1 Basic Concepts

1. Membership

Basic to set theory is the notion of *membership* or *elementhood*. Indeed, set theory can be formulated in its entirety as a formal theory in which the only non-logical notion (symbol) is membership. The remaining notions can all be defined in terms of membership using the standard machinery of first-order logic (including identity and definite descriptions).

Although set theory can be formulated exclusively in terms of membership, in actual practice, free use is made of defined expressions. Accordingly, very few set-theoretic theorems are actually displayed in primitive notation.

Membership is a two-place relation, and its syntactic counterpart is a two-place predicate – $\in$ (epsilon). Following the usual mathematical custom, we write this predicate in infix notation, rather than in prefix notation (as is customary in abstract logic and syntax). Specifically, in order to say that one thing is a member (element) of another thing, we write their respective names, and we infix the membership predicate, $\in$. Thus, the formula,[1]

$$x \in y$$

may be read in either of the following ways.

x is a member of *y*
x is an element of *y*

Infixing epsilon is analogous to the way the identity relation is expressed by the formula '$x=y$'.

Also, in analogy with the identity predicate, we simplify negated formulas in the natural way, according to the following definitions.

(d1) $\alpha \neq \beta \quad =_{df} \quad \sim[\alpha = \beta]$
(d2) $\alpha \notin \beta \quad =_{df} \quad \sim[\alpha \in \beta]$

Here, 'α' and 'β' stand for arbitrary singular terms. The expression '$=_{df}$' is a metalinguistic expression used to indicate that expressions of one form *abbreviate* expressions of another form.

2. Pure Set Theory versus Impure Set Theory

In the previous section, in saying that membership is the only non-logical notion of set theory, I mean *pure* set theory. *Pure set theory* may be formulated as a formal theory in a first-order language in which the only non-logical symbol is '$\in$'. This is because pure set theory talks about sets, *and nothing else*. Thus, in particular, in the formula '$x \in y$', both x and y are sets.

By contrast, *impure set theory* (for lack of a better name!) talks not only about sets, but also about other things, including *first elements* and *classes* (not necessarily both). Briefly, the notion of class is a generalization of the notion of set; sets are special kinds of classes (see below). Anything that isn't a class is a *first element*, sometimes called a *point*. First elements, or points, which are very important in informal set theory, have no internal structure from the viewpoint of set theory. Of course,

[1] It depends upon the author whether '$\in$' and '=' are treated like other two-place functors written in infix format, specifically whether the official formula has outer parentheses. We adopt the view that parentheses (or brackets) officially flank identity statements and membership statements, and that these are dropped when convenient.

first elements may have very interesting internal structure from the viewpoint of, say, biology! The point is that they have no internal *set theoretic* structure. In informal set theory, as used throughout modern mathematics, points include numbers, geometrical points, vectors, group elements, etc. Points also include all concrete objects, as well as all abstract objects (other than classes!)

In one particular formulation of *impure* set theory, there are two non-logical predicates; besides '$\in$' there is the one-place predicate '***C***' ('...is a class').

Three other predicates may then be defined in terms of '***C***' and '$\in$', as follows.

$$\boldsymbol{F}a =_{\text{df}} \sim\boldsymbol{C}a$$
$$\boldsymbol{S}a =_{\text{df}} \boldsymbol{C}a \;\&\; \exists y[a \in y]$$
$$\boldsymbol{P}a =_{\text{df}} \boldsymbol{C}a \;\&\; \sim\exists y[a \in y]$$

[Here, a is any singular-term, and y is any variable not free in a.] The predicate letters have the following intended readings.

$\boldsymbol{C}x$: x is a class;
$\boldsymbol{F}x$: x is a first element;
$\boldsymbol{S}x$: x is a set;
$\boldsymbol{P}x$: x is a proper class.

A proper class is a class that is not a set, by the following theorem, which follows immediately from the definitions.

$$\forall x(\boldsymbol{P}x \leftrightarrow [\boldsymbol{C}x \;\&\; \sim\boldsymbol{S}x])$$

Alternatively, we could introduce three primitive one-place predicates, '***C***', '***S***', and '***P***', and add the following as axioms.

$$\forall x(\boldsymbol{S}x \leftrightarrow [\boldsymbol{C}x \;\&\; \exists y[x \in y]]);$$
$$\forall x(\boldsymbol{P}x \leftrightarrow [\boldsymbol{C}x \;\&\; \sim\exists y[x \in y]]).$$

Other than elegance, nothing hinges on whether we introduce additional terms by adding definitions or by adding axioms.

Officially, this book concentrates on pure set theory.

Nonetheless, we will adopt certain conventions borrowed from informal set theory. The usual informal convention is to use lower-case Roman letters to denote points, upper-case Roman letters to denote sets whose elements are points, and upper-case script letters to denote sets whose elements are sets.

Some sort of convention like this is occasionally useful in visually clarifying the hierarchy of sets, and we will use such a convention when it is helpful. For example, we might write

$$a \in B \;\&\; B \in \mathcal{C}$$

even though the following is equally legitimate in pure set theory.

$$a \in b \;\&\; b \in c$$

In pure set theory, every element of every set is itself a set, so no simple-minded syntactic convention can possibly do justice to the richness of possible relations among sets.

Notice in set theory that upper-case letters and script letters are employed just like lower-case letters – as atomic singular-terms. This is quite different from elementary logic, where upper-case letters are used as predicates.

In this connection, bear in mind the following distinction between elementary logic and first-order theories, such as set theory. On the one hand, in EL, every kind of predicate, every kind of function-sign, and every kind of constant, is instantiated. On the other hand, in any given FOT, only a handful of non-logical signs are employed. For example, in pure set theory (ST), there are only two predicates, '$\in$' and '$=$' , there are *no* function-signs, and there are *no* proper names, at least in the primitive vocabulary.

So, for example, the following are formulas of ST.

$$\forall x \exists y \forall z(x \in y \rightarrow x \in z)$$
$$\exists x \forall y[x \in y] \rightarrow \exists x \forall y[y \in x].$$

On the other hand, their *logical forms* are formulas of EL, but they are *not* formulas of ST.

$$\forall x \exists y \forall z(Rxy \rightarrow Rxz)$$
$$\exists x \forall y Rxy \rightarrow \exists x \forall y Ryx$$

3. The Axiom of Extensionality

Sets are like clubs in having members. But, unlike clubs, in general at least, sets are completely determined by their membership. Two clubs can be quite distinct, yet have precisely the same members; clubs are individuated not only by their membership but also by their social function, among other things.

In contrast to clubs, sets are completely determined (individuated) by their membership. If set A has the same members as set B, then A and B are in fact identical (i.e., $A=B$); alternatively stated, if A and B are distinct ($A \neq B$), then they don't have the same members. For example, the set consisting of all even prime numbers and the set consisting exclusively of the number 2 are identical.

The individuation of sets by their members is known as the **Principle of Extensionality**, which is formally stated as follows.

(E) $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x=y]$

Since the converse

(E^c) $\forall x \forall y[x=y \rightarrow \forall z(z \in x \leftrightarrow z \in y)]$

is a theorem of first-order logic (exercise), (E) is logically equivalent to the following alternative axiom.

(E*) $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x=y]$

Whereas the converse of (E) is a logical truth, (E*) is *not* a logical truth; specifically, it is not a theorem of first-order logic.

The logical form of (E) is the following.

(Ef) $\forall x \forall y[\forall z(Rzx \leftrightarrow Rzy) \rightarrow x=y]$

To show that (Ef) is not a theorem of first-order logic, we need merely construct an interpretation of the predicate 'R' that falsifies (Ef). Informal examples are easy to imagine. For example, let the domain be people, and let 'Rxy' stand for 'x is not taller than y'. Under this interpretation, (Ef) says in effect that no two people have the same height, which is false.

Since (E) is not a theorem of first-order logic, and since it formulates a basic and important principle about sets, it must be included in the list of non-logical theorems of set theory. One way to accomplish this (not the only way, of course) is to add it as a non-logical *axiom*. So our first axiom is the Axiom of Extensionality.

(a1) $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x=y]$ [Axiom of **Extensionality**]

4. Inclusion and Exclusion

According to the Axiom of Extensionality, in order to demonstrate (show) that sets A and B are identical ($A=B$), it is sufficient to show that they have precisely the same elements. This naturally breaks into two parts: showing that every element of A is also an element of B, and showing that every element of B is also an element of A.

The two-place relation implicitly defined by the expression 'every element of A is an element of B' is sufficiently important in set theory that it is given a special name and symbol. In particular, we say that A *is included in* B precisely if every element of A is an element of B; we also describe this relation by saying that A *is a subset of* B.

The usual symbol for the inclusion relation is '$\subseteq$', which is a derivative (defined) two-place predicate, formally defined as follows.

(d3) $A \subseteq B \;=_{\text{df}}\; \forall x(x \in A \rightarrow x \in B)$

Here, A and B are to be understood as arbitrary singular terms, and x is understood as any variable that does not occur free in either A or B.

Given the definition of inclusion, the following theorems are easily proven.

(1) $A \subseteq A$
(2) $A \subseteq B \;\&\; B \subseteq C \;.\!\rightarrow\; A \subseteq C$
(3) $A \subseteq B \;\&\; B \subseteq A \;.\!\rightarrow\; A=B$

Two devices are used to avoid symbolic clutter. First, we occasionally drop universal quantifiers when they are the major logical operators. Recall universal derivation: in order to show '$\forall xFx$', one shows 'Fa', where 'a' is new. Second, we occasionally use periods to mark the relatively dominant connective in a formula. Thus, (1)-(3) are short for the following universally quantified theorems.

(1*) $\forall X(X \subseteq X)$
(2*) $\forall X \forall Y \forall Z([X \subseteq Y \;\&\; Y \subseteq Z] \rightarrow X \subseteq Z)$
(3*) $\forall X \forall Y([X \subseteq Y \;\&\; Y \subseteq X] \rightarrow X=Y)$

The properties (1)-(3) are customarily described by saying that the inclusion relation is *reflexive*, *transitive*, and *anti-symmetric*. These general notions are explained in Chapter 2. The reflexivity and transitivity of inclusion are logical theorems – when rewritten in primitive notation. The anti-symmetry of inclusion is not a logical theorem; indeed, it is a restatement of the Axiom of Extensionality.

As seen above, every set is included in itself. Often, we want to say that A is *properly included* in (a *proper subset* of) B, which is to say that A is included in B, but A is *not identical to* B. This leads to the following further definition.

(d4) $A \subset B =_{\text{df}} A \subseteq B \;\&\; A \neq B$

The following theorems about proper inclusion can be easily proven.

(1) $\sim[A \subset A]$
(2) $(A \subset B \;\&\; B \subset C) \rightarrow A \subset C$
(3) $A \subset B \rightarrow \sim[B \subset A]$
(4) $A \subset B \rightarrow \sim[B \subseteq A]$

The first three properties are customarily described by saying that the proper inclusion relation is *irreflexive*, *transitive*, and *asymmetric*. Notice that (3) logically implies (1), and (4) logically implies (3).

The inclusion and proper inclusion relations have natural converses: we say that B (properly) *includes* A precisely if A is (properly) included in B; alternatively, we say that B is a (proper) *superset* of A. The official definitions go as follows.

(d5) $A \supseteq B =_{\text{df}} B \subseteq A$
(d6) $A \supset B =_{\text{df}} B \subset A$

The inclusion relation also has a natural negative counterpart, which might be called *exclusion*. To say that A *excludes* B is to say that *no* element of A is an element of B, which is to say that A and B have no elements in common. This is also commonly described by saying that A and B are *disjoint*. The official definition is given as follows.

(d7) $A \perp B =_{\text{df}} \sim\exists x(x \in A \;\&\; x \in B)$

Once again, A and B are arbitrary terms, and x is any variable not free in either A or B.

The properties of the exclusion relation are as follows.

(1) $A \perp B \rightarrow B \perp A$
(2) $A \perp A \rightarrow \sim\exists x[x \in A]$

These can be described by saying that the exclusion relation is *symmetric* and *anti-reflexive*.

5. Set-Abstracts

Sets may be denoted in various ways. In the simplest cases at least, the elements can be listed, and the resulting list can be enclosed in curly braces, and the resulting expression denotes the set having precisely these elements. The following are examples.

(1) $\{a\}$
(2) $\{a,b\}$
(3) $\{a,b,c\}$

The first set has exactly one member – a; the second set has a and b as members, and nothing else; the third set has a, b, c as members and nothing else. In other words, we have the following principles.

