) $\{a,b\} \in (a,b)$.

These are all true, but they compose the irrelevant part of the definition (model) of ordered-pairs. Basically, the only relevant feature of an ordered-pair is that Principle (POP) is satisfied.

Having defined ordered-pairs, we turn briefly to ordered-triples, ordered-quadruples, etc., which are also useful in mathematics and logic. There are a number of approaches to these additional concepts. The following are a few.

(1)   One can introduce each as a further primitive (with its corresponding axiom).

(2)   One can define all of them in terms of ordered-pairs (taken as primitive or not).

(3)   One can give separate definitions for each in terms of unordered-pairs.

(4)   One can define them in terms of *sequences*, which are defined in terms of *functions*, which are defined in terms of *relations*, which are defined in terms of ordered-pairs.

We follow course (4), which is conceptually most economical (see Chapter 3). However, the reader is invited to propose analyses along the lines of (1), (2), and (3).

3.   The Cartesian Product

Once we have the notion of ordered-pair, we can define numerous other set-theoretic notions. The first one is the *Cartesian product*.

René Descartes (a.k.a. Cartesius) observed that the Euclidean plane can be coordinatized by the real numbers. This means that every spatial point can be labeled, i.e., assigned coordinates, which are simply ordered-pairs of real numbers. Beginning with this observation, Descartes showed how the Euclidean plane can be replaced by (reduced to) the set of ordered-pairs of real numbers. Thus came into being the field of analytic geometry, which was essential to the development of the differential and integral calculi, which were essential to the development of physics.

Indeed, the reduction of geometry to analysis is so natural that we are led to identify the Euclidean points *with* (not simply *by*) their Cartesian coordinates.

The set of ordered-pairs of real numbers is a prominent and historically important example of a Cartesian product. In particular, it is the Cartesian product of the set $\mathbb{R}$ of real numbers with itself.

More generally, for any sets A and B, we can form their Cartesian product in accordance with the following definition.

(D2) $A \times B =_{df} \{(x,y) : x \in A \ \& \ y \in B\}$.

This definition involves a *generalized set-abstract*. For a detailed general presentation of general set-abstract notion, see Appendix 1. Expanding this particular generalized set-abstract yields the following alternative definition in terms of the corresponding simple set-abstract.

(D2′) $A \times B =_{df} \{w : \exists xy[x \in A \ \& \ y \in B \ \& \ w=(x,y)]\}$

Alternative definitions of '$A \times B$' use the defined terms 'OP', '1st', and '2nd'.

(D3) $OP[a] =_{df} \exists x \exists y[a = (x,y)]$

(D4) $1st(a) =_{df} \imath x \exists y[a = (x,y)]$

(D5) $2nd(a) =_{df} \imath y \exists x[a = (x,y)]$

[Notice that if a is not an ordered-pair then it has neither a first nor a second component, in which case $1st(a)=✕$, and $2nd(a)=✕$.] (D3)-(D5) enable us to rewrite (D2) as follows.

(D2″) $A \times B =_{df} \{x : OP[x] \ \& \ 1st(x) \in A \ \& \ 2nd(x) \in B\}$

(D2″) provides the most explicit definition; it says that $A \times B$ consists of ordered-pairs whose respective first components are in A and whose respective second components are in B. For example, if $A=\{a,b\}$, and $B=\{c,d\}$, then $A \times B = \{(a,c), (a,d), (b,c), (b,d)\}$.

Insofar as the set-abstract is legitimate (which we haven't demonstrated yet), we can prove the following theorems.

(t1) $(a,b) \in A \times B \leftrightarrow. \ a \in A \ \& \ b \in B$

(t2) $p \in A \times B \leftrightarrow \exists xy[x \in A \ \& \ y \in B \ \& \ p=(x,y)]$

(t3) $p \in A \times B \leftrightarrow. \ OP[p] \ \& \ 1st(p) \in A \ \& \ 2nd(p) \in B$

The question remains whether $A \times B$ is a set. Can we prove the set-abstract in question is legitimate, or do we have to postulate it existence by way of an additional axiom. Well, as it turns out, given our definition of ordered-pairs, and given our earlier axioms – in particular the Axiom of Separation, the Axiom of Simple Union, and the Axiom of Powers – we can prove that the set-abstract in (D2) is legitimate.

(t4) $\Sigma u \exists xy[x \in A \ \& \ y \in B \ \& \ u=(x,y)]$

To see this obtains, suppose A and B are sets. Is there at least one set that contains all the ordered-pairs (a,b) for which $a \in A$ and $b \in B$? Suppose $a \in A$ and $b \in B$. Then (a,b) is by definition $\{\{a\},\{a,b\}\}$. Consider the set $\wp(\wp(A \cup B))$, which is legitimatized by the Axiom of Simple Unions and the Axiom of Powers. Now, $a \in A$, and $b \in B$, so $a,b \in A \cup B$, so $\{a\}, \{a,b\} \in \wp(A \cup B)$, so $\{\{a\},\{a,b\}\} \in \wp(\wp(A \cup B))$. Thus, every ordered-pair (a,b) for which $a \in A$ and $b \in B$ is an element of $\wp(\wp(A \cup B))$. There are many other elements in $\wp(\wp(A \cup B))$ besides, so appealing to the Axiom of Separation, we separate off those that have the appropriate property, thus obtaining $A \times B$.

By way of concluding this section, we observe that, algebraically speaking, the Cartesian product leaves much to be desired. For example, it is neither *commutative*, nor even *associative*: the following are *not* theorems.

(✳) $A \times B = B \times A$

(✳) $(A \times B) \times C = A \times (B \times C)$

On the other hand, the Cartesian product interacts with the earlier operations in a sensible way, as illustrated in the following theorems.

(t5) $A \times (B \cap C)$ $=$ $(A \times B) \cap (A \times C)$
(t6) $A \times (B \cup C)$ $=$ $(A \times B) \cup (A \times C)$
(t7) $A \times (B - C)$ $=$ $(A \times B) - (A \times C)$
(t8) $A \times \cap(C)$ $=$ $\cap \{A \times X : X \in C\}$
(t9) $A \times \cup(C)$ $=$ $\cup \{A \times X : X \in C\}$

4.   Relations

The next order of business in the development of set theory is the analysis of (binary) relations as set-theoretic objects. The basic intuition is that just as a property has an extension, which is a set, a (binary) relation has an extension, which is also a set. Of course, we are already wise to the fact that not every property has an extension (Russell's Paradox), so we should be constantly on the lookout for similar problems with extensions of relations.

Well, what is the extension of a relation? Consider the relation implicitly defined by the following open formula.

x is a man, and y is a woman, and x is married to y

The extension of this relation is the set of all those ordered-pairs that satisfy this formula, which is to say those ordered-pairs (a,b) such that a is a man, b is a woman, and a is married to b.

More generally, we have the following informal definition.

(d1) $ext(R)$ $=_{df}$ $\{(a,b) : a$ bears R to $b\}$

Now, set theory concentrates on extensions, so by a process of abstraction, set theory identifies a relation with its extension. In particular, set theory defines a relation to be (merely) a set of ordered-pairs. This idea is formally presented as follows.

$\mathcal{R}[A]$ $=:$ A is a relation
(D6) $\mathcal{R}[A]$ $=_{df}$ $\forall x(x \in A \rightarrow OP[x])$
(D6′) $\mathcal{R}[A]$ $=_{df}$ $\forall x(x \in A \rightarrow \exists y \exists z[x=(y,z)])$

This says that set A is a relation iff every element of A is an ordered-pair – more briefly, A is a set of ordered-pairs.

Given this definition of relations, to say that element a bears relation R to element b is just to say that the ordered-pair (a,b) is an element of R:

(d2) a bears R to b $=_{df}$ $(a,b) \in R$

The logical form of the *definiendum* involves the three-place predicate

$$\_\_ \text{ bears } \_\_ \text{ to } \_\_$$

which is filled by three singular-terms. It is common practice in set theory to simplify this notation vastly by entirely eliminating explicit display of the predicate, which leaves the singular-terms in stark juxtaposition, as follows.

>   (d3) aRb $=_{df}$ a bears R to b

Thus, we have the following official definition.

>   (D7) aRb $=_{df}$ $(a,b) \in R$

5.   When Relations Aren't Relations

The notation 'aRb', introduced in the previous section, is very fruitful, but can be the source of grammatical confusion. We should regard the notation as metaphorical; like most metaphors, it is suggestive but at the same time (potentially) misleading. It is useful because it *suggests* an intimate connection between set-theoretic relations and two-place predicates. It is potentially misleading because it *suggests* that they are *identical*. In particular, it suggests that 'aRb' has the same logical form as, say, '$a \in b$'. But this could not be farther from the truth.

Notwithstanding the suggestiveness of the typographical convention, the expression 'aRb' consists of three singular-terms simply juxtaposed. There is additionally a predicate, but it is suppressed, as noted earlier. From the viewpoint of elementary logic, whereas the logical form of '$a \in b$' is $E[a,b]$, the logical form of 'aRb' is $B[a,r,b]$.

To put this point in bold relief, observe carefully that the following is a perfectly valid instance of the definition.

>   (i) $a\{b\}c$ $=_{df}$ $(a,c) \in \{b\}$

Of course, the *definiens* is true only if $(a,c) = b$, which may or may not be true. In any case, (i) illustrates how 'aRb' has a logical form quite different from '$a \in b$'.

At this point, the reader is likely to ask: doesn't the formula '$x \in y$' implicitly define a binary relation, indeed, a very important set-theoretic relation. And, in particular, isn't the following principle true?

>   (p?) $a \in b \leftrightarrow a$ bears $\in$ to b

As it stands, (p?) ungrammatical. The problem is that '$\in$' is a predicate, yet it occurs in subject position on the right side. We need a *name* (singular-term) for the relation, not just a way of expressing that the relation holds between two things (which is what the predicate does). That seems easy enough, since we have set-abstracts. In particular, *one* name of the (extension of the) relation expressed by '$\in$' is just:

$$\{(x,y) : x \in y\}$$

So the proper reformulation of (p?) is:

>   (p?*) $a \in b \leftrightarrow a$ bears $\{(x,y) : x \in y\}$ to b

Unfortunately, there is a serious problem with the set-abstract in question. But let us ignore that problem for the moment. Even if the set-abstract is legitimate, our deliberations have merely shown that '$a \in b$' is set-theoretically *equivalent* to a formula of the form 'aRb' (i.e., '$B[a,r,b]$').

This does not prove that the formulas have the same logical form. All tautologies are logically equivalent; it does not follow that they all have the same logical form. Similarly, '$P\&(P \lor Q)$' is logically equivalent to 'P', but they do not have the same logical form; one is molecular, the other atomic.

