

1. Sentential Logic

Henceforth, \mathcal{A} , \mathcal{B} , \mathcal{C} , \mathcal{D} are closed formulas.

1. Inference Rules

&I	&O		\sim &I/O
$\frac{\mathcal{A}}{\mathcal{B}} \quad \frac{\mathcal{B}}{\mathcal{A} \& \mathcal{B}}$	$\frac{\mathcal{A} \& \mathcal{B}}{\mathcal{A}}$	$\frac{\mathcal{A} \& \mathcal{B}}{\mathcal{B}}$	$\frac{\sim(\mathcal{A} \& \mathcal{B})}{\mathcal{A} \rightarrow \sim \mathcal{B}}$

\vee I	\vee O		\sim \vee I/O
$\frac{\mathcal{A}}{\mathcal{A} \vee \mathcal{B}} \quad \frac{\mathcal{B}}{\mathcal{A} \vee \mathcal{B}}$	$\frac{\mathcal{A} \vee \mathcal{B} \quad \sim \mathcal{A}}{\mathcal{B}}$	$\frac{\mathcal{A} \vee \mathcal{B} \quad \sim \mathcal{B}}{\mathcal{A}}$	$\frac{\sim(\mathcal{A} \vee \mathcal{B})}{\sim \mathcal{A} \& \sim \mathcal{B}}$

\rightarrow I	\rightarrow O		\sim \rightarrow I/O
$\frac{\sim \mathcal{A}}{\mathcal{A} \rightarrow \mathcal{B}} \quad \frac{\mathcal{B}}{\mathcal{A} \rightarrow \mathcal{B}}$	$\frac{\mathcal{A} \rightarrow \mathcal{B} \quad \mathcal{A}}{\mathcal{B}}$	$\frac{\mathcal{A} \rightarrow \mathcal{B} \quad \sim \mathcal{B}}{\sim \mathcal{A}}$	$\frac{\sim(\mathcal{A} \rightarrow \mathcal{B})}{\mathcal{A} \& \sim \mathcal{B}}$

\leftrightarrow I	\leftrightarrow O		\sim \leftrightarrow I/O
$\frac{\mathcal{A} \rightarrow \mathcal{B} \quad \mathcal{B} \rightarrow \mathcal{A}}{\mathcal{A} \leftrightarrow \mathcal{B}}$	$\frac{\mathcal{A} \leftrightarrow \mathcal{B}}{\mathcal{A} \rightarrow \mathcal{B}}$	$\frac{\mathcal{A} \leftrightarrow \mathcal{B}}{\mathcal{B} \rightarrow \mathcal{A}}$	$\frac{\sim(\mathcal{A} \leftrightarrow \mathcal{B})}{\sim \mathcal{A} \leftrightarrow \mathcal{B}}$

\times I	\times O	DN
$\frac{\mathcal{A} \quad \sim \mathcal{A}}{\times}$	$\frac{\times}{\mathcal{A}}$	$\frac{\sim \sim \mathcal{A}}{\mathcal{A}}$

Note: The \sim O and \sim I rules are combined, using a long equals sign '==='. Henceforth, any rule that is displayed with '===' is a bi-directional rule, which can be used both as an in-rule and as an out-rule.

2. Strategic Rules

Direct Derivation (DD)	Indirect Derivation (ID)
$\begin{array}{l} \text{SHOW: } \mathcal{A} \\ \\ \mathcal{A} \end{array} \quad \text{DD}$	$\begin{array}{l} \text{SHOW: } \mathcal{A} \\ \\ \sim \mathcal{A} \\ \\ \text{SHOW: } \times \\ \end{array} \quad \begin{array}{l} \text{ID} \\ \text{As} \end{array}$

Conditional Derivation (CD)	Tilde Indirect Derivation (\sim D)
$\begin{array}{l} \text{SHOW: } \mathcal{A} \rightarrow \mathcal{B} \\ \\ \mathcal{A} \\ \\ \text{SHOW: } \mathcal{B} \\ \end{array} \quad \begin{array}{l} \text{CD} \\ \text{As} \end{array}$	$\begin{array}{l} \text{SHOW: } \sim \mathcal{A} \\ \\ \mathcal{A} \\ \\ \text{SHOW: } \times \\ \end{array} \quad \begin{array}{l} \sim \text{D} \\ \text{As} \end{array}$