(p1) $\forall x(x \in \{a\} \leftrightarrow x=a)$
(p2) $\forall x(x \in \{a,b\} \leftrightarrow (x=a \vee x=b))$
(p3) $\forall x(x \in \{a,b,c\} \leftrightarrow (x=a \vee x=b \vee x=c))$

These convey the obvious in logically precise terms; for example, (p1) says that a is an element of $\{a\}$ but nothing else is. This could also be formulated as follows.

(p1*) $a \in \{a\} \;\&\; \sim\exists x(x \neq a \;\&\; x \in \{a\})$

It is easy to show (exercise) that (p1) and (p1*) are logically equivalent.[2]

In developing the formal theory of sets, we could introduce an infinite series of function-signs s_1, s_2, s_3, etc., one for each number of arguments, and introduce a corresponding infinite series of axioms (p1), (p2), (p3), etc. This would be logically acceptable, but it would *not* be elegant.

A more general syntactic technique of designating sets is the *set-abstract* method, which employs a different kind of grammatical construction. Set-abstracts are common in informal set theory. For example, the following set-abstract

$\{x : x$ is a prime number & x is less than 13$\}$

denotes the set of prime numbers less than 13. Given the Axiom of Extensionality, we have the following identity.

$\{x : x$ is a prime number & x is less than 13$\} = \{1,2,3,5,7,11\}$

More generally, where $\mathbb{F}$ is any formula, and ν is any variable free in $\mathbb{F}$, the set-abstract

$\{\nu{:}\mathbb{F}\}$

is *intended* to denote the set of things that satisfy formula $\mathbb{F}$ – roughly, the set of $\mathbb{F}$’s. I say "intended" because $\{\nu{:}\mathbb{F}\}$ may be *denotationally-improper*; see below.

Using the resources of description logic, we can formulate this idea in the following definition.

(d8) $\{\nu{:}\mathbb{F}\} =_{df} \iota S \forall \nu(\nu \in S \leftrightarrow \mathbb{F})$

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[2] The equivalence actually depends upon which exact logical system we adopt. If we adopt free logic, then the formulas are equivalent provided the singular term ‘a’ is proper.

Here, ν is any variable, and 𝔽 is any formula in which S does not occur free. Usually, ν occurs free in 𝔽. Recall ‘ιx’ reads “the unique x such that...”. In other words, (d6) defines {ν:𝔽} to be *the unique* set whose elements are precisely those things that satisfy 𝔽 – the set of 𝔽s.

Definition d8 includes infinitely-many special cases, one for each particular formula 𝔽 and variable ν free in 𝔽. These special cases in turn permit us to offer further definitions using the general set-abstract notation. The following are examples important in set theory.

(d9.1) $\{a\} =_{df} \{x : x = a\}$

(d9.2) $\{a,b\} =_{df} \{x : x = a \vee x = b\}$

(d9.3) $\{a,b,c\} =_{df} \{x : x = a \vee x = b \vee x = c\}$

etc.

(d10) $\mathbb{U} =_{df} \{x : x = x\}$

(d11) $\varnothing =_{df} \{x : x \neq x\}$

(d12) $A \cap B =_{df} \{x : x \in A \;\&\; x \in B\}$

(d13) $A - B =_{df} \{x : x \in A \;\&\; x \notin B\}$

(d14) $A \cup B =_{df} \{x : x \in A \vee x \in B\}$

(d15) $A + B =_{df} \{x : x \in A \text{ xor } x \in B\}$[3]

(d16) $\bigcup(C) =_{df} \{x : \exists y(y \in C \;\&\; x \in y)\}$

(d17) $\bigcap(C) =_{df} \{x : \forall y(y \in C \rightarrow x \in y)\}$

(d18) $\wp(A) =_{df} \{X : X \subseteq A\}$

Each of these definitions corresponds to an important concept of set theory, and each will be examined in turn. But first, we discuss the general question whether the *putative* sets exist!

6. Whether Set-Abstracts Denote

The previous section ended with a series of definitions of various sets. Given that all the definitions involve set-abstracts, and given that set-abstraction is defined in terms of definite descriptions, the question remains whether the sets described by these definitions in fact exist.

Recall that the definite description ιν𝔽 is (denotationally) *proper* if and only if there is exactly one object in the domain satisfying the formula 𝔽, in which case ιν𝔽 denotes that object. If ιν𝔽 is not denotationally-proper, we have a number of choices concerning what to do with ιν𝔽.

(1) ιν𝔽 denotes a particular, but arbitrarily chosen, object in the domain, which all denotationally improper terms denote (Frege).

(2) ιν𝔽 is *syncategorimatic*, which amounts to saying that any expression involving it is short for another expression that does not involve it; the question of its denotation is dissolved (Russell).

[3]‘xor’ is exclusive disjunction. This definition uses a set abstract, in order to parallel the other Boolean operations. Actually, our *official* definition defines A+B to be (A–B)∪(B–A).

(3) $\iota\nu\mathbb{F}$ denotes nothing (free logic).

In what follows, we adopt Frege's position – (1); we arbitrarily choose one particular object in the domain to serve as the denotation for *all* improper descriptions. In this connection, we also employ a special singular term, '※', to denote this arbitrarily chosen object, whose identity is completely irrelevant. These ideas are summarized in the following principles.

(ι1) $\exists!\nu\mathbb{F} \rightarrow \forall\nu(\mathbb{F} \leftrightarrow \nu{=}\iota\nu\mathbb{F})$
(ι2) $\sim\exists!\nu\mathbb{F} \rightarrow \iota\nu\mathbb{F} = ※$

These principles employ a new quantifier expression, '$\exists!\nu$', which is read 'there is *exactly one* ν such that...', and which is officially defined as follows.

(d19) $\exists!\nu\mathbb{F} =_{df} \exists u\forall\nu(\mathbb{F} \leftrightarrow \nu{=}u)$

Here, u is any variable not free in $\mathbb{F}$.

In light of this, let us reconsider Definition D8.

(d8) $\{\nu{:}\mathbb{F}\} =_{df} \iota S\forall\nu(\nu \in S \leftrightarrow \mathbb{F})$

Applying (ι1) and (ι2) to the definiens of (d8), we obtain the following. [Note carefully the difference between $\mathbb{F}$ in (d8) and in (ι1) and (ι2).]

(ι1+) $\exists!S\forall\nu(\nu \in S \leftrightarrow \mathbb{F}) \rightarrow \forall\nu(\nu \in \iota S\forall\nu(\nu \in S \leftrightarrow \mathbb{F}) \leftrightarrow \mathbb{F}))$
(ι2+) $\sim\exists!S\forall\nu(\nu \in S \leftrightarrow \mathbb{F}) \rightarrow \iota S\forall\nu(\nu \in S \leftrightarrow \mathbb{F}) = ※$

Of course, the formulas would be much worse if we expanded them using D19! One way to simplify somewhat is to abbreviate this according to D8.

(ι1*) $\exists!S\forall\nu(\nu \in S \leftrightarrow \mathbb{F}) \rightarrow \forall\nu(\nu \in \{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F}))$
(ι2*) $\sim\exists!S\forall\nu(\nu \in S \leftrightarrow \mathbb{F}) \rightarrow \{\nu{:}\mathbb{F}\} = ※$

These can be further simplified by introducing the following definition (where S is any variable not free in $\mathbb{F}$).

(d20) $\Sigma\nu\mathbb{F} =_{df} \exists!S\forall\nu(\nu \in S \leftrightarrow \mathbb{F})$

According to this definition, $\Sigma\nu\mathbb{F}$ means that there is exactly one set whose elements are precisely the things that satisfy $\mathbb{F}$ – the set of $\mathbb{F}$s. Applying D20 to (ι1*) and (ι2*), we obtain the following restatements of the principles.

(Σ1) $\Sigma\nu\mathbb{F} \rightarrow \forall\nu(\nu \in \{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F}))$
(Σ2) $\sim\Sigma\nu\mathbb{F} \rightarrow \{\nu{:}\mathbb{F}\} = ※$

The first one says that, *if* there is exactly one set of $\mathbb{F}$s, *then* a thing is an element of that set iff it is an $\mathbb{F}$. The second one says that, *if* there is not exactly one set of $\mathbb{F}$s, *then* "the set of $\mathbb{F}$s" is the arbitrarily chosen object.

Before continuing, we should notice that (Σ1) and (Σ2) are logical truths, when rewritten in primitive notation. For example, we can prove the following logical theorems concerning the various sorts of sets alluded to in the previous section.

(t9.1) $\Sigma x[x{=}a] \rightarrow \forall x(x \in \{a\} \leftrightarrow x{=}a)$
$\sim\Sigma x[x{=}a] \rightarrow \{a\} = ⋇$

(t9.2) $\Sigma x[x{=}a \vee x{=}b] \rightarrow \forall x(x \in \{a,b\} \leftrightarrow x{=}a \vee x{=}b)$
$\sim\Sigma x[x{=}a \vee x{=}b] \rightarrow \{a,b\} = ⋇$

(t9.3) $\Sigma x[x{=}a \vee x{=}b \vee x{=}c] \rightarrow \forall x(x \in \{a,b,c\} \leftrightarrow x{=}a \vee x{=}b \vee x{=}c)$
$\sim\Sigma x[x{=}a \vee x{=}b \vee x{=}c] \rightarrow \{a,b,c\} = ⋇$

etc.

(t10) $\Sigma x[x{=}x] \rightarrow \forall x(x \in \mathbb{U} \leftrightarrow x{=}x)$
$\sim\Sigma x[x{=}x] \rightarrow \mathbb{U} = ⋇$

(t11) $\Sigma x[x{\neq}x] \rightarrow \forall x(x \in \varnothing \leftrightarrow x{\neq}x)$
$\sim\Sigma x[x{\neq}x] \rightarrow \varnothing = ⋇$

(t14) $\Sigma x[x \in A \;\&\; x \in B] \rightarrow \forall x(x \in A{\cap}B \leftrightarrow\!. \; x \in A \;\&\; x \in B)$
$\sim\Sigma x[x \in A \;\&\; x \in B] \rightarrow A{\cap}B = ⋇$

(t15) $\Sigma x[x \in A \;\&\; x \notin B] \rightarrow \forall x(x \in A{-}B \leftrightarrow\!. \; x \in A \;\&\; x \notin B)$
$\sim\Sigma x[x \in A \;\&\; x \notin B] \rightarrow A{-}B = ⋇$

(t16) $\Sigma x[x \in A \vee x \in B] \rightarrow \forall x(x \in A{\cup}B \leftrightarrow\!. \; x \in A \vee x \in B)$
$\sim\Sigma x[x \in A \vee x \in B] \rightarrow A{\cup}B = ⋇$

(t18) $\Sigma x[\exists y(y \in C \;\&\; x \in y)] \rightarrow \forall x(x \in \bigcup(C) \leftrightarrow \exists y(y \in C \;\&\; x \in y))$
$\sim\Sigma x[\exists y(y \in C \;\&\; x \in y)] \rightarrow \bigcup(C) = ⋇$

(t19) $\Sigma x[\forall y(y \in C \rightarrow x \in y)] \rightarrow \forall x(x \in \bigcap(C) \leftrightarrow \forall y(y \in C \rightarrow x \in y))$
$\sim\Sigma x[\forall y(y \in C \rightarrow x \in y)] \rightarrow \bigcap(C) = ⋇$

(t20) $\Sigma X[X \subseteq A] \rightarrow \forall x(x \in \wp(A) \leftrightarrow X \subseteq A)$
$\sim\Sigma X[X \subseteq A] \rightarrow \wp(A) = ⋇$

For example, (t9.1) says that, *if* there is exactly one set of objects satisfying the formula '$x{=}$a', *then* something is an element of that set iff it is identical to a. If '$\{a\}$' in fact denotes the set it purports to denote, then by George, something is in it iff that something is identical to a. On the other hand, if there is no such set, then '$\{a\}$' denotes the arbitrarily chosen object.

Logic cannot settle whether there are any sets, or whether there are sets of various kinds. Alternatively stated, whether the expression '$\{x : x{=}a\}$', or any other set abstract, properly denotes, is not a matter of logic. What is required are further (non-logical) axioms *postulating* what kinds of sets exist, or equivalently, what set-abstracts are proper.

7. Classical Set Theory

One approach to the postulation of sets is taken by Classical Set Theory (CST), which takes the most generous approach. Specifically, CST postulates that *every* set-abstract is proper.

The traditional formulation of CST is given by the following two axioms.