There is a more serious problem: the putative relation (set of ordered-pairs) $\{(x,y) : x \in y\}$ is ill-defined! The epsilon-relation is not a relation! Less flamboyantly speaking, the epsilon-relation may exist, but its extension does not. In as plain language as possible perhaps, the epsilon-relation may exist, but its extension is not a set!

The question is whether the set-abstract

$$\{(x,y) : x \in y\}$$

which we abbreviate by 'E' (short for 'epsilon') is proper. To see that it is illegitimate, let us assume otherwise. Next, we note the following theorem.

$$\forall x[x \in \{x\}]$$

Hence, letting $E = \{(x,y) : x \in y\}$, we have:

$$\forall x[(x,\{x\}) \in E]$$

But $\{a\} \in (a,b)$ (one of the odd results), so we obtain

$$\forall x[\{x\} \in \bigcup E]$$

from which we get

$$\forall x[x \in \bigcup\bigcup E]$$

from which we get:

$$\exists y \forall x[x \in y]$$

This contradicts an earlier theorem to the effect that there is no universal set. In other words, if E is a set then so is $\bigcup E$, and so is $\bigcup\bigcup E$, but our reasoning above shows that $\bigcup\bigcup E$ is the universal set. It follows that E (i.e., $\{(x,y) : x \in y\}$) is ill-defined.

We already know that not every property has an extension. For example, the property of being self-identical has no extension, and the property of being normal has no extension. We have now learned that not every relation has an extension.

Indeed, most of the relations described by set theory (identity, inclusion, exclusion, etc.) have no extensions. Moral: set theory analyzes mathematics, but in an important sense, it does not analyze itself. Set theory defines a relation to be a *set* of ordered-pairs, yet the very relations *used by* set theory (identity, membership, inclusion, etc.) are not, strictly speaking, relations *according to* (or *inside*) set theory.

Later, we see how to salvage relational concepts by relativizing them to various sets.

6.   Relations From, To, and On Sets

In the present section, we introduce further terminology that is common in the set-theoretic description of relations. Recall that the formal definition is that a relation is just a set of ordered-pairs.

$$A \text{ is a relation } =: \quad \mathcal{R}[A] \quad =_{df} \quad \forall x(x \in A \rightarrow \text{OP}[x])$$

The definition above follows the following schematic format.

$$\text{expression} \quad =: \quad \text{abbreviation} \quad =_{df} \quad \text{definition of expression}$$

We now define what it means to say that R is a relation *from* set A *to* set B.

Intuitively, to say that <u>R is a relation *from A to B*</u> is to say:

(1) R is a relation;
(2)   the first component of every pair in R is an element of A;
(3)   the second component of every pair in R is an element of B.

Formally, this may be written as follows.

$$R \text{ is a relation from } A \text{ to } B \quad =_{df} \quad \begin{array}{l} \mathcal{R}[R] \ \& \\ \forall p[p \in R \rightarrow 1(p) \in A] \ \& \\ \forall p[p \in R \rightarrow 2(p) \in B] \end{array}$$

Interestingly enough, the *definiens* can be greatly simplified to

$$R \subseteq A \times B$$

which allows us to offer the following very simple official definition.

(D8) R is a relation from A to B $=:$ $\mathcal{R}[R,A,B]$ $=_{df}$ $R \subseteq A \times B$

The definition has an obvious consequence, that if R is a relation from A to B, then R is a relation. The converse (properly formulated) is also true: if R is a relation, then it is a relation from one set to another. In other words, we have the following theorem.

(t1) $\mathcal{R}[A] \leftrightarrow \exists X \exists Y \mathcal{R}[A,X,Y]$

When the two sets A and B are the same, in which case we can call them both A, we have slightly specialized terminology. Specifically, a relation from A to A is called a relation *on* (or *in*) A. This is formally defined as follows.

(D9) R is a relation on A $=:$ $\mathcal{R}[R,A]$ $=_{df}$ $\mathcal{R}[R,A,A]$

As it turns out, every relation R is a relation *on* at least one set, which is to say the following.

(t2) $\mathcal{R}[A] \leftrightarrow \exists X \mathcal{R}[A,X]$

As a further notational convention, we define the set of all relations from A to B as follows.

(D10) $\mathcal{R}(A,B) =_{df} \{R : \mathcal{R}[R,A,B]\}$

It is easy to prove the following theorem (exercise).

(t3) $\mathcal{R}(A,B) = \wp(A\times B)$

Given a set A, there are two relatively uninteresting examples of relations on A. First of all, there is the *empty relation*, i.e., $\varnothing$, which is trivially a relation on every set. Second, there is the *universal relation* on A, which is simply $A\times A$. The following is the official definition.

(D11) the universal-relation on A $=:$ $\mathbb{U}_A$ $=_{df}$ $A\times A$

In addition to the empty relation and the universal relation (restricted to A), there is a slightly more interesting relation on A, called the *identity relation on A*, defined as follows.

(D12) the identity-relation on A $=:$ I_A $=_{df}$ $\{(x,x) : x \in A\}$

Observe the following obvious consequence.

(t4) $I_A = \{(x,y) : x \in A\ \&\ y \in A\ \&\ x=y\}$

Notice that although there is an identity relation relative to (restricted to) any given set A, there is no identity-relation *simpliciter*; for if there were such a set, then its union would be the universal set. Similarly, there is no membership relation *simpliciter*, as already remarked. On the other hand, there is a relativized membership relation (just as there is a relativized identity relation), defined as follows.

(d2) $\in_A$ $=_{df}$ $\{(x,y) : x \in A\ \&\ y \in A\ \&\ x \in y\}$

We can do the same with other relations used by set theory; for example, the relativized inclusion and exclusion relations are defined as follows.

(d3) $\subseteq_A\ =_{df}\ \{(x,y) : x \in A\ \&\ y \in A\ \&\ x \subseteq y\}$

(d4) $\perp_A\ =_{df}\ \{(x,y) : x \in A\ \&\ y \in A\ \&\ x \perp y\}$

More generally, where $\mathbb{F}[x,y]$ is any formula with two free variables, and A is any set, there is a *relativized relation* described by the following set-abstract.

(d5) $\{(x,y) : x \in A\ \&\ y \in A\ \&\ \mathbb{F}[x,y]\}$

We discuss this more in Chapter 3.

7.   The Domain, Range, and Field of a Relation

Recall the marriage relation described earlier. It is not true that everyone bears this relation to someone or other. This holds of a person if and only if that person is a married man. Similarly, it is not true that everyone has the relation born to him/her. This holds of a person if and on if that person is a married woman.

More generally, we speak of the *domain*, *range*, and *field* of a relation. An element is in the *domain* of a relation R if and only if it bears R to at least one thing. Correspondingly, an element is in the *range* of R if and only if it has the relation R born to it by at least one thing. Finally, an element is in the *field* of R if and only if it is in the domain and/or range of R. The following are the formal definitions.

(D13)	the domain of R	=:	$dom(R)$	$=_{df}$	$\{x : \exists y[xRy]\}$
(D14)	the range of R	=:	$ran(R)$	$=_{df}$	$\{y : \exists x[xRy]\}$
(D15)	the field of R	=:	$fld(R)$	$=_{df}$	$dom(R) \cup ran(R)$

Consider another example; let R be the following impure set.

$$\{(x,y) : x \text{ is living, } y \text{ is living, } x \text{ is a parent of } y\}$$

Then the domain of R is the set of people who are living and have at least one living offspring, whereas the range of R consists of living people who have at least one living parent. A living person x fails to be in the field of R precisely when x has neither a living parent nor a living offspring.

Now, one can prove the following theorems (exercise) .

(t1) $\mathcal{R}[R] \rightarrow dom(R) = \{1\text{st}(p) : p \in R\}$

(t2) $\mathcal{R}[R] \rightarrow ran(R) = \{2\text{nd}(p) : p \in R\}$

In other words, the domain of a relation R is the set of first components of pairs in R, whereas the range is the set of second components of pairs in R.

One can also prove the following theorems.

(t3) $\mathcal{R}[R] \rightarrow \mathcal{R}[R, dom(R), ran(R)]$

(t4) $\mathcal{R}[R] \rightarrow \mathcal{R}[R, fld(R)]$

In other words, every relation is a relation from its domain to its range, and every relation is a relation on its field.

Finally, one can prove the following theorem, granted our particular definition of ordered-pairs.

(t5) $\mathcal{R}[R] \rightarrow fld(R) = \bigcup\bigcup R$

8.   Images and Pre-Images

Closely related to the notions of domain and range are the notions of *image* and *pre-image*, which are defined as follows.

| (D16) | the image of A under R | =: | $R^{\rightarrow}(A)$ | $=_{df}$ | $\{y : \exists x(x \in A \ \& \ xRy)\}$[1] |
| (D17) | the pre-image of A under R | =: | $R^{\leftarrow}(A)$ | $=_{df}$ | $\{x : \exists y(y \in A \ \& \ xRy)\}$[2] |

$R^{\rightarrow}(A)$ consists of those things *to whom* the relation R is born by at least one element in A. For example, if R is the marriage relation discussed earlier, and A is the set of octogenarian men, then $R^{\rightarrow}(A)$ is the set of women *to whom* octogenarian men are married. On the other hand, $R^{\leftarrow}(A)$ consists of those things that bear the relation R to at least one element in A. For example, if R is the marriage relation, and A is the set of octogenarian women, then $R^{\leftarrow}(A)$ is the set of men who are married to octogenarian women.

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[1] Sometimes, the image of A under R is also denoted simply R(A).

[2] Sometimes, the pre-image is also called the *inverse image*, and it is also denoted R⁻¹(A).

Notice the following theorems.

(t1) $\mathcal{R}[R, A, B] \rightarrow R^{\rightarrow}(A) = ran(R)$

(t2) $\mathcal{R}[R, A, B] \rightarrow R^{\leftarrow}(B) = dom(R)$

The following are further theorems concerning images and pre-images.

(t3) $R^{\rightarrow}(dom(R)) = ran(R)$

(t4) $R^{\leftarrow}(ran(R)) = dom(R)$

(t5) $A \subseteq dom(R) \rightarrow A \subseteq R^{\leftarrow}(R^{\rightarrow}(A))$

(t6) $B \subseteq ran(R) \rightarrow B \subseteq R^{\rightarrow}(R^{\leftarrow}(B))$

(t7) $A \subseteq B \rightarrow R^{\rightarrow}(A) \subseteq R^{\rightarrow}(B)$

(t8) $A \subseteq B \rightarrow R^{\leftarrow}(A) \subseteq R^{\leftarrow}(B)$

9.   Inversion

Relations are all sets, so the (Boolean) algebra of sets applies equally to relations. In addition to the algebraic ideas that apply to sets in general (intersection, union, etc.), there are algebraic ideas that are peculiar to relations, the most important being *inversion* and *composition*.