Ampersand Derivation (&D)	Biconditional Derivation (\leftrightarrow D)
$\begin{array}{l} \text{SHOW: } \mathcal{A} \& \mathcal{B} \\ \\ \text{SHOW: } \mathcal{A} \\ \\ \text{SHOW: } \mathcal{B} \\ \end{array} \quad \& \text{D}$	$\begin{array}{l} \text{SHOW: } \mathcal{A} \leftrightarrow \mathcal{B} \\ \\ \text{SHOW: } \mathcal{A} \rightarrow \mathcal{B} \\ \\ \text{SHOW: } \mathcal{B} \rightarrow \mathcal{A} \\ \end{array} \quad \leftrightarrow \text{D}$

Wedge Indirect Derivation (\vee ID)	Separation of Cases (SC)
$\begin{array}{l} \text{SHOW: } \mathcal{D}_1 \vee \mathcal{D}_2 \vee \dots \vee \mathcal{D}_k \\ \\ \sim \mathcal{D}_1 \\ \\ \sim \mathcal{D}_2 \\ \\ \dots \\ \\ \sim \mathcal{D}_k \\ \\ \text{SHOW: } \times \\ \end{array} \quad \begin{array}{l} \vee \text{ID} \\ \text{As} \\ \text{As} \end{array}$	$\begin{array}{l} \mathcal{D}_1 \vee \mathcal{D}_2 \vee \dots \vee \mathcal{D}_k \\ \text{SHOW: } \mathcal{C} \\ \\ \text{c1: } \mathcal{D}_1 \\ \\ \text{SHOW: } \mathcal{C} \\ \\ \text{c2: } \mathcal{D}_2 \\ \\ \text{SHOW: } \mathcal{C} \\ \\ \cdot \\ \\ \cdot \\ \\ \text{ck: } \mathcal{D}_k \\ \\ \text{SHOW: } \mathcal{C} \\ \end{array} \quad \begin{array}{l} \text{SC} \\ \text{As} \\ \text{As} \\ \text{As} \end{array}$

2. Quantifier Logic (Free Logic Version)

1. Introduction

Classical first-order logic is based on the following two presuppositions:

- (C1) The domain (universe) of discourse is not empty; accordingly, the sentence ‘there is something’ is logically true, even though it is not necessarily true.
- (C2) Every singular term, no matter how silly, denotes an existing object (i.e., element of the domain).

In contrast to classical logic, there is *Free Logic*, of which there are two variants. The more radical version (Universally-Free Logic) denies both (C1) and (C2). The less radical version of Free Logic denies (C2), but accepts (C1). In what follows, we pursue the more radical variant.

2. Constants

In intro logic, the distinction between unquantified variables ("constants") (*a*, *b*, *c*, etc.) and proper nouns (*Jay*, *Kay*, *the U.S.*, etc.) is *not* important. By contrast, in free logic, the distinction is **very important**. In particular, whereas constants always denote existing objects (in the domain of quantification), proper nouns need not denote anything.

In doing derivations in free logic, one treats constants as purely intra-derivational symbols. In particular, we have the following definition.

A constant is an atomic singular-term that is introduced by UD or $\exists\text{O}$.
Ordinarily, any atomic singular-term that occurs in the premises or conclusion is regarded as a proper noun , not a constant. However, in mathematics (including set theory), one often does derivations in which universal quantification is taken for granted, so constants are allowed in premises and show-lines. These are understood as having been introduced by UD.
A constant counts as old precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as new . (as before)

3. Quantifier Rules

In what follows, Φ is a formula, in which v is the only variable (if any) that occurs free, and $\Phi[\varepsilon/v]$ is the formula that results when ε replaces every occurrence of v that is free in Φ . An expression is **closed** iff it contains no *free occurrence* of any variable. An occurrence of a variable v is **free** in expression \mathcal{E} iff that occurrence does not lie within the scope of an operator binding v – i.e., $\forall v$, $\exists v$, or $\neg v$.

Universal-Out ($\forall O$)	Existential-In ($\exists I$)
$\forall v\Phi$ <hr style="width: 20%; margin: auto;"/> $\Phi[o/v]$	$\Phi[o/v]$ <hr style="width: 20%; margin: auto;"/> $\exists v\Phi$
<i>o</i> is any old constant.	

Universal-Derivation (UD)	Existential-Out ($\exists O$)
SHOW: $\forall v\Phi$ SHOW: $\Phi[n/v]$ 	$\exists v\Phi$ <hr style="width: 20%; margin: auto;"/> $\Phi[n/v]$
<i>n</i> is any new constant.	