(AE) $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x{=}y)$ [Axiom of Extensionality]

(AC) $\exists S \forall x(x \in S \leftrightarrow \mathbb{F})$ [Axiom of Comprehension]
(S not free in $\mathbb{F}$)

The former is the **Axiom of Extensionality**. The latter is the **Axiom of Comprehension**; it is an axiom scheme, being short for an infinite list of axioms, one for each actual formula $\mathbb{F}$ (in which S does not occur free). Useful applications involve those $\mathbb{F}$'s in which 'x' is free.

Intuitively, (AC) says that, for any "property" $\mathbb{F}$, there is a set S that *comprehends* precisely those things that have property $\mathbb{F}$. This is summarized in the following principle

(p) Every property $\mathbb{F}$ has an *extension*.

The extension of a property is just the set of objects that have that property.

Before continuing, it is important to note that AE and AC logically entail the following.

(t1) $\exists! S \forall x(x \in S \leftrightarrow \mathbb{F})$

In other words, every property $\mathbb{F}$ has a *unique* extension. AC says that *at least one* set comprehends all $\mathbb{F}$s; AE entails that *at most one* set comprehends all $\mathbb{F}$s; hence, *exactly one* set does.

Recall the following definitions and theorems from the previous two sections.

(d8) $\{x{:}\mathbb{F}\} =_{\text{df}} \iota S \forall x(x \in S \leftrightarrow \mathbb{F})$
(d20) $\Sigma \nu \mathbb{F} =_{\text{df}} \exists! S \forall \nu(\nu \in S \leftrightarrow \mathbb{F})$
(PD1) $\exists! \nu \mathbb{F} \rightarrow \mathbb{F}[\iota \nu \mathbb{F}/\nu]$
(PA1) $\Sigma \nu \mathbb{F} \rightarrow \forall \nu(\nu \in \{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F})$

Combining (t1) with (d20) and (PA1), we obtain the following two principles of abstraction.

(t2) $\forall \nu(\nu \in \{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F})$
(t3) $a \in \{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F}[a/\nu]$

Here, a is any constant, and $\mathbb{F}[a/\nu]$ results from $\mathbb{F}$ by replacing every free occurrence of ν in $\mathbb{F}$ by a.

8. Russell's Paradox

The Axiom of Comprehension seems fairly innocuous. It posits an extension for every property. Since sets are abstract objects, we can call them into existence simply by positing them.

Or so it would seem! The difficulty with the Axiom of Comprehension was first noted by Russell around the turn of the $19^{th}/20^{th}$ century, and is known as *Russell's Paradox*.

Recall that $\mathbb{F}$ can be any formula (in which S is not free), so in particular, $\mathbb{F}$ can be the following formula.

$$x \notin x$$

As with all formulas with one free variable, this formula implicitly defines a property. In particular, let us call a set *normal* if it does not contain itself as a member; otherwise, we call it *abnormal*. Surely, most sets are normal; indeed, it is hard to picture any abnormal sets. There is at least one; everything is self-identical ($\forall x[x=x]$), so everything is an element of the universal set, i.e., $\{x{:}x=x\}$, so in particular, the universal set is an element of the universal set. So the universal set is abnormal.

Most sets are normal; at least one is abnormal. In any event, according to the Axiom of Comprehension, we can collect all the normal sets into a single set – the set of all normal sets, which is defined as follows.

$$N =_{\text{df}} \{x\colon x \notin x\}$$

Applying the Principle of Abstraction (Th.2) to this abstract, we obtain the following as a special case.

$$\forall x(x \in \{x\colon x \notin x\} \leftrightarrow x \notin x)$$

i.e.:

$$\forall x(x \in N \leftrightarrow x \notin x)$$

The latter is a universal formula, so it can be instantiated to any singular-term, including 'N', which yields the following.

$$N \in N \leftrightarrow N \notin N$$

The latter has the following sentential form

$$P \leftrightarrow \sim P$$

which is logically self-contradictory!

Thus, intuitively obvious or not, the Axiom of Comprehension is logically inconsistent. It is generally agreed that logically inconsistent theories are no good. Classical Set Theory is, on the whole, no good. Still, we want to salvage what we can.

9. Modern Set Theory

Classical Set Theory, in particular the Axiom of Comprehension, is logically inconsistent. In light of Russell's Paradox, when we write down a set-abstract $\{\nu:\mathbb{F}\}$, we are not guaranteed in advance that the putative collection is in fact well-defined, contrary to intuition.

Not every set abstract $\{\nu:\mathbb{F}\}$ is legitimate; not every instance of $\Sigma\nu\mathbb{F}$ can be a theorem, on pain of contradiction. So, which abstracts are legitimate, and which are not?

At least one abstract is legitimate – $\{x:x\in a\}$ – since the following formula follows from AE.

(t1) $\forall y\Sigma x[x\in y]$

This is not surprising; it just says that for any set y, the elements of y form a set (which of course is y!)

On the other hand, we know that $\{x:x\notin x\}$ is *not* legitimate, since the corresponding Σ-formula

(※) $\Sigma x[x\notin x]$

is logically self-contradictory.

How do we distinguish between legitimate and illegitimate abstracts? One approach (adopted by Russell) is purely syntactic, and leads to an alternative logical under-pinning of set theory, known as type theory.

Another approach, modern set theory (MST), postulates certain set-construction methods. One begins with the empty set, $\varnothing$, and one "constructs" the remaining sets by various techniques, each one associated with a particular axiom of the theory. This approach is sometimes called the *iterative* conception of sets, which is contrasted with the *logical* conception (basically CST).

MST can also be understood as beginning with the Axiom of Extensionality

(a1) $\forall x\forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x=y)$ [Axiom of Extensionality]

plus a single unconditional axiom

(a2) $\Sigma x[x\neq x]$

which postulates the existence of the empty set, and then adding a series of conditional axioms, of the following general form.

(i) $\Sigma\nu\mathbb{F}_1 \;\&\; \Sigma\nu\mathbb{F}_2 \;\&\; \ldots \;\&\; \Sigma\nu\mathbb{F}_m \;.\!\rightarrow\; \Sigma\nu\mathbb{F}_0$

Here, $\mathbb{F}_1$, $\mathbb{F}_2$, etc. are formulas. Formulas of form (i) say that, *if* the abstracts $\{\nu:\mathbb{F}_1\}$ – $\{\nu:\mathbb{F}_m\}$ are legitimate, *then* so is the abstract $\{\nu:\mathbb{F}_0\}$.

10. The Axiom of Separation

The first such constructive axiom is originally due to Ernst Zermelo (1871-1953), and is called the *Axiom of Separation*.

The basic idea is quite simple. In light of Russell's Paradox, we cannot in general form the set of things satisfying a formula $\mathbb{F}$. On the other hand, according to Zermelo's *Axiom of Separation*, if we already have a set, say A, then we can *separate off* those elements of A that satisfy condition $\mathbb{F}$, and the resulting collection is also a set. This idea is officially formulated in the following axiom scheme.

(a3) $\forall x \exists s \forall y(y \in s \leftrightarrow (y \in x \ \& \ \mathbb{F}))$ [Axiom of Separation]

Here, $\mathbb{F}$ is any formula in which 's' does not occur free. Also, $\mathbb{F}$ can have many free variables; any unbound variables in (a3) are understood to be universally quantified at the beginning of the formula; recall universal derivation.

The set postulated to exist in (a3) is designated by the following abstract.

$$\{y : y \in x \ \& \ \mathbb{F}\}$$

(a1) and (a3) imply the following

(a3*) $\forall x \exists ! s \forall y(y \in s \leftrightarrow (y \in x \ \& \ \mathbb{F}))$

which is equivalent to the following Σ-version of SEP

(a3+) $\forall x \Sigma y(y \in x \ \& \ \mathbb{F})$

which says that the abstract $\{y\text{: } y \in x \ \& \ \mathbb{F}\}$ is legitimate, for any set x, for any condition $\mathbb{F}$.

Another alternative formulation of Separation also employs the 'Σ' operator.

(a3′) $\Sigma \nu \mathbb{F}_1 \ \rightarrow \ \Sigma \nu(\mathbb{F}_1 \ \& \ \mathbb{F}_2)$

This says that, for any two formulas $\mathbb{F}_1$, $\mathbb{F}_2$, if at least one of them, $\mathbb{F}_1$, has an extension, then whether $\mathbb{F}_2$ does or doesn't have an extension, the conjunction $\mathbb{F}_1 \& \mathbb{F}_2$ has an extension.

The skeleton of the proof that (a3) follows from (a3′) and (a1) goes as follows.

(1)	$\Sigma y[y \in a]$	earlier Th.
(2)	$\Sigma y[y \in a] \rightarrow \ \Sigma y(y \in a \ \& \ \mathbb{F})$	a3′
(3)	$\Sigma y(y \in a \ \& \ \mathbb{F})$	1,2,SL
(4)	$\exists ! s \forall y(y \in s \leftrightarrow . \ y \in a \ \& \ \mathbb{F})$	3, Def Σ
(5)	$\exists s \forall y(y \in s \leftrightarrow . \ y \in a \ \& \ \mathbb{F})$	4, IL
(6)	$\forall x \exists s \forall y(y \in s \leftrightarrow . \ y \in x \ \& \ \mathbb{F})$	1-5, $\forall$I[a/x]

11. Russell's Paradox Revisited

The reader may wonder what happens if we apply Russell's technique to the Axiom of Separation. Well, although we do not obtain a logical contradiction (fortunately!), we do obtain a surprising and counter-intuitive result.

The Axiom of Separation allows us, given a set a, to form a subset of elements of a that satisfy formula $\mathbb{F}$, whether $\{\nu{:}\mathbb{F}\}$ is legitimate or not. In other words, the following is a theorem.

(SEP+) $\Sigma x(x \in a \ \& \ \mathbb{F})$

Once again, consider letting $\mathbb{F}$ be the formula '$x \notin x$', which yields the first formula in the following proof.

(1)	$\Sigma x(x \in a \ \& \ x \notin x)$	a1+, $x{\notin}x/\mathbb{F}$
(2)	$\exists! s \forall x(x \in s \leftrightarrow. \ x \in a \ \& \ x \notin x)$	1, Def Σ
(3)	$\exists s \forall x(x \in s \leftrightarrow. \ x \in a \ \& \ x \notin x)$	2, IL
(4)	$\forall x(x \in b \leftrightarrow. \ x \in a \ \& \ x \notin x)$	3, $\exists$O
(5)	$b \in b \leftrightarrow. \ b \in a \ \& \ b \notin b$	4, $\forall$O
(6)	$b \notin a$	5, SL
(7)	$\exists x[x \notin a]$	6, $\exists$I
(8)	$\forall x \exists y[y \notin x]$	7, $\forall$I[a/x]
(9)	$\sim\exists x \forall y[y \in x]$	8, QL

Recall the definition of the universal set $\mathcal{U}$, and the associated theorems.

(d) $\mathbb{U} =_{df} \{x : x{=}x\}$

(t1) $\Sigma x[x{=}x] \rightarrow \forall x(x \in \mathbb{U} \leftrightarrow x{=}x)$

(t2) $\sim\Sigma x[x{=}x] \rightarrow \mathbb{U}{=}⋇$

(t1) implies the following

$$\Sigma x[x{=}x] \rightarrow \forall x[x \in \mathbb{U}],$$

which implies

$$\Sigma x[x{=}x] \rightarrow \exists x \forall y[y \in x].$$

Combining the latter with line 8 above, we obtain

$$\sim\Sigma x[x{=}x],$$

which together with (t2) yields

$$\mathbb{U}{=}⋇$$

Thus, the abstract $\{x{:}x{=}x\}$ is not legitimate; there is no universal set!

12. The Empty Set

Recall that iterative set theory postulates the empty set together with various set-construction methods. Historically, the empty set axiom is written as follows.

(ES) $\exists x{\sim}\exists y[y \in x]$

ES together with extensionality entails the following theorem

(t1) $\exists! S \forall x(x \in S \leftrightarrow x \neq x)$

which is definitionally equivalent to the following Σ-version of ES.

(ES+) $\Sigma x[x \neq x]$

The latter, which says that the abstract $\{x{:}x \neq x\}$ is legitimate, is in fact interchangeable with ES. In particular, given the following theorem,

$$\Sigma x[x \neq x] \rightarrow \forall x(x \in \{x{:}x \neq x\} \leftrightarrow x \neq x),$$

and given that '$x \neq x$' is a self-contradiction, we can derive ES from ES+.

The Empty-Set Axiom is in fact a consequence of the Axiom of Separation, so long as we use first-order logic *with* identity. According to SEP, for any set A and any formula $\mathbb{F}$, we can form the set consisting of elements of A satisfying the formula $\mathbb{F}$, which is the set $\{z : z \in A \ \& \ \mathbb{F}\}$. So, in particular, we can let $\mathbb{F}$ be the formula '$z \neq z$'. Applying the Axiom of Separation to this formula yields the following proof.