First, inversion is defined as follows.

(D18)   the inverse of R =: R^{-1} $=_{df}$ $\{(x,y) : yRx\}$

R^{-1} is also called the *converse* of R. The basic idea is simple: a bears the converse relation R^{-1} to b if and only if b bears the original relation R to a.

Examples of converses are familiar. The converse of the relation of parenthood is the relation of offspringhood; a parents b iff b is an offspring of a. In arithmetic, the converse of the less-than relation is the greater-than relation.

Evidently, any relation is the converse of its converse, which is to say that either of the following logically equivalent formulas is a theorem.

(t1) $\mathcal{R}[A] \rightarrow (A^{-1})^{-1} = A$

(t2) $A = B^{-1} \leftrightarrow B = A^{-1}$

Notice the proviso in (t1); if A is not a relation, then $(A^{-1})^{-1} \subseteq A$, but the converse inclusion does not hold (exercise). Notice that (t2) has no such proviso; this is because the converse of any *set* is a relation (possibly empty). Thus, we have the following substitute theorem for (t1), which does not require assuming A is a relation.

(t3) $((A^{-1})^{-1})^{-1} = (A^{-1})^{-1}$

10.   Composition

In addition to inversion, which is a one-place operation on relations, there is the operation of *composition*, also called *relative* (or relational) *product*, which is a two-place operation, formally defined as follows.

(D19)   the composition of R and S =: $R \circ S$ $=_{df}$ $\{ (x,z) : \exists y[xRy \ \& \ ySz] \}$

$R \circ S$ is also called the (*relative*) *product* of R and S. From the definition we see that a bears the composite relation $R \circ S$ to b iff a bears the relation R to something that in turn bears the relation S to b. Or formally expressed:

(t4) $a[R \circ S]b \ \leftrightarrow \ \exists y[aRy \ \& \ yRb]$

Note that the square-brackets on the left-hand side are inserted for visual clarity.

Examples of relational-composition are common in the realm of kinship relations. For example, the grandparent-relation is the product of the parent-relation with itself; i.e., $G=P \circ P$. On the other hand, the aunt relation is the product of the sister relation and the parent relation; i.e., $A=S \circ P$. Examples of relational-composition abound.

Algebraically speaking, relational composition is fairly well-behaved.   For example, it is *associative*, which is to say that the following theorem holds.

(t5) $(R \circ S) \circ T = R \circ (S \circ T)$

Note carefully, however, that relational composition is not *commutative*, which is to say that:

(t6) $R \circ S \neq S \circ R$ (for some R,S)

To construct a counterexample to the commutativity of $\circ$, consider the sister relation S and the parent relation P; a bears $S \circ P$ to b iff a is a sister of someone who is a parent of b; by contrast, a bears $P \circ S$ to b iff a is a parent of someone who is a sister of b. If by 'sister' we mean full-sister, then $P \circ S \subseteq P$.

Next, we note the following theorem about the interaction between inversion and composition.

(t7) $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$

Consider an intuitive example from kinship relations. Let R be the mother-son relation, and let S be the father-daughter relation. Then R^{-1} is the son-mother relation, and S^{-1} is the daughter-father relation. Also, $R \circ S$ is the paternal grandmother-granddaughter relation, and $S^{-1} \circ R^{-1}$ is the granddaughter-paternal grandmother relation.

Finally, we note that every relational composite $R \circ S$ is defined, although not every composite is "sensible". For example, let R be the father-daughter relation, and let S be the brother-sister relation; then $R \circ S$ is null; i.e., $R \circ S = \varnothing$. In this connection, we have the following theorems.

(t8) $R \circ S \neq \varnothing \ \leftrightarrow \ ran(R) \cap dom(S) \neq \varnothing$
(t9) $R \circ S = \varnothing \ \leftrightarrow \ ran(R) \perp dom(S)$

11. Restriction

Next, we introduce a further two-place operation, called *restriction*, officially defined as follows.

(D20)   the restriction of R to A =: $R|A$ $=_{df}$ $\{(x,y) : xRy \ \& \ x,y \in A\}$

If R is a relation, then the restriction of R to set A consists of those ordered-pairs in R whose components are both elements of A. In other words, we have the following theorems.

(t10) $R|A \ = \ \{p : p \in R \ \& \ 1\text{st}(p) \in A \ \& \ 2\text{nd}(p) \in A\}$

(t11) $R|A \ = \ R \cap (A \times A)$

The notion of restriction is intuitively useful. We can, for example, begin with the *general* mother-child relation; we can then restrict this relation to obtain the mother-child relation *restricted to* humans, or we can restrict it to obtain the mother-child relation *restricted* to females.

12. Ancestors

If we think of relational composition as a form of multiplication, then it is natural to use exponential notation, borrowed from arithmetic, which yields the following definitions.[3]

(d1) $R^2 \ =_{df} \ R \circ R$
(d2) $R^3 \ =_{df} \ R \circ R \circ R$
(d3) $R^4 \ =_{df} \ R \circ R \circ R \circ R$
         etc.

Once we have the infinite sequence of relations R, R^2, R^3, etc., we can conceptually (if not formally) formulate another very important set-theoretic notion, that of *ancestor*. Let's begin with the concrete example that motivates using this particular word. To say that a is an ancestor of b is to say that one of the following infinite sequence of statements is true:

(1) a parents b
(2) a parents someone who parents b
(3) a parents someone who parents someone who parents b
         etc.

In other words, one of the following is true:

(1) $(a,b) \in P$
(2) $(a,b) \in P^2$
(3) $(a,b) \in P^3$
         etc.

If we could somehow wrangle all of the sets, P, P^2, P^3, P^4, etc., into a single collection, then we could define the ancestor relation simply to be the *union* of this collection, as follows.

(d4) $A \ =_{df} \ \cup\{P, P^2, \ P^3, \ P^4, ...\}$
(d5) $aAb \ =_{df} \ (a,b) \in \cup\{P, P^2, \ P^3, \ P^4, ...\}$

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[3] Since relational composition is associative, we can drop parentheses.

The collection in question is infinitely-large, so its existence may be questionable; the same can said about its union, *unless* we can obtain it by some other set-theoretic construction.

Set theory is not concerned with the ancestor relation *per se*. Rather, it is interested in the above construction for an arbitrary relation R. In particular, we can begin with any relation R, and we can talk about the corresponding ancestral relation associated with R, which we denote R^*, intuitively defined as follows.

(d6) $R^* =_{df} \cup\{R, R^2, R^3, R^4, ...\}$

In a later section, we will see that, irrespective of whether the set $\{R, R^2, R^3, R^4, ...\}$ is legitimate or not, the set R^* exists.

13. Special Types of Relations

Oftentimes, we are interested in relations with special properties, to which we give special names. The following is a list of some of the basic types of relations that are usually considered in set theory. In what follows, R is presumed to be a relation, but the definitions do not depend upon this; technically, R can be any set. We begin with three natural abbreviations.

(D21) $\forall uv =_{df} \forall u \forall v$

　　　　　$\exists uv =_{df} \exists u \exists v$

(D22)	$a,b \in S$	$=_{df}$	$a \in S \ \& \ b \in S$
(D23)	R is *reflexive* in A	$=_{df}$	$\forall x(x \in A \rightarrow xRx)$
(D24)	R is *irreflexive* in A	$=_{df}$	$\forall x(x \in A \rightarrow \sim xRx)$
(D25)	R is *transitive* in A	$=_{df}$	$\forall xyz(x,y,z \in A \rightarrow: xRy \ \& \ yRz . \rightarrow xRz)$
(D26)	R is *intransitive* in A	$=_{df}$	$\forall xyz(x,y,z \in A \rightarrow: xRy \ \& \ yRz . \rightarrow \sim xRz)$
(D27)	R is *symmetric* in A	$=_{df}$	$\forall xy(x,y \in A \rightarrow. xRy \rightarrow yRx)$
(D28)	R is *asymmetric* in A	$=_{df}$	$\forall xy(x,y \in A \rightarrow. xRy \rightarrow \sim yRx)$
(D29)	R is *weakly-asymmetric*	$=_{df}$	$\forall xy(x,y \in A \rightarrow: x \neq y \rightarrow. xRy \rightarrow \sim yRx)$
(D30)	R is *anti-symmetric* in A	$=_{df}$	$\forall xy(x,y \in A \rightarrow: xRy \ \& \ yRx . \rightarrow x=y)$
(D31)	R is *strongly-connected* in A	$=_{df}$	$\forall xy(x,y \in A \rightarrow. xRy \vee yRx)$
(D32)	R is *weakly-connected* in A	$=_{df}$	$\forall xy(x,y \in A \rightarrow. x=y \vee xRy \vee yRx)$

Notice that weak-asymmetry and anti-symmetry are logically equivalent. They are separately defined primarily because each definition suggests something different about the type of relation defined. The definition of weak-asymmetry shows how it is related to asymmetry. The definition of anti-symmetry shows how it is related to extensionality ($A \subseteq B \ \& \ B \subseteq A . \rightarrow A = B$).

Every predicate defined above is a two-place predicate, and expresses a relation between relation R and set A. We can define an associated one-place predicate by specifying A to be the field of R. This yields a corresponding series of definitions, as follows.

(D23′) R is *reflexive* $=_{df}$ R is reflexive in $fld(R)$

(D24′) R is *irreflexive* $=_{df}$ R is irreflexive in $fld(R)$

etc.

Notice that in many, but not all, cases the initial proviso about being elements of $fld(R)$ is redundant. For example, one can prove the following theorem about symmetry.

(t1) R is symmetric $\leftrightarrow \forall xy(xRy \rightarrow yRx)$

The reason is that if xRy then $x,y \in fld(R)$, which makes the latter redundant.

In addition to the basic properties of relations listed above, there are various combinations that are important. We list a few of the most important combinations.[4]

(D33) R is a quasi-order relation $=_{df}$ R is reflexive and transitive

(D34) R is a partial-order relation $=_{df}$ R is a quasi-order relation
                                                                          and R is anti-symmetric

(D35) R is a linear-order relation $=_{df}$ R is a partial-order relation,
                                                                          and R is strongly-connected

(D36) R is a strict-linear-order relation $=_{df}$ R is asymmetric,
                                                                          transitive, and
                                                                          weakly-connected

(D37) R is an equivalence relation $=_{df}$ R is reflexive,
                                                                          symmetric, and
                                                                          transitive

Notice that the following is a theorem (exercise).

(t2)   if R is symmetric and transitive, then R is reflexive

Thus, every symmetric, transitive relation is an equivalence relation. On the other hand, the following is not a theorem.

(✳)   if R is symmetric <u>in A</u> and transitive <u>in A</u>, then R is reflexive <u>in A</u>

For example, the empty relation $\varnothing$ is symmetric and transitive in every set (trivially), but it is reflexive only in its field, which is the empty set.