Quantifier-Negation (QN)	
$\sim\forall v\Phi$ <hr style="width: 20%; margin: auto;"/> $\exists v\sim\Phi$	$\sim\exists v\Phi$ <hr style="width: 20%; margin: auto;"/> $\forall v\sim\Phi$

Tilde-Universal-Out ($\sim\forall O$) $\sim\forall O = QN+\exists O$	Tilde-Existential-Out ($\sim\exists O$) $\sim\exists O = QN+\forall O$
$\sim\forall v\Phi$ <hr style="width: 20%; margin: auto;"/> $\sim\Phi[n/v]$	$\sim\exists v\Phi$ <hr style="width: 20%; margin: auto;"/> $\sim\Phi[o/v]$
<i>n</i> is any new constant.	<i>o</i> is any old constant.

3. Identity Logic

Reflexivity (R=)	Symmetry (S=)	Transitivity (T=)
$\frac{\emptyset}{\sigma = \sigma}$	$\frac{\sigma = \tau}{\tau = \sigma}$	$\frac{\begin{matrix} \rho = \sigma \\ \sigma = \tau \end{matrix}}{\rho = \tau}$
$\sigma, \tau,$ and ρ are any closed singular-terms.		
LL		
$\frac{\begin{matrix} \sigma = \tau \\ \Phi[\sigma/v] \end{matrix}}{\Phi[\tau/v]}$	$\frac{\begin{matrix} \sigma = \tau \\ \Phi[\tau/v] \end{matrix}}{\Phi[\sigma/v]}$	

4. Description Logic

Iota-Out (iO)	Iota-In (iI)
$\frac{c = \iota v \Phi}{\forall v(\Phi \leftrightarrow v=c)}$	$\frac{\forall v(\Phi \leftrightarrow v=c)}{c = \iota v \Phi}$
c must be a <i>constant</i> .	

5. Short-Cut Rules

1. The Immediate Show-Cancel Rule

If a show-line follows from available lines (**earlier** or **later!**) by a rule, then it can be cancelled by that rule. Annotation: cite the line number(s) and the rule.

2. The Conjunction Rule

Any available conjunctive line (with any number of conjuncts) can be treated as the appropriate number of separate lines, numbered (e.g.) 7a, 7b, 7c. And conversely, any number of available lines can be treated as the corresponding conjunction.

3. Rule-Multiplication

Any one-place rule can be multiplied, **provided** the particular rule also applies to the intermediate line. For example, $\forall O + \forall O = \forall O2$; $UD + UD = U2D$; $\sim \exists O + \sim \exists O = \sim \exists O2$.

4. Further Rule Combinations

$\forall O$, $\forall O2$, etc. can be combined with $\rightarrow O$ to produce $\forall \rightarrow O$, $\forall 2 \rightarrow O$, etc., and UD , $UD2$, etc., can be combined with CD to produce UCD , $U2CD$, etc. Similarly, UD , $UD2$, etc., can be combined with $\leftrightarrow D$ to produce UBD , $U2BD$, etc.

5. Contraposition Rule

For every *genuine* one-place rule \mathbb{R} , there is an associated contrapositive rule $\mathbb{R}(-)$, which is obtained by reversing and negating and the premise and conclusion.

NOTE CAREFULLY:

$\exists O$ is *not* a genuine *inference rule*, but is rather an *assumption rule*.

6. Alphabetic Variance (AV)

$$\frac{\Phi[u]}{\Phi[v]}$$

Here, u , v are variables, $\Phi[u]$ is a formula in which v does *not* occur, $\Phi[v]$ is a formula in which u does *not* occur, and $\Phi[v]$ results when every occurrence of u in $\Phi[u]$ is replaced by an occurrence of v .

7. The Rule SL

If a line can be derived from available lines using only SL rules, then it may be written down by the rule SL. This rule may be used in place of any combination of SL rules, including inference rules and show rules.

8. The Rule QL

If a line can be derived from previous available lines using only quantifier rules, but it cannot be derived using just SL rules, then it may be written down by the rule QL. This rule may be used in place of any QL inference rule, as well as any combination of QL rules, including inference rules and show rules.

NOTE CAREFULLY: The rule $\exists O$, and hence $\sim \forall O$, are **not** genuine inference rules, but **assumption rules**. No instance of $\exists O$ is valid by QL! So when you cite $\exists O$ or $\sim \forall O$, **do not** cite it as QL.