(1)	$\forall x \exists y \forall z(z \in y \leftrightarrow. \ z \in x \ \& \ z \neq z)$	Separation
(2)	$\exists y \forall z(z \in y \leftrightarrow. \ z \in a \ \& \ z \neq z)$	1, ∀O
(3)	$\forall z(z \in b \leftrightarrow. \ z \in a \ \& \ z \neq z)$	2, ∃O
(4)	$\forall z[z \notin b]$	3, IL
(4)	${\sim}\exists y[y \in b]$	4, QL
(5)	$\exists x{\sim}\exists y[y \in x]$	5, ∃I

As in a previous section, we officially define the empty set as follows.

(d11) $\varnothing =_{df} \{x : x \neq x\}$

The following are standard facts about the empty set.

(1) ${\sim}\exists x[x \in \varnothing]$
(2) $\forall x({\sim}\exists y[y \in x] \rightarrow x = \varnothing)$
(3) $\forall x[\varnothing \subseteq x]$
(4) $\forall x[\varnothing \perp x]$
(5) $\forall x(x \subseteq \varnothing \rightarrow x = \varnothing)$
(6) $\forall x(x \perp x \rightarrow x = \varnothing)$

13. Intersection and Set Difference

In this and the following section, we discuss finite operations on sets, which are customarily referred to as Boolean operations. Whereas the operations defined in the present section are guaranteed by the Axiom of Separation, those defined in the next section require one additional axiom.

The Σ-version of the Axiom of Separation is written as follows.

(SEP+) $\Sigma x(x \in A \ \&\ \mathbb{F})$

This says that abstracts of the form $\{x : x \in A \ \&\ \mathbb{F}\}$ are legitimate. We have already seen what happens when one substitutes the formulas '$x \neq x$' and '$x \notin x$'. We now consider the formulas '$x \in$ B' and '$x \notin$ B'. In the first case, we obtain the following theorem.

(t1) $\Sigma x(x \in A \ \&\ x \in B)$

Recall the associated definition,

(d12) $A \cap B =_{df} \{x : x \in A \ \&\ x \in B\}$

$A \cap B$ is called the *intersection* of A and B; the defined set purports to contain the elements common to A and B. The associated Σ-theorem,

$$\Sigma x(x \in A \ \&\ x \in B) \rightarrow \forall x(x \in A \cap B \leftrightarrow . \ x \in A \ \&\ x \in B),$$

together with (t1) yields the following theorem.

$$\forall x(x \in A \cap B \leftrightarrow . \ x \in A \ \&\ x \in B)$$

Grammatically, '$\cap$' is a two-place *function-sign*; it takes two singular-terms and yields a singular-term. By contrast, '$\in$' is a two-place *predicate*; it takes two singular-terms and yields a formula.

Alternatively, we can substitute '$x \notin$B' into the Axiom of Separation, to obtain:

(t2) $\Sigma x(x \in A \ \&\ x \notin B)$

The associated definition is:

(d13) $A - B =_{df} \{x : x \in A \ \&\ x \notin B\}$

This operation is called *set difference*. The set $A-B$ consists of those elements of A that are not elements of B. The following are the associated theorems.

$$\Sigma x(x \in A \ \&\ x \notin B) \rightarrow \forall x(x \notin A-B \leftrightarrow . \ x \in A \ \&\ x \notin B)$$
$$\forall x(x \in A-B \leftrightarrow . \ x \in A \ \&\ x \notin B).$$

14. Union and Boolean Sum

In addition to intersection, informal set theory has the notion of *union*, which is the set-theoretic counterpart of logical disjunction, much in the same way that intersection is the set-theoretic counterpart of logical conjunction. The official definition is given as follows.

(d14) $A \cup B =_{\text{df}} \{x : x \in A \lor x \in B\}$

The set $A \cup B$ is called the *union* of A and B; allegedly, $A \cup B$ contains something x iff x is an element of A and/or B.

The obvious question is whether the abstract is legitimate. It does not have a form subsumed by the Axiom of Separation, so its legitimacy is not guaranteed by *that* axiom. Indeed, it can be shown that the existence of unions cannot be deduced from the axioms we have so far postulated. Once again, in order to reconstruct as much of classical set theory as possible (short of inconsistency!), we add the appropriate axiom, which is called the Axiom of Simple Union. This is to be carefully distinguished from the Axiom of General Union, to be discussed later.

(ASU) $\forall x \forall y \exists z \forall w (w \in z \leftrightarrow. w \in x \lor w \in y)$ [Axiom of Simple Unions]

An alternative Σ-version of the Axiom of Simple Unions can be written in either of the following ways.

$$\Sigma x (x \in A \lor x \in B)$$

$$\Sigma x \mathbb{F}_1 \mathbin{\&} \Sigma x \mathbb{F}_2 \,.\!\rightarrow \Sigma x (\mathrm{F}_1 \lor \mathbb{F}_2)$$

In combination with the general Σ-theorem, the former yields the following associated theorem.

$$\forall x (x \in A \cup B \leftrightarrow. x \in A \lor x \in B)$$

Next, given the two set-operations union and difference, one can define what is sometimes called the *Boolean sum* operation, as follows.

(d15) $A+B =_{\text{df}} (A-B) \cup (B-A)$

The following can be shown.

$$A+B = (A \cup B)-(A \cap B)$$
$$A+B = \{x : x \in A \text{ xor } x \in B)$$

In other words, $A+B$ consists of those elements that are in A or B but not both, which is to say that + is the set-theoretic counterpart of the *exclusive disjunction* (xor). Accordingly, + has the same odd properties that the logical connective exclusive-or has. For example,

(odd) $\forall x [x \in A \mathbin{\&} x \in B \mathbin{\&} x \in C \,.\!\rightarrow x \in (A+B)+C]$

15. Absolute versus Relative Complementation

In informal set theory, it is natural to define a one-place operation of complementation as follows.

$$-A =_{\text{df}} \{x : x \notin A\}$$

$-A$ is called the *absolute complement* of A. Something is an element of $-A$ if and only if it is *not* an element of A, *provided* the abstract is legitimate; i.e.,

$$\Sigma x[x \notin A] \rightarrow \forall x(x \in -A \leftrightarrow x \notin A)$$

The difficulty with the concept of absolute set-complement is that, given our other set-theoretic principles, we can prove that $-A$ is not a set, that the above abstract is illegitimate.

Suppose that the complement of every set is also a set. Then in particular, the complement of the empty set is a set. Since nothing is in $\varnothing$, everything is in $-\varnothing$, which is to say that $-\varnothing$ is the universal set, which we have already disqualified.

This proves that $\{x{:}x{\notin}A\}$ is illegitimate for $A{=}\varnothing$. We can in fact prove something much stronger – that $\{x{:}x{\notin}A\}$ is illegitimate for any A, i.e.,

$$\forall y{\sim}\Sigma x[x \notin y]$$

We argue by *reductio* in the following proof sketch.

(1)	$\sim\forall y{\sim}\Sigma x[x \notin y]$	As
(2)	$\exists y\Sigma x[x \notin y]$	1, QL
(3)	$\Sigma x[x \notin a]$	2, ∃O
(4)	$\Sigma x[x \in a]$	prev. th.
(5)	$\Sigma x[x \in a \vee x \notin a]$	3,4, ASU (Σ-form)
(6)	$\exists! s\forall x(x \in s \leftrightarrow. x \in a \vee x \notin a)$	5, Def Σ
(7)	$\exists s\forall x(x \in s \leftrightarrow. x \in a \vee x \notin a)$	6, IL
(8)	$\forall x(x \in a \vee x \notin a)$	QL
(9)	$\exists s\forall x[x \in s]$	7,8, QL
(10)	$\sim\exists s\forall x[x \in s]$	prev. th.

Informally argued, if A and $-A$ are both sets, then by ASU, $A{\cup}{-A}$ is a set. But $x \in A{\cup}{-A}$ iff $x{\in}A$ or $x{\in}{-A}$, and $x{\in}{-A}$ iff $x \notin A$, so $x \in A{\cup}{-A}$ iff $x \in A$ or $x \notin A$. But the latter is a tautology, so everything is in $A{\cup}{-A}$, which contradicts the previous theorem that says that no set contains everything.

As we have seen, absolute set-complementation is not a legitimate set-forming operation. On the other hand, in many contexts, all the sets in question are subsets of a common set U, basically, the domain (or "universe") of discourse. For example, in arithmetic, U is the set of all natural numbers; in real analysis, U is the set of all real numbers. In restricted contexts such as these, we can talk about complementation relative to the antecedently understood "universe" U. Then the complement of a set A is the complement of A *relative* to U, i.e., $U{-}A$.

16. Singletons, Doubletons, etc.

In an earlier section, we gave the following definitions, which are an initial segment of an infinite list of definitions, one for each positive integer.

(d9.1) $\{a\} =_{df} \{x : x=a\}$
(d9.2) $\{a,b\} =_{df} \{x : x=a \vee x=b\}$
(d9.3) $\{a,b,c\} =_{df} \{x : x=a \vee x=b \vee x=c\}$

$\{a\}$ is called the *singleton* of a, $\{a,b\}$ is called the *doubleton* of a,b, and $\{a,b,c\}$ is called the *tripleton* of a,b,c.

It is natural to suppose that such sets are legitimate. However, once we abandon the Axiom of Comprehension, the existence of these sets is no longer assured. The axioms we have thus far introduced in place of Comprehension are not adequate to prove the existence of singletons, doubletons, etc.

We accordingly add a further axiom to our list, the Axiom of Singletons, formulated as follows.

AS $\forall x \exists y \forall z(z \in y \leftrightarrow z = x)$ [Axiom of Singletons]

In other words, for any set x there is a set y that has x and nothing else as an element. The sigma-version of this axiom is as follows.

AS+ $\forall x \Sigma y[y=x]$

The Axiom of Singletons in combination with the Axiom of Simple Unions yields the existence doubletons, tripletons, etc., in light of the following theorems.

(t1) $\{a,b\} = \{a\} \cup \{b\}$
(t2) $\{a,b,c\} = \{a\} \cup \{b\} \cup \{c\}$
etc.

Once we have singletons, etc., we can produce a rather bewildering array of sets, based merely on the empty set $\varnothing$. For example, we have the following sets.

$\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, etc.
$\{\varnothing,\{\varnothing\}\},\{\varnothing,\{\varnothing\},\{\{\varnothing\}\}\}$
$\{\varnothing,\{\varnothing\},\{\{\varnothing\}\},\{\{\{\varnothing\}\}\}\}$
$\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$

At this point, it is important to make sure one is completely clear about the difference between the membership relation $\in$ and the inclusion (subset) relation $\subseteq$. For example, the empty set $\varnothing$ is included in (is a subset of) every set. On the other hand, although $\varnothing$ is an element of $\{\varnothing\}$, it is not an element of $\{\{\varnothing\}\}$, nor of $\{\{\{\varnothing\}\}\}$, nor of $\{\{\{\{\varnothing\}\}\}\}$. Still other combinations are possible, $\varnothing$ is both an element and a subset of the following sets: $\{\varnothing\}$, $\{\varnothing,\{\varnothing\}\}$, $\{\varnothing,\{\{\varnothing\}\}\}$.

Occasionally a set, say $\varnothing$, is both an element and a subset of a given set. For example, $\varnothing \in \{\varnothing\}$ & $\varnothing \subseteq \{\varnothing\}$. Or turning our attention to $\{\varnothing\}$, it has the following property – every element is also a subset. Sets with this property are of sufficient interest to be given a special name; they are called *transitive sets*. More about them in Chapter 4.

17. General Union and Intersection

The operations of union and intersection defined so far are fundamentally *finite* in character. Union and intersection are binary (two-place) operations on sets, the official expressions being binary (two-place) function-signs. Given *two* sets A,B, we can form their union $A\cup B$, and we can form their intersection $A\cap B$.

Of course, we can take binary union/intersection and define *finite* union/intersection. For any *finite* collection of sets $\{A_1, A_2, ..., A_n\}$, we can form their *joint union*, as well as their *common intersection*, intuitively defined as follows.

$$\bigcup\{A_1, A_2, ..., A_n\} =_{df} A_1\cup A_2\cup...\cup A_n$$
$$\bigcap\{A_1, A_2, ..., A_n\} =_{df} A_1\cap A_2\cap...\cap A_n$$

Of course, strictly speaking, both expressions on the right require parentheses. But union and intersection are associative, so the parentheses can be dropped without resulting in ambiguity. It is completely analogous to addition and multiplication.