The definitions of the various order relations above make no reference to a special set A. There are corresponding definitions that involve A, being schematically presented as follows.

(D′) R is a $\mathbb{K}$ *relation <u>on A</u>* $=_{df}$ R is a $\mathbb{K}$ relation, and $fld(R)=A$

Here, the schematic letter '$\mathbb{K}$' stands in place of the various adjective phrases from above.

Before continuing, notice carefully, for example, that an equivalence relation on A is not just a relation on A that is an equivalence relation. The following is *not* a theorem.

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[4] A quasi/partial/linear order relation is also called a quasi/partial/linear *ordering*.

(✶)   equ[R,A] ↔. $\mathcal{R}[R,A]$ & equ[R]

For example, the empty relation $\varnothing$ is a relation on A that is an equivalence relation, but it is not an equivalence relation on A, unless $A=\varnothing$.

By way of concluding this section, we note that many of the properties of relations can be formulated algebraically using inversion and composition. The following are examples; R is presumed to be a relation.

(t3) R is symmetric   ↔ $R = R^{-1}$

(t4) R is transitive   ↔ $R \circ R \subseteq R$

(t5) R is an equivalence relation   ↔ $R \circ R^{-1} = R$

(t6) R is asymmetric   ↔ $R \perp R^{-1}$

(t7) R is anti-symmetric   ↔ $R \cap R^{-1} \subseteq I_{fld(R)}$

14.   Ancestors Revisited

Recall that the *ancestral* relation R^* associated with a given relation R is *informally* defined as follows.

(d1) aR^*b $=_{df}$ aRb or aR^2b or aR^3b or ...

It is informal because the *definiens* is not a formula of set theory, since it is infinitely-long. A more nearly formal definition is the following

(d2) R^* $=_{df}$ $\cup\{R, R^2, R^3, ...\}$

but the existence of the set $\{R, R^2, R^3, ...\}$ – which is infinitely-large – is not established yet. Indeed, as it currently stands, we cannot establish the existence of *any* infinitely-large set.

In the present section, we see that the existence of R^* may be deduced from the axioms we already have. This requires producing an alternative definition, which is given informally as follows.

(d3) R^* $=_{df}$ the smallest transitive relation that includes R.

In the realm of sets, the *smallest so-and-so* is defined formally as follows.

(D38) σν𝔽 $=_{df}$ ıs(𝔽[s/v] & ∀v(𝔽 → $s\subseteq v$))

Intuitively, the smallest 𝔽 is the unique thing that is 𝔽 and that is included in every 𝔽. Thus, (d3) amounts to the following.

(d3*) R^* $=_{df}$ the unique relation that:

            (*i*)   is a transitive relation that includes R, and
            (*ii*)   is included in every transitive relation that includes R

The obvious question is whether there is such a relation. As it turns out, one can prove the following.

(t1) $R^* = \cap\{R' : R' \subseteq fld(R) \times fld(R)$ & transitive[R'] & $R \subseteq R'\}$

Notice that the set-abstract has a form that is legitimized by the Axiom of Separation and the Axiom of Power Sets. We are not off the hook yet; the intersection $\cap C$ of a collection C is well-defined if and only if C is non-empty. So, we must make sure that the set

$$\{R' : R' \subseteq fld(R) \times fld(R) \ \& \ \text{transitive}[R'] \ \& \ R \subseteq R'\}$$

is not empty. Given any set relation R, even $R = \varnothing$, there is at least one transitive relation on $fld(R)$ – namely, the identity relation on $fld(R)$. Thus, this set is non-empty, and its intersection is well-defined.

So, the relation exists. The remaining question is whether it is the ancestral relation, as defined earlier. Unfortunately, this cannot be adequately demonstrated until we have a formal development of numbers, including proof by mathematical induction. This happens in a later chapter.

15.   Equivalence-Relations and Equivalence-Classes

Among the special types of relations, equivalence relations are perhaps the most important, so we devote two sections to them.

Let A be any set. Then there are two relatively trivial equivalence relations on A, defined as follows.

$$I_A \ =_{df} \ \{(x,x) : x \in A\}$$
$$U_A \ =_{df} \ \{(x,y) : x,y \in A\}$$

The former is called the *identity relation* on A, and the latter is called the *universal relation* on A. The former deems two elements of A to be equivalent iff they are identical. The latter deems every pair of elements of A to be equivalent.

Evidently, I_A is a subset of every equivalence relation on A, and every equivalence relation on A is a subset of U_A, as stated in the following theorem.

(t1) $\forall R(\ equ[R, A] \rightarrow . \ I_A \subseteq R \ \& \ R \subseteq U_A\ )$
       where:
       $equ[R, A] \quad =_{df} \quad R$ is an equivalence relation on A

Next, if R and S are both equivalence relations on A, then $R \cap S$ is also an equivalence relation on A (exercise). Something stronger can be said: if C is any collection of equivalence relations on A, then the intersection $\cap C$ is also an equivalence relation on A (exercise).

(t2) $equ[R, A] \ \& \ equ[S, A] \ . \rightarrow \ equ[R \cap S, A]$

(t3) $\forall R(R \in C \ \rightarrow \ equ[R, A]) \ \rightarrow \ equ[\cap C, A]$

The same cannot be said for union; the union of two equivalence relations need not be an equivalence relation (exercise).

Next, we introduce the important notion of an *equivalence class*. Given any equivalence relation R on set A, and given any element e of A, the equivalence class of e with respect to R and A is defined to be the set of elements that are equivalent to e (according to R). Formally, we have the following definition.

(D39) $[a]_R$ $=_{df}$ $\{x : xRa\}$
         $[a]_R$ $=:$ the equivalence class[5] of a (with respect to R)

In other words, $[a]_R$ is just the set of things equivalent to a according to R. Of course, the formal definition does not require that R is an equivalence relation, nor even a relation for that matter! But the *intended* application of the definition is to the case where R is in fact an equivalence relation.

Let us consider some simple examples. For example, if R is the identity-relation on A, and a is an element of A, then $[a]_R = \{a\}$. On the other hand, if R is the relation of parallelism on the set of Euclidean lines, and a is a particular line, then $[a]_R$ is the set of all the lines parallel to a. Finally, if R is the relation of having-the-same-height among humans, and a is a particular human, then $[a]_R$ is the set of humans who have the same height as a.

16.   Partitions and Equivalence-Relations

A concept that is intimately associated with equivalence-relations is the concept of a *partition*. Intuitively, a partition of a set A divides A into non-empty subsets (called *cells*). Most importantly, the cells are *mutually-exclusive*, which means that no two cells overlap, and they are *jointly-exhaustive*, which means that every element of A is an element of at least one cell. In other words, no cell is empty, and every element of A is an element of *exactly one* cell. The formal definition is given as follows.

(D40) C is a partition on A $=_{df}$

   (1) $\forall X(X \in C \rightarrow X \neq \varnothing)$
   (2) $\forall XY(X,Y \in C \rightarrow (X \perp Y \vee X = Y))$
   (3) $\cup C = A$

Note that conditions (1) and (2) can be replaced by a single condition.

   (1*) $\forall XY(X,Y \in C \rightarrow (X \perp Y \text{ xor } X = Y))$

For example, humans may be partitioned in numerous ways: according to age, gender, income, political affiliation, education, etc. At a more mundane level, ones silverware can be partitioned into various sections of the silverware drawer.

The partitions of a set are naturally ordered by the relation defined as follows.

(D41) $P_1 \leq P_2$ $=_{df}$ $\forall X(X \in P_1 \rightarrow \exists Y(Y \in P_2 \ \& \ X \subseteq Y))$

In other words $P_1 \leq P_2$ iff every cell in P_1 is included in at least one cell in P_2. This is customarily described by saying that the partition P_1 is *at least as fine as* partition P_2.

We can also define the associated asymmetrical relation "_ is finer than _" in the usual way.

(D42) $P_1 < P_2$ $=_{df}$ $P_1 \leq P_2 \ \& \ \sim[P_2 \leq P_1]$

As an example of fineness of partitions, the set $\{\{x\} : x \in A\}$ of singletons of elements of A is the finest partition on A, whereas $\{A\}$ itself is the coarsest partition on A. In between are the all the remaining partitions. If one has a partition C, one obtains a finer partition by taking each cell in C and

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[5] The term 'class' here has no significance, but is simply a tradition in set theory.

partitioning it, possibly trivially. On the other hand, one obtains a coarser partition by taking at least two cells and "fusing" them into a single cell. These ideas are formally presented in the following theorems.

(t1)   If C partitions A, and every element of $\mathcal{B}$ partitions exactly one element of C, then $\cup\mathcal{B}$ partitions A, and $\cup\mathcal{B} \leq C$.

(t2)   If C partitions A, and every element of $\mathcal{B}$ is the union of exactly one subset of C, and furthermore $\cup\mathcal{B}=A$, then $\mathcal{B}$ partitions A, and $C \leq \mathcal{B}$.

The following are theorems about the fineness relation.

(t3) $A \leq A$
(t4) $A \leq B$ & $B \leq C$.$\rightarrow$ $A \leq C$
(t5) $A \leq B$ & $B \leq A$.$\rightarrow$ $A=B$
(t6) $A < B$ & $B < C$.$\rightarrow$ $A < C$
(t7) $A < B \rightarrow \sim[B < A]$

In other words, the fineness relation partially orders the collection of partitions of a given set A; it is reflexive, transitive, and anti-symmetric. On the other hand, the associated strict relation is transitive and asymmetric.

Next, we consider the connection between partitions and equivalence relations. First of all, every collection C of subsets of A induces an associated relation $\mathcal{E}_C$ on A, according to the following definition.

(D43) $\mathbb{E}_C =_{df} \{ (x,y) : \exists Z(Z \in C$ & $x,y \in Z) \}$

One can prove the following theorem (exercise).

(t8)   if C is a partition of A, then $\mathcal{E}_C$ is an equivalence relation on A.

Thus, every partition induces an associated equivalence relation.

The converse is also true: Every equivalence relation induces an associated partition. In order to see this, we recall the notion of *equivalence class*.

$$[a]_R =_{df} \{x : xRa\}$$

In constructing the partition associated with relation R, first, for each element a we can form its equivalence class $[a]_R$. Next, we collect all of the respective equivalence classes and form a collection, defined as follows (where $A=fld(R)$).

(D44) $A/R =_{df} \{ [x]_R : x \in A \}$

This collection of subsets of A is called "A modulo R" or "A divided by R".

Now, one can prove the following theorem (exercise).

(t9)   if R is an equivalence relation on A, then A/R is a partition of A.