9. The Rule IL

If a line can be derived from previous available lines using only identity rules, but it cannot be derived using just QL rules, then it may be written down by the rule IL. This rule may be used in place of any IL inference rule, as well as any combination of IL rules, including inference rules and show rules.

7. Axioms

- a1. $\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x=y]$ [Extensionality]
a2. $\exists x \sim \exists y [y \in x]$ [Empty Set]
a3. $\forall x \exists y \forall z [z \in y \leftrightarrow (z \in x \ \& \ \mathbb{F})]$ [[y nor free in \mathbb{F}] Separation]
a4. $\forall x \forall y \exists z \forall w (w \in z \leftrightarrow [w \in x \vee w \in y])$ [Simple Unions]
a5. $\forall x \exists y \forall z (z \in y \leftrightarrow z=x)$ [Singletons]
a6. $\forall x \exists y \forall z [z \in y \leftrightarrow \exists w (w \in x \ \& \ z \in w)]$ [General Unions]
a7. $\forall x \exists y \forall z [z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x)]$ [Power Sets]

8. Definitions

- d0.1 $a \neq b =_{df} \sim [a = b]$ [negation]
d0.2 $a \notin b =_{df} \sim [a \in b]$ [negation]
d1. $\exists! v \mathbb{F} =_{df} \exists x \forall v (\mathbb{F} \leftrightarrow v=x)$ [unique existence]
d2. $\Sigma v \mathbb{F} =_{df} \exists! S \forall v (v \in S \leftrightarrow \mathbb{F})$ [legitimacy]
d3. $A \subseteq B =_{df} \forall x (x \in A \rightarrow x \in B)$ [inclusion]
d4. $A \subset B =_{df} A \subseteq B \ \& \ A \neq B$ [proper inclusion]
d5. $A \supseteq B =_{df} B \subseteq A$ [converse inclusion]
d6. $A \supset B =_{df} B \subset A$ [converse proper inclusion]
d7. $A \perp B =_{df} \sim \exists x (x \in A \ \& \ x \in B)$ [exclusion]
d8. $\{v:\mathbb{F}\} =_{df} \iota S \forall v (v \in S \leftrightarrow \mathbb{F})$ [set-abstract]
d9.1 $\{a\} =_{df} \{x : x=a\}$ [singleton]
d9.2 $\{a,b\} =_{df} \{x : x=a \vee x=b\}$ [doubleton]
d9.3 $\{a,b,c\} =_{df} \{x : x=a \vee x=b \vee x=c\}$ [tripleton]
etc.
d10. $\mathbb{U} =_{df} \{x : x=x\}$ [universal set]
d11. $\emptyset =_{df} \{x : x \neq x\}$ [empty set]
d12. $A \cap B =_{df} \{x : x \in A \ \& \ x \in B\}$ [simple intersection]
d13. $A - B =_{df} \{x : x \in A \ \& \ x \notin B\}$ [set-difference]
d14. $A \cup B =_{df} \{x : x \in A \vee x \in B\}$ [simple union]
d15. $A + B =_{df} (A - B) \cup (B - A)$ [Boolean sum]
d16. $\cup C =_{df} \{x : \exists y (y \in C \ \& \ x \in y)\}$ [general union]
d17. $\cap C =_{df} \{x : \forall y (y \in C \rightarrow x \in y)\}$ [gen intersection]
d18. $\mathcal{P}(A) =_{df} \{X : X \subseteq A\}$ [power set]

9. Definitions (Sentential Versions)

- cd8. $a \in \{v:\mathbb{F}\} =_{df} \mathbb{F}[a/v]$
cd9.1 $a \in \{b\} =_{df} a=b$
cd9.2 $a \in \{b,c\} =_{df} a=b \vee a=c$
cd9.3 $a \in \{b,c,d\} =_{df} a=b \vee a=c \vee a=d$
etc.
cd11. $a \in \emptyset =_{df} a \neq a$
cd12. $a \in A \cap B =_{df} a \in A \ \& \ a \in B$
cd13. $a \in A - B =_{df} a \in A \ \& \ a \notin B$
cd14. $a \in A \cup B =_{df} a \in A \vee a \in B$
cd16. $a \in \cup C =_{df} \exists X (X \in C \ \& \ a \in X)$
cd17. $a \in \cap C =_{df} \forall X (X \in C \rightarrow a \in X)$ [provided $C \neq \emptyset$]
cd18. $A \in \mathcal{P}(B) =_{df} A \subseteq B$