What we would like is to be able to take joint unions and common intersections even when the number of sets involved is infinite. The *intuitive* (but not official!) definitions would look very similar to the above.

$$\bigcup\{A_1, A_2, ...\} =_{df} A_1\cup A_2\cup...$$
$$\bigcap\{A_1, A_2, ...\} =_{df} A_1\cap A_2\cap...$$

The difficulty is that the '...' in the latter expressions cannot be made logically correct using only the resources of binary union and intersection. Rather, we must introduce further, more general, notions of union and intersection, which will capture the intuition in the last intuitive definitions, and which will be logically precise. This is accomplished as follows.

(d16) $\cup C =_{df} \{x : \exists Y(Y \in C \ \&\ x \in Y)\}$
(d17) $\cap C =_{df} \{x : \forall Y(Y \in C \rightarrow x \in Y)\}$

$\cup C$ is called the union of collection C, whereas $\cap C$ is called the intersection of collection C. Note carefully, that 'C' replaces the intuitive expression '$\{A_1, A_2, ...\}$'.

Now, according to the first definition, insofar as the abstract is legitimate, something is an element of the union of C if and only if it is a element of at least one element of C. According to the second definition, insofar as the abstract is legitimate, something is a member of the intersection of C if and only if it is a member of every member of C. In other words, we have the following.

$$\Sigma x\exists Y(Y\in C \ \&\ x\in Y) \rightarrow \forall x(x \in \cup C \leftrightarrow \exists Y(Y \in C \ \&\ x \in Y))$$
$$\Sigma x\forall Y(Y\in C \rightarrow x\in Y) \rightarrow \forall x(x \in \cap C \leftrightarrow \forall Y(Y \in C \rightarrow x \in Y))$$

Before considering whether the abstracts are legitimate, notice that the collection C can have any number of elements. In the case C has two, three, etc., elements, we obtain the previously defined finite union and finite intersection, as seen in the following theorems.

(t1) $\bigcup\{A,B\} = A\cup B$
(t2) $\bigcup\{A,B,C\} = A\cup B\cup C$
etc.
(t3) $\bigcap\{A,B\} = A\cap B$
(t4) $\bigcap\{A,B,C\} = A\cap B\cap C$
etc.

But C can also have just one element, in which case we obtain the following theorems.

(t5) $\bigcup\{A\} = A$
(t6) $\bigcap\{A\} = A$

Note carefully the difference between $\bigcup\{A\}$ and $\bigcup(A)$!

Indeed, C can contain no elements at all, in which case $C=\varnothing$, and we obtain the following theorems.

(t7) $\bigcup\varnothing = \varnothing$
(t8) $\bigcap\varnothing = ⋇$

What (t8) entails, of course, is that general intersection is not always legitimate. However, its illegitimacy is well-confined, since $\bigcap C$ is ok so long as $C\neq\varnothing$. This is summarized in the following theorem, which can be proven using only Separation and Extensionality.

(t9) $C\neq\varnothing \rightarrow \Sigma z\forall w(w \in x \rightarrow z \in w)$
(t9+) $C\neq\varnothing \rightarrow \forall x(x \in \bigcap C \leftrightarrow \forall Y(Y \in C \rightarrow x \in Y))$

By contrast, the legitimacy of $\bigcup C$ cannot be proven using the axioms we have postulated thus far. We accordingly introduce yet another axiom, the Axiom of General Unions, which is officially formulated as follows.

(AGU) $\forall x\exists y\forall z(z \in y \leftrightarrow \exists w(w \in x \;\&\; z \in w))$ [Axiom of General Union]

The sigma-version of AGU goes as follows.

(AGU+) $\forall x\Sigma z\exists w(w \in x \;\&\; z \in w)$

By way of concluding this section, it is worth noting that the Axiom of General Unions affords a simplification of our axiom system. As it stands, we cannot prove the Axiom of Simple Unions as a logical consequence of the Axiom of General Unions! In order to do that we need doubletons, but we prove the existence of doubletons using the Axiom of Simple Unions! At this point, what we can do instead is go back and replace both the Axiom of Simple Unions and the Axiom of Singletons by the following single axiom, the Axiom of Pairs. This is the usual way in which the axioms of set theory are presented.

(AP) $\forall x\forall y\exists z\forall w(w \in z \leftrightarrow (w=x \vee w=y))$ [Axiom of Pairs]

Note, for example, that in the special case that $x=y$ we can prove the existence of singletons.

18. Power Sets

Consider a set A. By a *subset* of A is meant any set X that is included in A (which is to say that every element of X is an element of A). It is thought plausible that the subsets of a given set A can be collected together and the result is also a set. This set is customarily called the *power set* of A, which is defined as follows.

(d18) $\wp(A) =_{df} \{X : X \subseteq A\}$

No principle so far introduced ensures the legitimacy of this definition. Accordingly, we introduce a further axiom, called the *Axiom of Power Sets*, as follows.

(POW) $\forall x \exists y \forall z(z \in y \leftrightarrow z \subseteq x)$ [Axiom of Power Sets]

The following is the Σ-version of the Axiom of Powers.

(POW+) $\forall x \Sigma z[z \subseteq x]$

This just says that for any set x, there is a set y whose elements are all and only the subsets of x. In light of the definition of '$\wp$', and POW+, we have the following theorem.

(t1) $\forall X(X \in \wp(A) \leftrightarrow X \subseteq A)$

The power set $\wp(A)$ has more elements than A itself. Observe the following.

$\wp(\varnothing) = \{\varnothing\}$
$\wp(\{a\}) = \{\varnothing,\{a\}\}$
$\wp(\{a,b\}) = \{\varnothing,\{a\},\{b\},\{a,b\}\}$
$\wp(\{a,b,c\}) = \{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$

The following is the general theorem for finite sets, although we cannot prove it officially without the definition of numbers.

if A has n elements, then $\wp(A)$ has 2^n elements.

In other words, one can obtain a very large set by taking the power set of a fairly small set. For example, a set with only 270 elements has a power set with over 10^{80} elements, the approximate number of nucleons in the universe!

Or, consider another kind of example, but one that is completely analogous. Suppose you match coins with a friend, suppose you play double or nothing, and suppose you start by betting one penny. How long a string of wins does it take for you to win one *trillion* dollars? Well, believe it or not, a string of 47 wins nets 1.4 trillion dollars. Of course, a string of 47 wins is overwhelmingly improbable.

The power set of a set is always bigger than the set itself, at least in the case of finite sets, as noted above. In a later Chapter, we will see that, even in the case of infinite sets, the power set is bigger than the set itself. This is significant, especially in view of other theorems; for example, that the of positive integers (1,2,3,..) is *not* bigger than the set of positive even integers (2,4,6,...); worse, the set of rational numbers is *not* bigger than the set of integers!

19. Fields of Sets

For any set A, we have the following facts.

(1) $\varnothing \in \wp(A)$
(2) $A \in \wp(A)$
(3) for any sets X,Y, if X and Y are elements of $\wp(A)$, then so are:
$X\cap Y$, $X\cup Y$, $X-Y$, $Y-X$, $X+Y$, $A-X$, $A-Y$.

The properties (1)-(3) are sufficiently interesting in their own right that they have been abstracted from the notion of a power set and generalized to the notion of a *field of sets*. The following is the informal definition, followed by a small theorem.

(DEF) Let S be any set. Then a *field of subsets* of S is any collection C of subsets of S satisfying the following conditions.

(1) $\varnothing \in C$;
(2) $S \in C$;
(3) if $X \in C$, then $S-X \in C$;
(4) if $X,Y \in C$, then $X\cap Y \in C$, $S-A \in C$, $S-B \in C$.

(TH) Let C be a field of subsets of S. Then

(5) if $X,Y \in C$, then $X\cup Y \in C$;
(6) if $X,Y \in C$, then $X-Y \in C$;
(7) if $X,Y \in C$, then $X+Y \in C$.

Notice that, for any set S, $\wp(S)$ is a field of subsets of S, but so are many collections of sets, including $\{\varnothing,S\}$.

The importance of fields of sets rests more in meta-mathematics than in general set theory. Specifically, in 1935, Marshall Stone proved his famous theorem about Boolean algebras, a corollary of which is that every Boolean algebra is isomorphic to a field of subsets of some set. Henkin later proved that this is equivalent to the completeness theorem for classical logic.

20. The Principle of Set-Abstraction and Contextual Definitions

Although many of the definitions given above are written using set-abstraction, which is defined in terms of definite descriptions, in actual practice it is usually much convenient to bypass explicit definitions in favor of their affiliated implicit definitions.

1. Principle of Set-Abstraction

To see how the affiliated implicit definitions are produced, we begin by noting the **Principle of Set-Abstraction**, which is a fundamental theorem (schema) of set theory.

PSA $\exists S[S = \{\nu{:}\mathbb{F}\}] \quad \rightarrow \quad \forall\nu(\, \nu\in\{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F}\,)$

PSA employs a set-abstract, $\{\nu{:}\mathbb{F}\}$, which is officially defined as follows.

Def $\{\nu{:}\mathbb{F}\} \quad =_{df} \quad \iota S\forall\nu[\nu\in S \leftrightarrow \mathbb{F}]$

Note also that PSA is a *theorem schema*, which stands for infinitely-many theorems of the form displayed. In particular, ν is any variable, $\mathbb{F}$ is any formula, and S any set-denoting variable not free in $\mathbb{F}$.

The following is a *schematic* proof of PSA.

(1)	~~SHOW~~: $\exists S[S = \{\nu{:}\mathbb{F}\}] \rightarrow \forall\nu(\, \nu\in\{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F}\,)$	CD
(2)	$\exists S[S = \{\nu{:}\mathbb{F}\}]$	As
(3)	~~SHOW~~: $\forall\nu(\, \nu\in\{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F}\,)$	DD
(4)	$C = \{\nu{:}\mathbb{F}\}$	2,∃O
(5)	$C = \iota S\forall\nu[\, \nu\in S \leftrightarrow \mathbb{F}\,]$	4,Def $\{\nu{:}\mathbb{F}\}$
(6)	$\forall S\, [\, \forall\nu[\, \nu\in S \leftrightarrow \mathbb{F}\,] \leftrightarrow S{=}C\,]$	5,ιO
(7)	$C{=}C$	IL
(8)	$\forall\nu[\, \nu\in C \leftrightarrow \mathbb{F}\,]$	6,7,QL
(9)	$\forall\nu[\, \nu\in\{\nu{:}\mathbb{F}\} \leftrightarrow \mathbb{F}\,]$	4,8,IL

Note the use of the iota-out rule – from description logic – which is given as follows.

ιO $c = \iota\nu\mathbb{F} \quad / \quad \forall\nu\,[\,\mathbb{F} \leftrightarrow \nu{=}c\,]$

The corresponding iota-in rule is simply the converse argument form.

ιI $\forall\nu\,[\,\mathbb{F} \leftrightarrow \nu{=}c\,] \quad / \quad c = \iota\nu\mathbb{F}$

As with all rules of inference, these rules are schematic; in particular, c is any constant, ν is any variable, and $\mathbb{F}$ is any formula.

2. The Move from Explicit Definitions to Implicit Definitions

Let's now consider how to use PSA to produce the various implicit definitions. Let us consider a specific example – binary set-union. Officially, binary set-union is defined as follows.

$A \cup B$	$=_{df}$	$\{x : x \in A \vee x \in B\}$

As with most definitions in formal theories, the above definition is *schematic*, which means that it stands for infinitely-many instances [just as *modus ponens* in intro logic has infinitely-many instances]. In particular, in the above, A and B can be replaced by any *closed singular-terms*.[4] The following are examples of instances of the above schema.

$$\varnothing \cup \{\varnothing\} \quad =_{df} \quad \{x : x \in \varnothing \vee x \in \{\varnothing\}\}$$
$$\{a\} \cup \{a,b\} \quad =_{df} \quad \{x : x \in \{a\} \vee x \in \{a,b\}\}$$

Since union is defined by way of a set-abstract, we can apply PSA as follows.

(1) $\exists S\,[S = \{x: x \in A \vee x \in B\}] \rightarrow \forall x[x \in \{x: x \in A \vee x \in B\} \leftrightarrow. x \in A \vee x \in B]$ PSA
(2) $\exists S\,[S = A \cup B] \rightarrow \forall x\,[\, x \in A \cup B \leftrightarrow. x \in A \vee x \in B\,]$ 1,Def $\cup$

So, if the set $A \cup B$ exists, then an item x is a member of $A \cup B$ if and only if x is a member of A and/or of B.