Thus, every partition induces an equivalence relation, and every equivalence relation induces a partition. In order to demonstrate that equivalence relations and partitions are "practically identical", we

need to prove that the two processes – forming the equivalence relation from the partition, and forming the partition from the equivalence relation – are mutually consistent.

In order to do this, we introduce further notation.

(d1) $\mathbb{E}(C)$ $=_{df}$ $\{ (x,y) : \exists Z(Z \in C \; \& \; x,y \in Z) \}$

(d2) $\Pi(R)$ $=_{df}$ $\{ [x]_R : x \in fld(R) \}$

Here, $\mathbb{E}(C)$ is the relation induced by collection C, and $\Pi(R)$ is the collection induced by the relation R. Notice that:

$$\mathbb{E}(C) \quad = \quad \mathbb{E}_C$$
$$\Pi(R) \quad = \quad fld(R)/R$$

Now, the question of mutual consistency amounts to this. Suppose we start with equivalence relation R, and we form the associated partition $\Pi(R)$; then, suppose we take the partition $\Pi(R)$ and form the associated equivalence relation $\mathbb{E}(\Pi(R))$. Question: does R equal $\mathbb{E}(\Pi(R))$?

There is a corresponding question. Suppose we start with a partition C on A, and we form the associated equivalence relation $\mathbb{E}(C)$, and then take this equivalence relation and form the associated partition $\Pi(\mathbb{E}(C))$. Question: does C equal $\Pi(\mathbb{E}(C))$?

The answer to both questions is the same – yes. In particular, we have the following theorem, where the tacit proviso is that C is a partition and R is an equivalence relation.

(t10) $R = \mathbb{E}(C) \qquad \leftrightarrow \qquad C = \Pi(R)$

Given the natural correspondence between partitions and equivalence relations, we can transfer certain concepts from one to the other. For example, we already observed that the collection of all partitions on a given set A is partially ordered by the fineness relation. This means there is a corresponding partial ordering of the equivalence relations on A. Indeed that partial ordering is none other than the inclusion relation (suitably relativized)! In particular, we have the following theorems.

(t11) $P_1 \leq P_2 \quad \leftrightarrow \quad \mathbb{E}(P_1) \subseteq \mathbb{E}(P_2)$

(t12) $E_1 \subseteq E_2 \quad \leftrightarrow \quad \Pi(E_1) \leq \Pi(E_2)$

(t13) $P_1 < P_2 \quad \leftrightarrow \quad \mathbb{E}(P_1) \subset \mathbb{E}(P_2)$

(t14) $E_1 \subset E_2 \quad \leftrightarrow \quad \Pi(E_1) < \Pi(E_2)$

17.   Appendix 1: Generalized Set-Abstracts

Certain sets are easier to describe if we modify our notation slightly. Suppose we want to denote the set of all singletons of members of set A; call this set S temporarily. We have the following fact about S.

$$\forall x(x \in S \leftrightarrow \exists y(y \in A \ \& \ x = \{y\}))$$

In other words, $x \in S$ iff x is the singleton of some element of A. The official set-abstract is written as follows.

$$\{x : \exists y(y \in A \ \& \ x = \{y\})\}$$

A more natural notation, which uses a *generalized set-abstract*, goes as follows.

$$\{\{x\} : x \in A\}$$

This expression does not qualify as well-formed according to our official grammar, so we need some way to define it, and expressions like it, in terms of official notation.

The fundamental set-abstract notation is given by the following scheme,

$$\{\nu : \mathbb{F}\}$$

where ν is any variable, and $\mathbb{F}$ is any formula. On the other hand, generalized set-abstract notation is given by the following scheme,

$$\{\tau : \mathbb{F}\}$$

where τ is any *singular-term*, and $\mathbb{F}$ is any formula.

At this point, it might be useful to review what exactly constitutes a singular-term in the language of set theory. The general syntactic principles are presented schematically as follows.

(s1)   every *variable* is a singular-term;

(s2)   every *constant* is a singular-term;

(s3)   every *proper name* is a singular-term;

(s4)   the application of a *function-sign* to the appropriate number of singular-terms yields another singular-term;

(s5)   the application of a *subnective* to the appropriate number of formulas yields a singular-term;

(s6)   nothing else is a singular-term.

Variables include 'x', 'y', 'z', 'X', 'Y', 'Z', etc. Constants include 'a', 'b', 'c', 'A', 'B', 'C'; these are used primarily in derivations for universal derivation and existential elimination. They also often stand in proxy for arbitrary singular-terms in definitions. Proper names include '$\varnothing$'. Function-signs include '{...}' (singletons, doubletons, etc.), '$\cap$', '$\cup$', etc. The expression '(,)' is also a two-place function-sign. Subnectives include the set-abstraction operators $\{x:\_\}$ $\{y:\_\}$, etc., which are all one-place subnectives.

What we need is a definition of generalized set-abstracts of the form $\{\tau:\mathbb{F}\}$ in terms of simple set-abstracts of the form $\{v:\mathbb{F}\}$.

In order to do this we require additional notational machinery in the metalanguage of set theory. In general, both singular-term τ and formula $\mathbb{F}$ will involve free variables, and nearly always τ and $\mathbb{F}$ will share free variables in common. For example, in the following expressions

$$x \cap y \qquad\qquad y \in z$$

'x' and 'y' are free in the singular-term, 'y' and 'z' are free in the formula, and 'y' is the only common free variable. A more complicated example: in the following expressions

$$\{x : x \subseteq y\} \cup \{x : x \subseteq z\} \qquad \forall y(y \in z)$$

'y' and 'z' are free in the singular-term, and 'z' is free in the formula. In general, it is visually easier if we don't combine these two expressions, since 'y' is free in the singular-term but bound in the formula. Rather, the bound variable should be changed to, say, 'x'.

With this in mind, we define generalized set-abstraction as follows.

(D) $\{\tau : \mathbb{F}\}$ $=_{df}$ $\{\, v_0 : \exists v_1 \ldots \exists v_k[\mathbb{F} \ \& \ v = \tau]\,\}$

where $v_1 \ldots v_k$ are all the variables free in both τ and $\mathbb{F}$
and v_0 is any variable not free in either τ or $\mathbb{F}$

A few examples will help one understand the abstract definition. The first two come from arithmetic.

$\{x^2 : x \text{ is even}\}$	$=_{df}$	$\{\, y : \exists x(x \text{ is even } \& \ y = x^2)\,\}$
$\{x+y : y \text{ is even}\}$	$=_{df}$	$\{\, z : \exists y(y \text{ is even } \& \ z = x+y)\,\}$
$\{x \cap y : y \in z\}$	$=_{df}$	$\{\, w : \exists y(y \in z \ \& \ w = x \cap y)\,\}$
$\{\{x,y\} : x \in z \ \& \ y \in z\}$	$=_{df}$	$\{\, w : \exists x \exists y(x \in z \ \& \ y \in z \ \& \ w = \{x,y\})\,\}$
$\{\{x:x\subseteq y\} : y \in z\}$	$=_{df}$	$\{\, w : \exists y(y \in z \ \& \ w = \{x:x\subseteq y\})\,\}$

Before continuing, it is important to note that the legitimacy of the generalized set-abstract $\{\tau:\mathbb{F}\}$ reduces to the legitimacy of the corresponding simple set-abstract. Thus, some will be ok, and others will not. Later, we will add a special axiom (The Axiom of Replacement) for set-abstracts of the form $\{\tau(x) : x \in A\}$. For the moment, however, we do not presume such an axiom.

18.   Appendix 2: Computationally-Primitive Expressions

When asked to *compute* the value of an expression, what counts as a legitimate answer?   It depends on the context. For example, in arithmetic, if asked to compute 2+3, one cannot answer "1+4" , for although it is surely true that 2+3 **is** 1+4, this sort of expression is not acceptable. The acceptable answer is "5".

Likewise, in set theory, if asked to *compute*, say, $\{a,b\}\cup\{c,d\}$, the following is *not* an acceptable answer: $\{x : x \in \{a,b\}$ or $x \in \{c,d\}\}$. The acceptable answer is: $\{a,b,c,d\}$. Although the latter is not a primitive expression in the formal version of set theory we have adopted, it is nonetheless "computationally-primitive"; it is intuitively primitive.

The following is our official definition of *computationally-primitive*, which is a definition in the metalanguage.

(1)   Every constant and every proper name is computationally-primitive.
(2)   If $\varepsilon_1, \varepsilon_2, ..., \varepsilon_k$ are all computationally-primitive, then so is: $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_k\}$.
(3)   If ε_1 and ε_2 are computationally-primitive, then so is: $(\varepsilon_1,\varepsilon_2)$.
(4)   Nothing else is computationally-primitive.

Thus, when asked to compute the value of a singular-term τ, a correct answer is any computationally-primitive expression e such that $\ulcorner\tau=e\urcorner$ is true. The following are examples.

Where $R = \{ (1,4), (1,2), (2,2), (3,3), (4,3) \}$,

(1) $dom(R) = \{1,2,3,4\}$ [note: numerals are proper names.]
(2) $ran(R) = \{2,3,4\}$
(3) $R^{\rightarrow}\{2,3,4\} = \{2,3\}$
(4) $R^{\leftarrow}\{2,3\} = \{1,2,3,4\}$
(5) $R\circ R^{-1} = \{ (1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4) \}$

Similarly, if asked to give an example of a such-and-such, then one is expected to write an expression in computationally-primitive notation. The following are examples.

Supposing $A = \{1,2,3\}$, $B = \{1,2,3,4,5\}$:

(1)   an example of a *relation* on A that is *reflexive* but not *transitive*:
        $\{ (1,1), (2,2), (3,3), (2,3), (2,1) \}$

(2)   an example of a *function from B onto A*:
        $\{ (1,1), (2,2), (3,3), (4,3), (5,3) \}$

(3)   an example of a *partition C* on B such that $A \in C$:
        $\{ \{1,2,3\}, \{4,5\} \}$

19.   Definitions For Chapter 2

1.   Explicit Definitions

0. $\{\tau : \mathbb{F}\}$ $=_{df}$ $\{v_0 : \exists v_1 \ldots \exists v_k (\mathbb{F}\ \&\ v_0 = \tau\}$ [generalized set-abstract]
       [here, v_0 is not free in τ or $\mathbb{F}$;
       $v_1, \ldots, v_k$ are all the free variables common to τ and $\mathbb{F}$]

1. (a,b) $=_{df}$ $\{\{a\},\{a,b\}\}$ [ordered-pair]

2. $A \times B$ $=_{df}$ $\{(x,y) : x \in A\ \&\ y \in B\}$ [Cartesian Product]

3. $OP[a]$ $=_{df}$ $\exists x \exists y [a = (x,y)]$ [ordered-pair]

4. $1st(a)$ $=_{df}$ $\imath x \exists y [a = (x,y)]$ [1st component]

5. $2nd(a)$ $=_{df}$ $\imath y \exists x [a = (x,y)]$ [2nd component]

6. $\mathcal{R}[A]$ $=_{df}$ $\forall x (x \in A \rightarrow \exists y \exists z [x = (y,z)])$ [_ is a relation]

7. aRb $=_{df}$ $(a,b) \in R$ [_ bears _ to _]

8. $\mathcal{R}[R,A,B]$ $=_{df}$ $R \subseteq A \times B$ [_ is a relation from _ to _]

9. $\mathcal{R}[R,A]$ $=_{df}$ $\mathcal{R}[R,A,A]$ [_ is a relation on _]

10. $\mathcal{R}(A,B)$ $=_{df}$ $\mathcal{R}(A \times B)$ [the relations from _ to _]

11. $\mathbb{U}_A$ $=_{df}$ $A \times A$ [universal relation on A]

12. $\mathbb{I}_A$ $=_{df}$ $\{(x,x) : x \in A\}$ [identity relation on A]

13. $dom(R)$ $=_{df}$ $\{x : \exists y [xRy]\}$ [domain]

14. $ran(R)$ $=_{df}$ $\{y : \exists x [xRy]\}$ [range]

15. $fld(R)$ $=_{df}$ $dom(R) \cup ran(R)$ [field]

16. $R^{\rightarrow}(A)$ $=_{df}$ $\{y : \exists x (x \in A\ \&\ xRy)\}$ [image]

17. $R^{\leftarrow}(A)$ $=_{df}$ $\{x : \exists y (y \in A\ \&\ xRy)\}$ [pre-image]

18. R^{-1} $=_{df}$ $\{(x,y) : yRx\}$ [converse]

19. $R \circ S$ $=_{df}$ $\{(x,z) : \exists y [xRy\ \&\ ySz]\}$ [composition]

20. $R|A$ $=_{df}$ $\{(x,y) : xRy\ \&\ x,y \in A)$ [restriction]

21. $\forall xy$ $=_{df}$ $\forall x \forall y$
       $\exists xy$ $=_{df}$ $\exists x \exists y$ [quantifier abbr.]

22. $a,b \in S$ $=_{df}$ $a \in S\ \&\ b \in S$ [&-$\in$ abbr.]

23. $ref[R,A]$ $=_{df}$ $\forall x (x \in A \rightarrow xRx)$ [reflexive]
       $ref[R]$ $=_{df}$ $ref[R,fld(R)]$

24. $irr[R,A]$ $=_{df}$ $\forall x(x \in A \rightarrow \sim[xRx])$ [irreflexive]
        $irr[R]$ $=_{df}$ $irr[R,fld(R)]$

25. $tr[R,A]$ $=_{df}$ $\forall xyz(x,y,z \in A \rightarrow [xRy \ \& \ yRz . \rightarrow xRz])$ [transitive]
        $tr[R]$ $=_{df}$ $tr[R,fld(R)]$

26. $int[R,A]$ $=_{df}$ $\forall xyz(x,y,z \in A \rightarrow [xRy \ \& \ yRz . \rightarrow \sim xRz])$ [intransitive]
        $int[R]$ $=_{df}$ $int[R,fld(R)]$

27. $sym[R,A]$ $=_{df}$ $\forall xy(x,y \in A \rightarrow [xRy \rightarrow yRx])$ [symmetric]
        $sym[R]$ $=_{df}$ $sym[R,fld(R)]$

28. $asy[R,A]$ $=_{df}$ $\forall xy(x,y \in A \rightarrow [xRy \rightarrow \sim yRx])$ [asymmetric]
        $asy[R]$ $=_{df}$ $asy[R,fld(R)]$

29. $wasym[R,A]$ $=_{df}$ $\forall xy(x,y \in A \rightarrow [x \neq y \rightarrow . \ xRy \rightarrow \sim yRx])$ [weakly asymmetric]
        $wasym[R]$ $=_{df}$ $wasym[R,fld(R)]$

30. $ant[R,A]$ $=_{df}$ $\forall xy[xRy \ \& \ yRx . \rightarrow x=y]$ [anti-symmetric]
        $ant[R]$ $=_{df}$ $ant[R,fld(R)]$

31. $sc[R,A]$ $=_{df}$ $\forall xy[x,y \in fld(R) \rightarrow . \ xRy \vee yRx$ [strongly connected]
        $sc[R]$ $=_{df}$ $sc[R,fld(R)]$

32. $wc[R,A]$ $=_{df}$ $\forall xy[x,y \in fld(R) \rightarrow . \ xRy \vee yRx \vee x=y$ [weakly connected]
        $wc[R]$ $=_{df}$ $wc[R,fld(R)]$

33. $qo[R]$ $=_{df}$ $ref[R] \ \& \ tr[R]$ [quasi-order]
        $qo[R,A]$ $=_{df}$ $qo[R] \ \& \ fld(R)=A$

34. $po[R]$ $=_{df}$ $ref[R] \ \& \ tr[R] \ \& \ ant[R]$ [partial-order]
        $po[R,A]$ $=_{df}$ $po[R] \ \& \ fld(R)=A$

35. $lo[R]$ $=_{df}$ $po[R] \ \& \ sc[R]$ [linear-order]
        $lo[R,A]$ $=_{df}$ $lo[R] \ \& \ fld(R)=A$

36. $slo[R]$ $=_{df}$ $asym[R] \ \& \ tr[R] \ \& \ wc[R]$ [strict linear-order]
        $slo[R,A]$ $=_{df}$ $slo[R] \ \& \ fld(R)=A$

37. $equ[R]$ $=_{df}$ $ref[R] \ \& \ sym[R] \ \& \ tr[R]$ [equivalence relation]
        $equ[R,A]$ $=_{df}$ $equ[R] \ \& \ fld(R)=A$

38. $\sigma v \mathbb{F}$ $=_{df}$ $\imath s(\mathbb{F}[s/v] \ \& \ \forall v(\mathbb{F} \rightarrow v \subseteq s))$ [smallest v such that $\mathbb{F}$]

39. $[a]_R$ $=_{df}$ $\{x : aRx\}$ [equivalence class]

40. $\Pi[C,A]$ $=_{df}$ $\cup C=A \ \& \ \forall X(X \in C \rightarrow X \neq \varnothing) \ \&$
        $\forall XY(X,Y \in C \rightarrow . \ X \perp Y \vee X=Y)$ [partition]

41. $\mathcal{B} \leq \mathcal{C}$ $=_{df}$ $\forall X(X \in \mathcal{C} \rightarrow \exists Y(Y \in \mathcal{B} \ \& \ Y \subseteq X))$ [finer than]

42. $\mathcal{B} < \mathcal{C}$ $=_{df}$ $\mathcal{B} \leq \mathcal{C} \ \& \ \sim[\mathcal{C} \leq \mathcal{B}]$ [strictly finer than]

43. $\mathbb{E}(C)$ $=_{df}$ $\{ (x,y) : \exists X(X \in C \ \& \ x,y \in X \}$ [equivalence relation induced by C]

44. A/R $=_{df}$ $\{ [x]_R : x \in A \}$ [A modulo R]

2.   Contextual Definitions

1. $p \in A \times B$ $=_{df}$ $\exists xy(x \in A \ \& \ y \in B \ \& \ p=(x,y))$

2. $(a,b) \in A \times B$ $=_{df}$ $a \in A \ \& \ b \in B$

3. $a \in dom(R)$ $=_{df}$ $\exists x[aRx]$

4. $a \in ran(R)$ $=_{df}$ $\exists x[xRa]$

5. $a \in fld(R)$ $=_{df}$ $a \in dom(R) \ \lor \ a \in ran(R)$

6. $(a,b) \in \breve{R}$ $=_{df}$ $(b,a) \in R$

7. $aR^{-1}b$ $=_{df}$ bRa

8. $a \in R^{\rightarrow}(A)$ $=_{df}$ $\exists x(x \in A \ \& \ xRa)$

9. $a \in R^{\leftarrow}(A)$ $=_{df}$ $\exists x(x \in A \ \& \ aRx)$

10. $(a,b) \in R \circ S$ $=_{df}$ $\exists x(aRx \ \& \ xSb)$

11. $aR \circ Sb$ $=_{df}$ $\exists x(aRx \ \& \ xSb)$

12. $(a,b) \in \mathbb{E}(C)$ $=_{df}$ $\exists X(X \in C \ \& \ a,b \in X\}$

13. $b \in [a]_R$ $=_{df}$ aRb

14. $A \in \Pi(R)$ $=_{df}$ $\exists x(x \in fld(R)\} \ \& \ A=[x]_R)$

20.   Theorems for Chapter 2

1. $\{a,b\}=\{p,q\} \;\rightarrow\; (a{=}p \vee a{=}q \;.\&.\; b{=}p \vee b{=}q)$

2. $a{=}b \;\rightarrow\; \{a,b\}{=}\{a\}$

3. $\{a\}{=}\{b\} \;\rightarrow a{=}b$

4. $(a,b){=}(p,q) \;\rightarrow.\; a{=}p \;\&\; b{=}q$

5. $(a,b) \in A{\times}B \;\leftrightarrow.\; a \in A \;\&\; b \in B$

6. $A{\times}B = \varnothing \;\leftrightarrow.\; A{=}\varnothing \vee B{=}\varnothing$

7. $A{\times}B = B{\times}A \;\leftrightarrow.\; A{=}\varnothing \vee B{=}\varnothing \vee A{=}B$

8. $B \subseteq C \;\rightarrow\; A{\times}B \subseteq A{\times}C$

9. $A{\times}(B{\cap}C) = (A{\times}B) \cap (A{\times}C)$

10. $A{\times}(B{\cup}C) = (A{\times}B) \cup (A{\times}C)$

11. $A{\times}(B{-}C) = (A{\times}B) - (A{\times}C)$

12. $A \times \cap(C) \;=\; \cap\{A{\times}X : X \in C\}$

13. $A \times \cup(C) \;=\; \cup\{A{\times}X : X \in C\}$

14. $dom(R{\cup}S) \;=\; dom(R) \cup dom(S)$

15. $ran(R{\cup}S) \;=\; ran(R) \cup ran(S)$

16. $dom(R{\cap}S) \;\subseteq\; dom(R) \cap dom(S)$

17. $ran(R{\cap}S) \;\subseteq\; ran(R) \cap ran(S)$

18. $dom(R) - dom(S) \;\subseteq\; dom(R{-}S)$

19. $ran(R) - ran(S) \;\subseteq\; ran(R{-}S)$

20.   the negations of the converses of 16-19.