Many axioms of modern set theory posit the existence of certain kinds of sets. For example, the **axiom of simple-union** posits the existence of simple (binary) unions. It may be written as follows.

(asu) $\forall X \forall Y$ [X is a set & Y is a set $.\!\rightarrow\!.$ $X \cup Y$ is a set]

3. Contextual Definitions

Whereas an explicit (non-contextual) definition of an expression $\mathcal{E}$ provides a means of eliminating *any* occurrence of $\mathcal{E}$, an implicit (contextual) definition of $\mathcal{E}$ provides only a means of eliminating occurrences of $\mathcal{E}$ that appear in particular contexts.

The following is our official contextual definition of $A \cup B$.

$c \in A \cup B$	$=_{df}$	$c \in A \vee c \in B$

Here, A, B, and c are arbitrary *closed singular terms*. This definition can be used exactly like a bi-directional rule; specifically, if one has a formula of the left-side form, one can deduce a formula of the right-side form, and conversely. Indeed, it seems appropriate to rewrite the above as a bi-directional rule, as follows.

$c \in A \cup B$	//	$c \in A \vee c \in B$

[4] In some presentations of formal theories, definitions have open instances, but we stick to closed instances for the sake of simplicity.

The following is a schematic-proof that the above implicit definition is *admissible* in set theory, where we grant from the outset that A and B are sets. In this A, b, and c are arbitrary closed singular-terms.

(1)	$c \in A \cup B$	Pr
(2)	~~SHOW~~: $c \in A \vee c \in B$	DD
(3)	$\exists S\ [\ S = A \cup B\ ]$	Axiom of Simple Unions
(4)	$\exists S[S = \{x : x \in A \vee x \in B\}]$	3,Def ∪
(5)	$\forall x\ [\ x \in \{x : x \in A \vee x \in B\} \leftrightarrow.\ x \in A \vee x \in B]$	4,PSA
(6)	$\forall x\ [\ x \in A \cup B \leftrightarrow.\ x \in A \vee x \in B]$	5,Def ∪
(7)	$c \in A \vee c \in B$	1,6,QL

(1)	$c \in A \vee c \in B$	Pr
(2)	~~SHOW~~: $c \in A \cup B$	DD
(3)	$\exists S\ [\ S = A \cup B\ ]$	Axiom of Simple Unions
(4)	$\exists S[S = \{x : x \in A \vee x \in B\}]$	3,Def ∪
(5)	$\forall x\ [\ x \in \{x : x \in A \vee x \in B\} \leftrightarrow.\ x \in A \vee x \in B]$	4,PSA
(6)	$\forall x\ [\ x \in A \cup B \leftrightarrow.\ x \in A \vee x \in B]$	5,Def ∪
(7)	$c \in A \cup B$	1,6,QL

This means that we don't actually require the implicit definition of ∪. This is because, every time we apply we apply the left-right half of Def(∪), we can replace the step by a corresponding instance of the first derivation-schema, and every time we apply the right-left half of Def(∪), we can replace the step by a corresponding instance of the second derivation-schema.

4. An Aside on Contextuality

Although we say that our original definition of ∪ is explicit, and context-free, it is not completely *context-free*. In particular, the definition of ∪

$$A \cup B \quad =_{\text{df}} \quad \{x : x \in A \vee x \in B\}$$

includes the defined expression '∪' in a particular context. A completely context-free definition of ∪ would have the following form.

$$\cup \quad =_{\text{df}} \quad \text{blah, blah, blah}$$

What would such a definition look like? Well, we would have to enlarge our underlying logical structure to include lambda-abstraction, in which case the following would be an appropriate candidate.

$$\cup \quad =_{\text{df}} \quad \lambda xy\ \{z : z \in x \vee z \in y\}$$

However, in proofs in ordinary first-order set theory, the function-sign '∪' *never* occurs *except* in contexts of the form $\ulcorner A \cup B \urcorner$. Accordingly, we can safely define '∪' as we do, for we will never encounter a situation in which we cannot eliminate a given occurrence of '∪'.

21. Axioms for Chapter 1

The following are the axioms of set theory relevant to Chapter 1.

a1. $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x{=}y]$ [Extensionality]

a2. $\exists x {\sim} \exists y[y \in x]$ [Empty Set]

a3. $\forall x \exists y \forall z[z \in y \leftrightarrow (z \in x \,\&\, \mathbb{F}]$ [Separation]
($\mathbb{F}$ is any formula in which y is not free)

a4. $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow [w \in x \vee w \in y])$ [Simple Unions]

a5. $\forall x \exists y \forall z(z \in y \leftrightarrow z{=}x)$ [Singletons]

a6. $\forall x \exists y \forall z[z \in y \leftrightarrow \exists w(w \in x \,\&\, z \in w)]$ [General Unions]

a7. $\forall x \exists y \forall z[z \in y \leftrightarrow \forall w(w \in z \rightarrow w \in x)]$ [Power Sets]

The unquantified Σ-forms of the axioms go as follows.

a2+. $\Sigma x\,[x{\neq}x]$
a3+. $\Sigma x\,(x \in a \,\&\, \mathbb{F})$
a4+. $\Sigma x\,(x \in a \vee x \in b)$
a5+. $\Sigma x\,[x{=}a]$
a6+. $\Sigma x\,\exists y(y \in a \,\&\, x \in y)$
a7+. $\Sigma x\,\forall y(y \in x \rightarrow y \in a)$

Note that, first-order logic as ordinarily conceived presupposes a non-empty domain of discourse; $\forall xFx$ entails $\exists xFx$. In this case, one can deduce A2 from A3. In logics without existential presuppositions (free logics), $\forall xFx$ does not entail $\exists xFx$, so one cannot deduce A2 from A3. For that reason, and for historical reasons, A2 is included.

Note also that an equivalent axiom system results if we replace A4 and A5 by the following single axiom, as is customary in the presentation of modern axiomatic set theory.

AP. $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow .\, w{=}x \vee w{=}y)$ [Pairs]

In particular, one can deduce AP from A4 and A5, one can deduce A5 from AP, and one can deduce A4 from AP and A6.

By contrast with modern axiomatic set theory, classical set theory is based on the following two axioms.

AE. $\forall x \forall y[\forall z(z \in x \leftrightarrow z \in y) \rightarrow x{=}y]$ [Extensionality]

AC. $\exists S \forall x(x \in y \leftrightarrow \mathbb{F}[x])$ [Comprehension]

Recall that, in spite of its intuitive plausibility, AC is logically inconsistent.

22. Definitions for Chapter 1

1. Official (Explicit) Definitions

d0.1 $a \neq b =_{df} \sim[a = b]$ [negation]

d0.2 $a \notin b =_{df} \sim[a \in b]$ [negation]

etc.

d1. $\exists!\nu\mathbb{F} =_{df} \exists x\forall\nu(\mathbb{F} \leftrightarrow \nu=x)$ [unique existence]

d2. $\Sigma\nu\mathbb{F} =_{df} \exists!S\forall\nu(\nu \in S \leftrightarrow \mathbb{F})$ [legitimacy]

d3. $A \subseteq B =_{df} \forall x(x \in A \rightarrow x \in B)$ [inclusion]

d4. $A \subset B =_{df} A \subseteq B \;\&\; A \neq B$ [proper inclusion]

d5. $A \supseteq B =_{df} B \subseteq A$ [converse inclusion]

d6. $A \supset B =_{df} B \subset A$ [converse proper inclusion]

d7. $A \perp B =_{df} \sim\exists x(x \in A \;\&\; x \in B)$ [exclusion]

d8. $\{\nu{:}\mathbb{F}\} =_{df} \iota S\forall\nu(\nu \in S \leftrightarrow \mathbb{F})$ [set-abstract]

d9.1 $\{a\} =_{df} \{x : x=a\}$ [singleton]

d9.2 $\{a,b\} =_{df} \{x : x=a \vee x=b\}$ [doubleton]

d9.3 $\{a,b,c\} =_{df} \{x : x=a \vee x=b \vee x=c\}$ [tripleton]

etc.

d10. $\mathbb{U} =_{df} \{x : x=x\}$ [universal set]

d11. $\varnothing =_{df} \{x : x\neq x\}$ [empty set]

d12. $A\cap B =_{df} \{x : x \in A \;\&\; x \in B\}$ [simple intersection]

d13. $A-B =_{df} \{x : x \in A \;\&\; x \notin B\}$ [set-difference]

d14. $A\cup B =_{df} \{x : x \in A \vee x \in B\}$ [simple union]

d15. $A+B =_{df} (A-B)\cup(B-A)$ [Boolean sum]

d16. $\bigcup C =_{df} \{x : \exists y(y \in C \;\&\; x \in y)\}$ [general union]

d17. $\bigcap C =_{df} \{x : \forall y(y \in C \rightarrow x \in y)\}$ [gen intersection]

d18. $\wp(A) =_{df} \{X : X \subseteq A\}$ [power set]

2. Grammatical Categories

In examining the definitions, note carefully the grammatical category of each expression; is it a predicate? a function-sign? a subnective? In this connection, is the *definiens* (resp., the *definiendum*) a formula or a singular-term?

Singular-terms: $\varnothing$, $\mathbb{U}$.

2-place predicates: $=, \in, \neq, \notin, \subseteq, \subset, \supseteq, \supset, \perp$.

1-place function-signs: $\wp, \bigcap, \bigcup$.

2-place function-signs: $\cap, \cup, -, +$.

Anadic function-sign: { }

Variable-binding term-forming operator: { _ : _ }

Variable-binding formula-forming operators: $\exists!$ and Σ.

3. Contextual Definitions

Many definitions have alternative forms that are much more useful in doing proofs. These may be used pretty much like the official definitions. Each contextual definition is simply a combination of the official definition, together with the principle of abstraction and the relevant existence theorem [which is given in the rightmost column].

cd8.	$a \in \{\nu{:}\mathbb{F}\}$	$=_{df}$	$\mathbb{F}[a/\nu]$	provided $\vdash \Sigma\nu\mathbb{F}$
cd9.1	$a \in \{b\}$	$=_{df}$	$a=b$	$\vdash \Sigma x[x=b]$
cd9.2	$a \in \{b,c\}$	$=_{df}$	$a=b \vee a=c$	$\vdash \Sigma x[x=b \vee x=c]$
cd9.3	$a \in \{b,c,d\}$	$=_{df}$	$a=b \vee a=c \vee a=d$	$\vdash \Sigma x[x=b \vee x=c \vee x=d]$
	etc.			
cd11.	$a \in \varnothing$	$=_{df}$	$a \neq a$	$\vdash \Sigma x[x \neq x]$
cd12.	$a \in A \cap B$	$=_{df}$	$a \in A \ \& \ a \in B$	$\vdash \Sigma x[x \in A \ \& \ x \in B]$
cd13.	$a \in A - B$	$=_{df}$	$a \in A \ \& \ a \notin B$	$\vdash \Sigma x[x \in A \ \& \ x \notin B]$
cd14.	$a \in A \cup B$	$=_{df}$	$a \in A \vee a \in B$	$\vdash \Sigma x[x \in A \vee x \in B]$
cd16.	$a \in \bigcup C$	$=_{df}$	$\exists X(X \in C \ \& \ a \in X)$	$\vdash \Sigma x \exists Y[Y \in C \ \& \ x \in Y]$
cd17.	$a \in \bigcap C$	$=_{df}$	$\forall X(X \in C \rightarrow a \in X)$ provided $C \neq \varnothing$	$\vdash C \neq \varnothing \rightarrow \Sigma x \forall Y[Y \in C \rightarrow x \in Y]$
cd18.	$A \in \wp(B)$	$=_{df}$	$A \subseteq B$	$\vdash \Sigma X[X \subseteq B]$

Example (although this example is a bit silly, it illustrates how definitions go both directions)**:**

(1)	$q \in \{r,s\}$	supposition
(2)	$q=r \vee q=s$	1, Def { }
(3)	$q \in \{r,s\}$	2, Def { }

23. Theorems for Chapter 1

Note 1: Unless otherwise specified, every theorem is to be understood as universally quantified over variables corresponding to the constants ‘A’, ‘B’, ‘C’, etc.; recall universal derivation.