21. $R \in \mathcal{R}(A,B) \;\rightarrow\; R^{-1} \in \mathcal{R}(B,A)$

22. $\mathcal{R}[R] \;\rightarrow\; ((R)^{-1})^{-1} \;=\; R$

23. $(R{\cap}S)^{-1} \;=\; R^{-1} \cap S^{-1}$

24. $(R{\cup}S)^{-1} \;=\; R^{-1} \cup S^{-1}$

25. $(R{-}S)^{-1} \;=\; R^{-1} - S^{-1}$

26. $R{\circ}(S{\circ}T) \;=\; (R{\circ}S){\circ}T$

27. $R \in \mathcal{R}(A,B) \;\&\; S \in \mathcal{R}(B,C) \;.\rightarrow\; R{\circ}S \in \mathcal{R}(A,C)$

28. $R \in \mathcal{R}(A,B) \;\&\; S \in \mathcal{R}(B,C) \;\&\; B{\perp}C \;.\rightarrow\; R{\circ}S = \varnothing$

29. $(R{\circ}S)^{-1} \;=\; S^{-1}{\circ}R^{-1}$

30. $dom(R^{-1}) \;=\; ran(R)$

31. $ran(R^{-1}) \;=\; dom(R)$

32. $dom(R{\circ}S) \subseteq dom(R)$

33. $ran(R{\circ}S) \subseteq ran(S)$

34. $\sim\forall XY[dom(X) \subseteq dom(X{\circ}Y)]$

35. $\sim\forall XY[ran(Y) \subseteq ran(X{\circ}Y)]$

36. $R \in \mathcal{R}(A,B) \rightarrow R^{\rightarrow}(A) = ran(R)$

37. $R \in \mathcal{R}(A,B) \rightarrow R^{\leftarrow}(B) = dom(R)$

37. $(R^{-1})^{\rightarrow}(A) = R^{\leftarrow}(A)$

38. $A \subseteq dom(R) \rightarrow A \subseteq R^{\leftarrow}(R_{\_}(A))$

39. $R^{\rightarrow}(S^{\rightarrow}(A) = (S \circ R)^{\rightarrow}(A)$

40. $A \subseteq B \rightarrow R^{\rightarrow}(A) \subseteq R^{\rightarrow}(B)$

41. $A \subseteq B \rightarrow R^{\leftarrow}(A) \subseteq R^{\leftarrow}(B)$

42. $R^{\leftarrow}(A \cup B) = R^{\leftarrow}(A) \cup R^{\leftarrow}(B)$

43. $R^{\leftarrow}(A \cap B) \subseteq R^{\leftarrow}(A) \cap R^{\leftarrow}(B)$

44. $\sim \forall XY[R^{\leftarrow}(A) \cap R^{\leftarrow}(B) \subseteq R^{\leftarrow}(A \cap B)]$

In the following, $I = \mathbb{I}_{fld(R)}$, and $U = \mathbb{U}_{fld(R)}$.

45. $ref[R] \leftrightarrow I \subseteq R$

46. $irr[R] \leftrightarrow I \perp R$

47. $sym[R] \leftrightarrow R \subseteq R^{-1}$

48. $asy[R] \leftrightarrow R \perp R^{-1}$

49. $tr[R] \leftrightarrow R \circ R \subseteq R$

50. $int[R] \leftrightarrow R \circ R \perp R$

51. $ant[R] \leftrightarrow R \cap R^{-1} \subseteq I$

52. $sc[R] \leftrightarrow R \cup R^{-1} = U$

53. $wc[R] \leftrightarrow R \cup R^{-1} \cup I = U$

54. $dom(R) = A \leftrightarrow I_A \subseteq R \circ R^{-1}$

55. $ran(R) = B \leftrightarrow I_B \subseteq R^{-1} \circ R$

56. $o\text{-}m[R] \leftrightarrow R \circ R^{-1} \subseteq I_A$

57. $m\text{-}o[R] \leftrightarrow R^{-1} \circ R \subseteq I_B$

58. $m\text{-}o[R] \,\&\, ran(R)=B \leftrightarrow R^{-1} \circ R = I_B$

59. $o\text{-}m[R] \,\&\, dom(R)=A \leftrightarrow R \circ R^{-1} = I_B$

60. $asy[R] \rightarrow irr[R]$

61. $sym[R] \,\&\, ant[R] .\rightarrow \exists X[R = I_X]$

62. $A \cap \cup C = \cup\{A \cap X : X \in C\}$

63. $A \cup \cap C = \cap\{A \cup X : X \in C\}$

78. $A - \cup C = \cup\{A - X : X \in C\}$

79. $A - \cap C = \cup\{A - X : X \in C\}$

80. $\cup \mathcal{B} \cap \cup C = \cup\{X \cap Y : X \in \mathcal{B} \,\&\, Y \in C\}$

81. $\cap \mathcal{B} \cup \cap C = \cap\{X \cup Y : X \in \mathcal{B} \,\&\, Y \in C\}$

82. $\cup\cup\mathcal{B} = \cup\cup C \rightarrow \cup\mathcal{B} = \cup C$

83. $\cup\cup\mathcal{B} = \cup\cup C \rightarrow \cap\mathcal{B} = \cap C$

21.   Exercises for Chapter 2

1.   Part 1

Supposing $R = \{(1,4), (1,2), (2,2), (3,3), (4,3)\}$, compute the following; i.e., convert each expression into a computationally-primitive expression.

       (1) $dom(R)$
       (2) $ran(R)$
       (3) $R^{\rightarrow}(\{2,3,4\})$
       (4) $R^{\leftarrow}(\{2,3\})$
       (5) R^{-1}
       (6) $R \circ R^{-1}$
       (7) $R^{-1} \circ R$
       (8) $R \circ R$

2.   Part 2

Supposing $A = \{1,2,3\}$, $B = \{1,2,3,4,5\}$:

(1)   Construct an example of a *relation* R on A such that
       R is *reflexive* but not *transitive*.
(2)   Construct an example of a *relation* on A that is *reflexive*, and *symmetric*, but not *transitive*.
(3)   Construct an example of a *relation* on A that is *irreflexive* and *transitive*.
(4)   Construct an example of an equivalence relation on B; what is the corresponding partition?
(5)   Construct an example of a *partition* C on B such that $A \in C$; what is the corresponding equivalence relation?

3.   Part 3

Symbolize each of the following expressions in the language of first-order logic; i.e., for each expression, display its logical form relative to first-order logic.

       (1) aRb
       (2) $\{a\}R\{a,b\}$
       (3) $ran(R) \cup dom(R)$
       (4) $[a]_R = [b]_R \cap [c]_R$
       (5) $R^{-1} \circ R$

Use the following schemes:

    Atomic singular-terms: lower case letters.
    Function-signs: lower case letters and parentheses; e.g.:
    $f(\alpha)$, $s(\alpha,\beta)$, $p(\alpha,\beta,\gamma)$.
    Predicate signs: upper case letters and square brackets; e.g.:
    $P[\alpha]$, $F[\alpha,\beta]$, $B[\alpha,\beta,\gamma]$.

4.   Part 4

Convert each of the following generalized set-abstracts into the corresponding simple set-abstract according to the general definition of generalized set-abstracts.

(1) $\{ \{x\} : x \in Y \}$
(2) $\{ X{-}Y : X \in Z \ \& \ Y \in W \}$
(3) $\{ A{\times}X : X \in B \}$
(4) $\{ (x,y) : x \in y \}$
(5) $\{ \cup\mathcal{A} \cap \cup\mathcal{B} : \mathcal{A} \subseteq \mathcal{B} \}$

5.   Part 5

Convert each of the following expressions into *primitive notation* (logical notation, plus '$\in$' and '$\{v{:}\mathbb{F}\}$'), in accordance with the *explicit definitions*.

(1) (a,b)
(2) $A{\times}B$
(3) aRb
(4) $a[R{\circ}S]b$
(5) R^{-1}

6.   Part 6

Convert each of the following into "relatively primitive notation", in accordance with the *explicit definitions*. This includes primitive notation, plus all defined symbols from Chapter 1, plus ordered-pair notation. In particular, convert every generalized set-abstract into the corresponding simple set-abstract.

(1) $\{X{\cap}Y : Y \in Z\}$
(2) $dom(R)$
(3) $R^{\leftarrow}(A)$
(4) R is an equivalence relation on S
(5) C is a partition on S

22. Answers to Exercises for Chapter 2

1. Part 1

Supposing $R = \{(1,4), (1,2), (2,2), (3,3), (4,3)\}$,

        (1) $dom(R)$ $= \{1,2,3,4\}$
        (2) $ran(R)$ $= \{2,3,4\}$
        (3) $R^{\rightarrow}(\{2,3,4\})$ $= \{2,3\}$
        (4) $R^{\leftarrow}(\{2,3\})$ $= \{1,2,3,4\}$
        (5) R^{-1} $= \{(4,1), (2,1), (2,2), (3,3), (3,4)\}$
        (6) $R \circ R^{-1}$ $= \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (3,4), (4,3)\}$
        (7) $R^{-1} \circ R$ $= \{(2,2), (3,3), (4,4), (2,4), (4,2)\}$
        (8) $R \circ R$ $= \{(1,2), (1,3), (2,2), (3,3), (4,3)\}$

2. Part 2

Supposing $A = \{1,2,3\}$, $B = \{1,2,3,4,5\}$,

        (1)   an example of a <u>relation</u> R on A such that
                 R is <u>reflexive</u> but not <u>transitive</u>:
                 $\{(1,1), (1,2), (2,3), (2,2), (3,3)\}$

(2)   an example of a <u>relation</u> on A that is <u>reflexive</u>, and <u>symmetric</u>, but not <u>transitive</u>:
       $\{(1,1), (1,2), (2,1), (2,3), (3,2), (2,2), (3,3)\}$

(3)   an example of a <u>relation</u> on A that is <u>irreflexive</u> and <u>transitive</u>:
       $\{(1,2), (2,3), (1,3)\}$

(4)   an example of an equivalence relation on B; the corresponding partition:
       $\{(1,1), (2,2), (3,3), (4,4), (5,5)\}$
       $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$

(5)   an example of a <u>partition</u> C on B such that $A \in C$; the corresponding equivalence relation:
       $\{\{1,2,3\},\{4\},\{5\}\}$
       $\{(1,1),(2,2),(3,3),(4,4),(5,5),(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)\}$

3. Part 3

Symbolization in the language of first-order logic.

        (1) aRb $B[a,r,b]$
        (2) $\{a\}R\{a,b\}$ $B[s(a),r,d(a,b)]$
        (3) $ran(R) \cup dom(R)$ $u(f(r),g(r))$
        (4) $[a]_R = [b]_R \cap [c]_R$ $e(a,r)=i(e(b,r),e(c,r))$
        (5) $R^{-1} \circ R$ $c(i(r),r)$

4.   Part 4

Conversion of generalized set-abstracts into a simple set-abstracts.