Note 2: Occasionally, instead of extra parentheses (which clog up formulas), I use periods to mark the dominant connective. For example:

$$P \,\&.\, Q \vee R =_{\text{df}} P \,\&\, (Q \vee R)$$
$$P \,\&\, Q \,.\!\vee R =_{\text{df}} (P \,\&\, Q) \vee R.$$

1. $A=B \rightarrow \forall x(x \in A \leftrightarrow x \in B)$
2. $A \neq B \leftrightarrow \exists x([x \in A \,\&\, x \notin B] \vee [x \in B \,\&\, x \notin A])$
3. $A \subseteq A$
4. $A \subseteq B \,\&\, B \subseteq A \,.\!\rightarrow A=B$
5. $A \subseteq B \,\&\, B \subseteq C \,.\!\rightarrow A \subseteq C$
6. $\sim[A \subset A]$
7. $A \subset B \rightarrow B \not\subseteq A$
8. $A \subset B \,\&\, B \subset C \,.\!\rightarrow A \subset C$
9. $A \subset B \rightarrow B \not\subseteq A$
10. $A \subset B \leftrightarrow.\ A \subseteq B \,\&\, B \not\subseteq A$
11. $A \subseteq B \,\&\, B \perp C \,.\!\rightarrow A \perp C$
12. $A \perp B \leftrightarrow \sim\exists x[x \in A \,\&\, x \in B]$
13. $A \perp A \leftrightarrow \sim\exists x[x \in A]$
14. $\sim\exists x[x \in A] \rightarrow A \subseteq B$
15. $\sim\exists x[x \in A] \rightarrow A \perp B$
16. $\sim\exists x[x \in \varnothing]$
17. $\varnothing \subseteq A$
18. $\varnothing \perp A$
19. $\sim\exists x[x \in A] \leftrightarrow A=\varnothing$
20. $A \subseteq \varnothing \rightarrow A=\varnothing$
21. $A \cap A = A$
22. $A \cap B = B \cap A$
23. $A \cap (B \cap C) = (A \cap B) \cap C$
24. $A \cap B = A \leftrightarrow A \subseteq B$
25. $A \cap \varnothing = \varnothing$
26. $A \subseteq B \,\&\, A \subseteq C \,.\!\rightarrow A \subseteq B \cap C$
27. $A \perp B \leftrightarrow A \cap B = \varnothing$
28. $A \cup A = A$
29. $A \cup B = B \cup A$
30. $A \cup (B \cup C) = (A \cup B) \cup C$
31. $A \cup B = B \leftrightarrow A \subseteq B$
32. $A \cup \varnothing = A$
33. $A \subseteq C \,\&\, B \subseteq C \,.\!\rightarrow A \cup B \subseteq C$
34. $A \cap (A \cup B) = A$
35. $A \cup (A \cap B) = A$
36. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
37. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
38. $(A \cap B) \cup (A \cap C) \cup (B \cap C) = (A \cup B) \cap (A \cup C) \cap (B \cup C)$
39. $A - A = \varnothing$

40. $A-(A-B) = A\cap B$
41. $A-(A\cap B) = A-B$
42. $(A-B)-C = (A-C)-B$
43. $A-(B-A) = A$
44. $A-B \perp B$
45. $A \subseteq B \leftrightarrow A-B = \varnothing$
46. $A \perp B \leftrightarrow A-B = A$
47. $A-(B\cap C) = (A-B)\cup(A-C)$
48. $A-(B\cup C) = (A-B)\cap(A-C)$
49. $A+A = \varnothing$
50. $A+B = B+A$
51. $A+(B+C) = (A+B)+C$
52. $A+B = \varnothing \rightarrow A \subseteq B$
53. $A+B = \varnothing \leftrightarrow A=B$
54. $A\cup B = (A+B)+(A\cap B)$
55. $A\cap B = (A+B)+(A\cup B)$
56. $A+B = (A\cup B)+(A\cap B)$
57. $(A\cap B)\cap C \subseteq (A+B)+C$
58. $A+B = A\cup B \leftrightarrow A \perp B$
59. $A\cap(A+B) \subseteq A-B$
60. $a \in \{a\}$
61. $a \in \{a,b\}$ & $b \in \{a,b\}$
62. $\{a,b\}=\{c\} \rightarrow a=b$
63. $a=b \rightarrow \{a,b\}=\{a\}$
64. $\{a\}\cup\{b\} \subseteq \{c\} \rightarrow a=b$
65. $\{a,b\} = \{c,d\} \leftrightarrow. (a=c \,\&\, b=d) \vee (a=d \,\&\, b=c)$
66. $\bigcup\{A\} = A$
67. $\bigcap\{A\} = A$
68. $\bigcup\{A,B\} = A\cup B$
69. $\bigcap\{A,B\} = A\cap B$
70. $\bigcup\varnothing = \varnothing$
71. $\bigcup\{\varnothing\} = \varnothing$
72. $\bigcap\varnothing =$ ※
73. $\bigcap\{\varnothing\} = \varnothing$
74. $\bigcup(A\cup B) = \bigcup A \cup \bigcup B$
75. $\bigcap(A\cup B) = \bigcap A \cap \bigcap B$
76. $\forall X(X \in C \rightarrow X \subseteq B) \rightarrow \bigcup C \subseteq B$
77. $\forall X(X \in C \rightarrow X \perp B) \rightarrow \bigcup C \perp B$
78. $\forall X(X \in C \rightarrow B \subseteq X) \rightarrow B \subseteq \bigcap C$
79. $\varnothing \in \wp(A)$
80. $A \in \wp(A)$
81. $A \subseteq B \leftrightarrow \wp(A) \subseteq \wp(B)$
82. $\wp(A\cap B) = \wp(A) \cap \wp(B)$
83. $\wp(A) \cup \wp(B) \subseteq \wp(A\cup B)$
84. $\bigcup\wp(A) = A$
85. $\bigcap\wp(A) = \varnothing$
86. $A \in C \rightarrow A \subseteq \bigcup C$
87. $A \in C \rightarrow \bigcap C \subseteq A$

24. Exercises for Chapter 1

1. Part 1:

For each of the following, say whether the statement is true or false.

1. $\varnothing \in \{\varnothing\}$
2. $\varnothing \subseteq \{\varnothing\}$
3. $\varnothing = \{\varnothing\}$
4. $\{\varnothing\} \in \{\varnothing\}$
5. $\{\varnothing\} \subseteq \{\varnothing\}$
6. $\{\varnothing\} \in \{\varnothing,\{\varnothing\}\}$
7. $\{\varnothing\} \subseteq \{\varnothing,\{\varnothing\}\}$
8. $\{\varnothing\} \subset \{\varnothing,\{\varnothing\}\}$
9. $\{\varnothing,\varnothing\} = \{\varnothing\}$
10. $\{\varnothing\} = \{\{\varnothing\}\}$
11. $\{\varnothing\} \in \{\{\{\varnothing\}\}\}$
12. $\{\varnothing\} \subseteq \{\{\{\varnothing\}\}\}$
13. $\{\varnothing\} \subseteq \varnothing$
14. $\{\{\varnothing\}\} \subseteq \{\{\{\varnothing\}\}\}$

2. Part 2:

1. Give a few examples of sets in which every element is also a subset.
2. Give a few examples of sets in which no element is a subset.
3. Give an example of a set in which at least one element is a subset and at least one element is not a subset.

3. Part 3:

Supposing that $A = \{\varnothing,\{\varnothing\}\}$ and $B = \{\{\varnothing\},\{\{\varnothing\}\}\}$. Compute the following.

1. $A \cup B$
2. $A \cap B$
3. $A - B$
4. $A + B$
5. $(A + B) + A$
6. $A - (A - B)$

4. Part 4:

Compute the following.

1. $\wp(\varnothing)$
2. $\wp(\wp(\varnothing))$
3. $\wp(\wp(\wp(\varnothing)))$
4. $\bigcup\wp(\varnothing)$
5. $\bigcup\wp(\wp(\varnothing))$
6. $\bigcup\wp(\wp(\wp(\varnothing)))$
7. $\bigcap\wp(\varnothing)$
8. $\bigcap\wp(\wp(\varnothing))$
9. $\bigcap\wp(\wp(\wp(\varnothing)))$
10. $\bigcup\bigcup\wp(\varnothing)$
11. $\bigcup\bigcup\wp(\wp(\varnothing))$
12. $\bigcup\bigcup\wp(\wp(\wp(\varnothing)))$
13. $\bigcap\bigcup\wp(\varnothing)$
14. $\bigcap\bigcup\wp(\wp(\varnothing))$
15. $\bigcap\bigcup\wp(\wp(\wp(\varnothing)))$
16. $\wp(\bigcup\wp(\varnothing))$

5. Part 5.

For each theorem, construct a formal derivation.

25. Answers to Exercises for Chapter 1

1. Part 1:

1.	T	8.	T
2.	T	9.	T
3.	F	10.	F
4.	F	11.	F
5.	T	12.	F
6.	T	13.	F
7.	T	14.	F

2. Part 2:

1. ∅, {∅}, {∅, {∅},{{∅}}}, {∅,{∅},{{∅}},{{{∅}}}}
{∅,{∅},{∅,{∅}}},{∅,{∅},{∅,{∅}}

2. {{∅}},{{{∅}}},{{{{∅}}}},{{∅},{{∅}}}

3. {∅,{∅}}

3. Part 3:

1.	{∅,{∅},{{∅}}}	4.	{∅,{{∅}}}
2.	{{∅}}	5.	{{∅},{{∅}}}
3.	{∅}	6.	{{∅}}

4. Part 4:

1.	{∅}	9.	∅
2.	{∅,{∅}}	10.	∅
3.	{∅,{∅},{{∅}},{∅,{∅}}}	11.	∅
4.	∅	12.	{∅}
5.	{∅}	13.	∅
6.	{∅,{∅}}	14.	∅
7.	∅	15.	∅
8.	∅	16.	{∅}

26. Examples of Derivations of Theorems

#1:

(1)	~~SHOW~~: $A=B \rightarrow \forall x(x \in A \leftrightarrow x \in B)$	CD
(2)	$A=B$	As
(3)	~~SHOW~~: $\forall x(x \in A \leftrightarrow x \in B)$	UD
(4)	~~SHOW~~: $a \in A \leftrightarrow a \in B$	5,8,SL
(5)	~~SHOW~~: $a \in A \rightarrow a \in B$	CD
(6)	$a \in A$	As
(7)	$a \in B$	2,6,IL
(8)	~~SHOW~~: $a \in B \rightarrow a \in A$	CD
(9)	$a \in B$	As
(10)	$a \in A$	2,9,IL

#9:

(1)	~~SHOW~~: $A \subset B \rightarrow B \nsubseteq A$	CD
(2)	$A \subset B$	As
(3)	~~SHOW~~: $B \nsubseteq A$	ID
(4)	$B \subseteq A$	As
(5)	~~SHOW~~: ※	8,9,SL
(6)	$A \subseteq B$ & $A \neq B$	2,Def $\subset$
(7)	$A \subseteq B$	6,SL
(8)	$A \neq B$	6,SL
(9)	$A=B$	4,7,T4

The last line appeals to an **earlier theorem**. Theorems are stated (and proved) in a particular order, so we can appeal to ones we have (presumably) already done. In this connection keep in mind that theorems are all universally quantified over the constants. Also, when we appeal to an earlier theorem, we also implicitly appeal to logic, although we don't always say that, if the reasoning is fairly straightforward. For example, in line (9), the inference involves quantifier logic.

#17:

(1)	~~SHOW~~: $\varnothing \subseteq A$	Def $\subseteq$
(2)	~~SHOW~~: $\forall x(x \in \varnothing \rightarrow x \in A)$	UCD
(3)	$a \in \varnothing$	As
(4)	~~SHOW~~: $a \in A$	ID
(5)	$a \notin A$	As
(6)	~~SHOW~~: ※	7,8,SL
(7)	$a \neq a$	3,Def $\varnothing$
(8)	$a=a$	IL

In line (7), we appeal to the definition of $\varnothing$, in particular the contextual definition. Whether we appeal to an explicit definition [e.g., $\varnothing =_{df} \ldots$] or a contextual definition [e.g., $a \in \varnothing =_{df} \ldots$], we use the same notation.

#19a:

(1)	~~SHOW~~: $\sim\exists x[x \in A] \rightarrow A=\varnothing$	CD
(2)	│ $\sim\exists x[x \in A]$	As
(3)	│ ~~SHOW~~: $A=\varnothing$	AE
(4)	│ ~~SHOW~~: $\forall x(x \in A \leftrightarrow x \in \varnothing)$	UD
(5)	│ │ ~~SHOW~~: $a \in A \leftrightarrow a \in \varnothing$	6,9,SL
(6)	│ │ │ ~~SHOW~~: $a \in A \rightarrow a \in \varnothing$	CD
(7)	│ │ │ │ $a \in A$	As
(8)	│ │ │ │ ~~SHOW~~: $a \in \varnothing$	2,7,QL
(9)	│ │ │ ~~SHOW~~: $a \in \varnothing \rightarrow a \in A$	CD
(10)	│ │ │ │ │ $a \in \varnothing$	As
(11)	│ │ │ │ │ ~~SHOW~~: $a \in A$	12,IL
(12)	│ │ │ │ │ │ $a \neq a$	10, Def $\varnothing$

19b:

(1)	~~SHOW~~: $A=\varnothing \rightarrow \sim\exists x[x \in A]$	CD
(2)	│ $A=\varnothing$	As
(3)	│ ~~SHOW~~: $\sim\exists x[x \in A]$	ID
(4)	│ │ $\exists x[x \in A]$	As
(5)	│ │ ~~SHOW~~: $\times$	8,9,SL
(6)	│ │ │ $a \in A$	4,∃O
(7)	│ │ │ $a \in \varnothing$	2,6,IL
(8)	│ │ │ $a \neq a$	7,Def $\varnothing$
(9)	│ │ │ $a=a$	IL

#24a:

(1)	~~SHOW~~: $A \cap B=A \rightarrow A \subseteq B$	CD
(2)	│ $A \cap B=A$	As
(3)	│ ~~SHOW~~: $A \subseteq B$	Def $\subseteq$
(4)	│ ~~SHOW~~: $\forall x(x \in A \rightarrow x \in B)$	UCD
(5)	│ │ $a \in A$	As
(6)	│ │ ~~SHOW~~: $a \in B$	DD
(7)	│ │ │ $a \in A \cap B$	2,5,IL
(8)	│ │ │ $a \in A$ & $a \in B$	7,Def $\cap$
(9)	│ │ │ $a \in B$	8,SL

#27a:

(1)	~~SHOW~~: $A \perp B \rightarrow A \cap B = \varnothing$	CD
(2)	│ $A \perp B$	As
(3)	│ ~~SHOW~~: $A \cap B = \varnothing$	T19
(4)	│ ~~SHOW~~: $\sim\exists x(x \in A \cap B)$	ID
(5)	│ │ $\exists x(x \in A \cap B)$	As
(6)	│ │ ~~SHOW~~: ※	9,10,SL
(7)	│ │ │ $a \in A \cap B$	5,∃O
(8)	│ │ │ $a \in A$ & $a \in B$	7, Def $\cap$
(9)	│ │ │ $\exists x(x \in A$ & $x \in B)$	8,QL
(10)	│ │ │ $\sim\exists x(x \in A$ & $x \in B)$	2, Def $\perp$

#27b:

(1)	~~SHOW~~: $A \cap B = \varnothing \rightarrow A \perp B$	CD
(2)	$A \cap B = \varnothing$	As
(3)	~~SHOW~~: $A \perp B$	Def $\perp$
(4)	~~SHOW~~: $\sim\exists x(x \in A \ \& \ x \in B)$	ID
(5)	$\exists x(x \in A \ \& \ x \in B)$	As
(6)	~~SHOW~~: ※	10,11,SL
(7)	$a \in A \ \& \ a \in B$	5, $\exists$O
(8)	$a \in A \cap B$	7, Def $\cap$
(9)	$a \in \varnothing$	2,8,IL
(10)	$a \neq a$	9, Def $\varnothing$
(11)	$a = a$	IL

#45a:

(1)	~~SHOW~~: $A \subseteq B \rightarrow A - B = \varnothing$	CD
(2)	$A \subseteq B$	As
(3)	~~SHOW~~: $A - B = \varnothing$	T19
(4)	~~SHOW~~: $\sim\exists x[x \in A - B]$	ID
(5)	$\exists x[x \in A - B]$	As
(6)	~~SHOW~~: ※	10,12,SL
(7)	$a \in A - B$	5,$\exists$O
(8)	$a \in A \ \& \ a \notin B$	7,Def –
(9)	$a \in A$	8,SL
(10)	$a \notin B$	8,SL
(11)	$\forall x(x \in A \rightarrow x \in B)$	2,Def $\subseteq$
(12)	$a \in B$	9,11,QL

#58a:

(1)	~~SHOW~~: $A + B = A \cup B \rightarrow A \perp B$	CD
(2)	$A + B = A \cup B$	As
(3)	~~SHOW~~: $A \perp B$	Def $\perp$
(4)	~~SHOW~~: $\sim\exists x(x \in A \ \& \ x \in B)$	ID
(5)	$\exists x(x \in A \ \& \ x \in B)$	As
(6)	~~SHOW~~: ※	7,13,SL
(7)	$a \in A \ \& \ a \in B$	5, $\exists$O
(8)	$a \in A \vee a \in B$	7,SL
(9)	$a \in A \cup B$	8,Def $\cup$
(10)	$a \in A + B$	2,9,IL
(11)	$a \in (A - B) \cup (B - A)$	10,Def +
(12)	$a \in A - B \vee a \in B - A$	11,Def $\cup$
(13)	$a \in A \ \& \ a \notin B \ .\!\vee\!. \ a \in B \ \& \ a \notin A$	12,Def –

#61a:

(1)	~~SHOW~~: $a \in \{a,b\}$	DD
(2)	$a = a$	IL
(3)	$a = a \vee a = b$	2,SL
(4)	$a \in \{a,b\}$	3 Def {}

#61b:

(1)	~~SHOW~~: $b \in \{a,b\}$	DD
(2)	$b=b$	IL
(3)	$b=a \vee b=b$	2,SL
(4)	$b \in \{a,b\}$	3,Def {}

#62:

(1)	~~SHOW~~: $\{a,b\}=\{c\} \rightarrow a=b$	CD
(2)	$\{a,b\} = \{c\}$	As
(3)	~~SHOW~~: $a=b$	DD
(4)	$a \in \{a,b\}$	T61
(5)	$a \in \{c\}$	2,4,IL
(6)	$a=c$	5,Def {}
(7)	$b \in \{a,b\}$	T61
(8)	$b \in \{c\}$	2,7,IL
(9)	$b=c$	8,Def {}
(10)	$a=b$	6,9,IL

#69a:

(1)	~~SHOW~~: $\bigcap\{A,B\} \subseteq A\cap B$	Def $\subseteq$
(2)	~~SHOW~~: $\forall x(x \in \bigcap\{A,B\} \rightarrow x \in A\cap B)$	UCD
(3)	$a \in \bigcap\{A,B\}$	As
(4)	~~SHOW~~: $a \in A\cap B$	Def $\cap$
(5)	~~SHOW~~: $a \in A$ & $a \in B$	DD
(6)	$\forall X(X \in \{A,B\} \rightarrow a \in X)$	3,Def $\bigcap$
(7)	$A \in \{A,B\}$	T61
(8)	$a \in A$	6,7,QL
(9)	$B \in \{A,B\}$	T61
(10)	$a \in B$	6,9,QL
(11)	$a \in A$ & $a \in B$	8,10,SL

#75a:

(1)	~~SHOW~~: $\bigcap(A\cup B) \subseteq \bigcap A \cap \bigcap B$	Def $\subseteq$
(2)	~~SHOW~~: $\forall x[x \in \bigcap(A\cup B) \rightarrow x \in \bigcap A\cap\bigcap B]$	UCD
(3)	$a \in \bigcap(A\cup B)$	As
(4)	~~SHOW~~: $a \in \bigcap A\cap\bigcap B$	Def $\cap$
(5)	~~SHOW~~: $a \in \bigcap A$ & $a \in \bigcap B$	6,15,SL
(6)	~~SHOW~~: $a \in \bigcap A$	Def $\bigcap$
(7)	~~SHOW~~: $\forall X(X \in A \rightarrow a \in X)$	UCD
(8)	$D \in A$	As
(9)	~~SHOW~~: $a \in D$	DD
(10)	$\forall X(X \in A\cup B \rightarrow a \in X)$	3,Def $\cup$
(11)	$\forall X(X \in A \vee X \in B \ .\!\rightarrow a \in X)$	10,Def $\cup$
(12)	$D \in A \vee D \in B \ .\!\rightarrow a \in D$	11,QL
(13)	$D \in A \vee D \in B$	8,SL
(14)	$a \in D$	12,13,SL
(15)	~~SHOW~~: $a \in \bigcap(B)$	Def $\cup$
(16)	~~SHOW~~: $\forall X(X \in B \rightarrow a \in X)$	DD
(17)	$\forall X(X \in A\cup B \rightarrow a \in X)$	3,Def $\cup$
(18)	$\forall X(X \in A \vee X \in B \ .\!\rightarrow a \in X)$	17,Def $\cup$
(19)	$\forall X(X \in B \rightarrow a \in X)$	18,QL

#81:

(1)	~~SHOW~~: $A \subseteq B \leftrightarrow \wp(A) \subseteq \wp(B)$	2,14,SL
(2)	~~SHOW~~: $A \subseteq B \rightarrow \wp(A) \subseteq \wp(B)$	CD
(3)	$A \subseteq B$	As
(4)	~~SHOW~~: $\wp(A) \subseteq \wp(B)$	Def $\subseteq$
(5)	~~SHOW~~: $\forall X[X \in \wp(A) \rightarrow X \in \wp(B)]$	UCD
(6)	$D \in \wp(A)$	As
(7)	~~SHOW~~: $D \in \wp(B)$	Def $\wp$
(8)	~~SHOW~~: $D \subseteq B$	Def $\subseteq$
(9)	~~SHOW~~: $\forall x(x \in D \rightarrow x \in B\}$	DD
(10)	$D \subseteq A$	6,Def $\wp$
(11)	$\forall x(x \in D \rightarrow x \in A)$	10,Def $\subseteq$
(12)	$\forall x(x \in A \rightarrow x \in B)$	3,Def $\subseteq$
(13)	$\forall x(x \in D \rightarrow x \in B)$	11,12,QL
(14)	~~SHOW~~: $\wp(A) \subseteq \wp(B) \rightarrow A \subseteq B$	CD
(15)	$\wp(A) \subseteq \wp(B)$	As
(16)	~~SHOW~~: $A \subseteq B$	Def $\subseteq$
(17)	~~SHOW~~: $\forall x(x \in A \rightarrow x \in B)$	DD
(18)	$\forall X[X \in \wp(A) \rightarrow X \in \wp(B)]$	15,Def $\subseteq$
(19)	$A \in \wp(A) \rightarrow A \in \wp(B)$	18,QL
(20)	$\forall x(x \in A \rightarrow x \in A)$	QL
(21)	$A \subseteq A$	20,Def $\subseteq$
(22)	$A \in \wp(A)$	21,Def $\wp$
(23)	$A \in \wp(B)$	19,22,SL
(24)	$A \subseteq B$	23,Def $\wp$

#84:

(1)	~~SHOW~~: $\bigcup\wp(A) = A$	2,13,T4
(2)	~~SHOW~~: $\bigcup\wp(A) \subseteq A$	Def $\subseteq$
(3)	~~SHOW~~: $\forall x(x \in \bigcup\wp(A) \rightarrow x \in A)$	UCD
(4)	$a \in \bigcup\wp(A$	As
(5)	~~SHOW~~: $a \in A$	DD
(6)	$\exists X(X \in \wp(A)\ \&\ a \in X)$	4,Def $\bigcup$
(7)	$B \in \wp(A)\ \&\ a \in B$	6,$\exists$O
(8)	$B \in \wp(A)$	7,SL
(9)	$a \in B$	7,SL
(10)	$B \subseteq A$	8,Def $\wp$
(11)	$\forall x(x \in B \rightarrow x \in A)$	10,Def $\subseteq$
(12)	$a \in A$	9,11,QL
(13)	~~SHOW~~: $A \subseteq \bigcup\wp(A)$	Def $\subseteq$
(14)	~~SHOW~~: $\forall x[x \in A \rightarrow x \in \bigcup\wp(A)]$	UCD
(15)	$a \in A$	As
(16)	~~SHOW~~: $a \in \bigcup\wp(A)$	Def $\bigcup$
(17)	~~SHOW~~: $\exists X(X \in \wp(A)\ \&\ a \in X)$	Def $\wp$
(18)	~~SHOW~~:: $\exists X(X \subseteq A\ \&\ a \in X)$	DD
(19)	$\forall x(x \in A \rightarrow x \in A)$	QL
(20)	$A \subseteq A$	19,Def $\subseteq$
(21)	$A \subseteq A\ \&\ a \in A$	15,20,SL
(22)	$\exists X(X \subseteq A\ \&\ a \in X)$	21,QL