(1) $\{\{x\} : x \in Y\}$ $\{ w : \exists x(x \in Y \ \& \ w=\{x\}) \}$
(2) $\{X\text{-}Y : X \in Z \ \& \ Y \in W\}$ $\{ U : \exists X \exists Y(X \in Z \ \& \ Y \in W \ \& \ U=X\text{-}Y) \}$
(3) $\{A \times X : X \in B\}$ $\{ Y : \exists X(X \in B \ \& \ Y=A \times X) \}$
(4) $\{(x,y) : x \in y\}$ $\{ z : \exists x \exists y(x \in y \ \& \ z=(x,y)) \}$
(5) $\{\cup\mathcal{A} \cap \cup\mathcal{B} : \mathcal{A} \subseteq \mathcal{B}\}$ $\{ X : \exists\mathcal{A}\exists\mathcal{B}(\mathcal{A} \subseteq \mathcal{B} \ \& \ X = \cup\mathcal{A} \cap \cup\mathcal{B}) \}$

5.   Part 5

Definitions in primitive notation.

(1) (a,b) $=_{df}$ $\{ x : x=\{x:x=a\} \vee x=\{x:x=a \vee x=b\} \}$
(2) $A \times B$ $=_{df}$ $\{ x : \exists y \exists z(y \in A \ \& \ z \in B \ \& \ x=\{x : x=\{x:x=y\} \vee x=\{x:x=y \vee x=z\}\} \}$
(3) aRb $=_{df}$ $\{ x : x=\{x:x=a\} \vee x=\{x:x=a \vee x=b\}\} \} \in R$
(4) $a[R \circ S]b$ $=_{df}$ $\{ x : x=\{x:x=a\} \vee x=\{x:x=a \vee x=b\}\} \} \in$
                    $\{ w : \exists y(\{x:x=\{x:x=a\} \vee x=\{x:x=a \vee x=w\}\} \} \in R \ \&$
                    $\{ w : \exists y(\{x:x=\{x:x=w\} \vee x=\{x:x=w \vee x=b\}\} \} \in S\}$
(5) R^{-1} $=_{df}$ $\{ w : \exists y \exists z(\{x:x=\{x:x=y\} \vee x=\{x:x=y \vee x=z\}\}\} \in R \ \& \ w=(z,y)) \}$

6.   Part 6

Conversion into "relatively primitive notation".

(1) $\{X \cap Y : Y \in Z\}$ $=_{df}$ $\{ W : \exists Y(Y \in Z \ \& \ W=X \cap Y) \}$
(2) $dom(R)$ $=_{df}$ $\{ x : \exists y[(x,y) \in R] \}$
(3) $R^{\leftarrow}(A)$ $=_{df}$ $\{ x : \exists y(y \in A \ \& \ (x,y) \in R) \}$
(4) $equ[R,S]$ $=_{df}$ $ref[R] \ \& \ sym[R] \ \& \ tr[R] \ \& \ fld(R)=S$
                $=_{df}$ $\forall x(x \in fld(R) \rightarrow xRx) \ \&$
                        $\forall xy(x,y \in fld(R) \rightarrow. \ xRy \rightarrow yRx) \ \&$
                        $\forall xyz(x,y,z \in fld(R) \rightarrow. \ (xRy \ \& \ yRz) \rightarrow xRz) \ \&$
                        $fld(R)=S$
        $=_{df}$ $\forall x(x \in fld(R) \rightarrow (x,x) \in R) \ \&$
            $\forall xy(x,y \in fld(R) \rightarrow. \ (x,y) \in R \rightarrow (x,y) \in R) \ \&$
            $\forall xyz(x,y,z \in fld(R) \rightarrow. \ ((x,y) \in R \ \& \ (y,z) \in R) \rightarrow (x,z) \in R) \ \& \ fld(R)=S$

            where
            $fld(R) =_{df} \{x : \exists y[(x,y) \in R]\} \cup \{x : \exists y[(y,x) \in R]\}$

(5) $C[C,S]$ $=_{df}$ $\cup C=S \ \&$
                $\forall X(X \in C \rightarrow X \neq \varnothing) \ \& \ \forall XY(X,Y \in C \rightarrow. \ X \perp Y \vee X=Y)$

23.   Examples of Derivations for Chapter2

#1:

(1)	S̶H̶O̶W̶: $\{a,b\}=\{p,q\} \rightarrow: a=p \vee a=q\ .\&.\ b=p \vee b=q$	CD
(2)	$\{a,b\}=\{p,q\}$	As
(3)	S̶H̶O̶W̶: $a=p \vee a=q\ .\&.\ b=p \vee b=q$	6,9,SL
(4)	$a \in \{a,b\}$	T61/ch1
(5)	$a \in \{p,q\}$	2,4,IL
(6)	$a=p \vee a=q$	5,Def { }
(7)	$b \in \{a,b\}$	T1:61
(8)	$b \in \{p,q\}$	2,7,IL
(9)	$b=p \vee b=q$	8,Def { }

#2:

(1)	S̶H̶O̶W̶: $a=b \rightarrow \{a\}=\{a,b\}$	CD
(2)	$a=b$	As
(3)	S̶H̶O̶W̶: $\{a\}=\{a,b\}$	A1
(4)	S̶H̶O̶W̶: $\forall x(x \in \{a\} \leftrightarrow x \in \{a,b\})$	5,10,SL
(5)	$[\rightarrow]\ c \in \{a\}$	As
(6)	S̶H̶O̶W̶: $c \in \{a,b\}$	Def { }
(7)	S̶H̶O̶W̶: $c=a \vee c=b$	DD
(8)	$c=a$	5,Def { }
(9)	$c=a \vee c=b$	8,SL
(10)	$[\leftarrow]\ c \in \{a,b\}$	As
(11)	S̶H̶O̶W̶: $c \in \{a\}$	Def { }
(12)	S̶H̶O̶W̶: $c=a$	DD
(13)	$c=a \vee c=b$	10,Def { }
(14)	$c=a$	2,13,IL

#3:

(1)	S̶H̶O̶W̶: $\{a\}=\{b\} \rightarrow a=b$	CD
(2)	$\{a\}=\{b\}$	As
(3)	S̶H̶O̶W̶: $a=b$	DD
(4)	$a \in \{a\}$	T60/ch1
(5)	$a \in \{b\}$	2,4,IL
(6)	$a=b$	5,Def { }

#10a: [first half]

(1)	S̶H̶O̶W̶: $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$	Def $\subseteq$ + UCD
(2)	$p \in A \times (B \cap C)$	As
(3)	S̶H̶O̶W̶: $p \in (A \times B) \cap (A \times C)$	Def $\cap$
(4)	S̶H̶O̶W̶: $p \in A \times B\ \&\ p \in A \times C$	5,14,SL
(5)	S̶H̶O̶W̶: $p \in A \times B$	Def $\times$
(6)	S̶H̶O̶W̶: $\exists x \exists y(x \in A\ \&\ y \in B\ \&\ p=(x,y))$	DD
(7)	$\exists x \exists y(x \in A\ \&\ y \in B \cap C\ \&\ p=(x,y))$	2,Def $\times$
(8)	$a \in A\ \&\ b \in B \cap C\ \&\ p=(a,b)$	7,$\exists$O
(9)	$b \in B \cap C$	8,SL
(10)	$b \in B\ \&\ b \in C$	9,Def $\cap$
(11)	$a \in A\ \&\ b \in B\ \&\ p=(a,b)$	8,10,SL
(12)	$\exists x \exists y(x \in A\ \&\ y \in B\ \&\ p=(x,y))$	11,QL
(13)	S̶H̶O̶W̶: $p \in A \times C$	DD
(14)	S̶H̶O̶W̶: $\exists x \exists y(x \in A\ \&\ y \in C\ \&\ p=(x,y))$	DD
(15)	....	...

#6:

(1)	SHOW: $A \times B = \varnothing \leftrightarrow. A = \varnothing \vee B = \varnothing$	2,14,SL
(2)	[→] $A \times B = \varnothing$	As
(3)	SHOW: $A = \varnothing \vee B = \varnothing$	∨D
(4)	$A \neq \varnothing$	As
(5)	$B \neq \varnothing$	As
(6)	SHOW: ※	DD
(7)	$\exists x[x \in A]$	4,T19/ch1
(8)	$\exists x[x \in B]$	5,T19/ch1
(9)	$a \in A$	7,∃O
(10)	$b \in B$	8,∃O
(11)	$(a,b) \in A \times B$	9,10,T5
(12)	$A \times B \neq \varnothing$	11,T19/ch1
(13)	※	2,12,SL
(14)	[←] $A = \varnothing \vee B = \varnothing$	As
(15)	SHOW: $A \times B = \varnothing$	ID
(16)	$A \times B \neq \varnothing$	As
(17)	SHOW: ※	DD
(18)	$\exists x[x \in A \times B]$	16,T19/ch1
(19)	$p \in A \times B$	18,∃O
(20)	$\exists x \exists y[x \in A \ \& \ y \in B \ \& \ p = (x,y)]$	19,Def $\times$
(21)	$a \in A \ \& \ b \in B \ \& \ p = (a,b)$	20,∃O
(23)	$A \neq \varnothing$	21a,T19/ch1
(24)	$B \neq \varnothing$	21b,T19/ch1
(25)	※	14,23,24,SL

#7a:

(1)	SHOW: $A \times B = B \times A \ \to. A = \varnothing \vee B = \varnothing \vee A = B$	CD
(2)	$A \times B = B \times A$	As
(3)	SHOW: $A = \varnothing \vee B = \varnothing \vee A = B$	∨D
(4a)	$A \neq \varnothing$	As
(4b)	$B \neq \varnothing$	As
(4c)	$A \neq B$	As
(5)	SHOW: ※	4c,6,SL
(6)	SHOW: $A = B$	T4/ch1
(7)	SHOW: $A \subseteq B$	Def $\subseteq$ + UCD
(8)	$a \in A$	As
(9)	SHOW: $a \in B$	DD
(10)	$\exists x[x \in B]$	4b,T19/ch1
(11)	$b \in B$	10,∃O
(12)	$(a,b) \in A \times B$	8,11,T5
(13)	$(a,b) \in B \times A$	2,11,IL
(14)	$a \in B \ \& \ b \in A$	13,T5
(15)	$a \in B$	14,SL
(16)	SHOW: $B \subseteq A$	Def $\subseteq$ + UCD
(17)	$b \in B$	As
(18)	SHOW: $b \in A$	DD
(19)	$\exists x[x \in A]$	4a,T19/ch1
(20)	$a \in A$	19,∃O
(21)	$(a,b) \in A \times B$	17,20,T5
(22)	$(a,b) \in B \times A$	2,21,IL
(23)	$a \in B \ \& \ b \in A$	22,T5
(24)	$b \in A$	23,SL