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The Semantic Characterization of Logic

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1. Introduction

We have now examined Classical Sentential Logic (CSL). As we are well aware, CSL is not all there is to logic. At the very least, CSL has nothing to say about a large number of very important logical concepts, including quantification, identity, descriptions, etc. Second, it has nothing to say about non-truth functional connectives, such as the subjunctive conditional and the multitude of modal operators. Third, it does not speak directly about all truth-functional connectives, but only the privileged five. Finally, CSL is not universally accepted as the correct sentential logic. In opposition to CSL are a host of "deviant" logics that have been proposed as alternatives to CSL. These include multi-valued logic, super-valuational logic, intuitionistic logic, relevance logic, quantum logic, fuzzy logic, to name the most prominent examples.

Given that the subject of logic is not monolithic, it is useful to distinguish between 'Logic' and 'logic'. The former refers to a scholarly discipline – the "science of reasoning"; the latter refers to an abstract kind, and can be sensibly pluralized to form 'logics'. This leads to the following slogan.

Logic studies logics.

Here, we can read this aloud as:

logic with a capital-L studies logics with a little-L.

Once we have the general term 'logics', we must ask the following question.

what is a logic?

We propose that,

at a minimum,
a logic specifies a class of valid argument forms.

This means that a logic \mathcal{L} can be (minimally) characterized as a structure (L, A) , where L is a formal language over which \mathcal{L} is written, and A is a set of argument forms in L – namely those argument forms deemed valid by \mathcal{L} . [Henceforth, we use the script letter ' \mathcal{L} ' as a schematic name for logics, and we use ordinary ' L ' as a schematic name for formal languages.]

We already know what a formal language is. So what is it to be an argument (form) in a (formal) language L ? This is defined officially as follows.

Df

Let L be a formal language, and let S be the sentences (sentence forms; formulas) of L . Then an argument (form) in L is, by definition, any (ordered) pair (Γ/α) such that Γ is a subset of S , and α is an element of S .

Notice that we use the forward slash symbol ‘/’ to delineate the items of the ordered pair. This particular choice is made in order to remind us that, in canonical form, an argument consists of premises followed by the word ‘therefore’ followed by the conclusion. Thus, the slash-symbol has two roles: formally, it is simply a punctuation mark; however, heuristically, it is short for ‘therefore’.

Before continuing, it is important to realize our definition of argument deviates from ordinary usage in two ways. In ordinary usage, an argument has one or more premises, and has only finitely many premises. According to our official definition, for abstract purposes, an argument can have any number of premises, including zero-many and infinitely-many

2. The Semantic Characterization of Logics

So far, we have said that a logic \mathcal{L} specifies a class of \mathcal{L} -valid argument forms. We haven’t said how this is accomplished. We do have a model of this process, however – Classical Sentential Logic.

Recall that CSL is characterized in two steps. First, one specifies the formal language L over which CSL is written to be the "standard" sentential language (ZOL) based on the five standard SL connectives. Second, one specifies the class V of CSL-admissible valuations, with respect to which all the usual logical notions are defined. A valuation is deemed CSL-admissible if and only if it complies with the truth-functional interpretation of the five standard SL connectives.

This serves as a general model for the semantic specification of a logic.

A logic \mathcal{L} is semantically specified by a pair (L,V) , where L is a formal language, and V is a set of valuations on L .

Recall the definition of valuation on formal language L .

Df
Let L be a formal language, and let S be the set of sentences (formulas) of L . Then a valuation on L is, by definition, any function from S into $\{T,F\}$.

In other words, a valuation on L assigns a truth-value, T or F , to every sentence of L .

Once we have the class V of admissible valuations, we are in a position to define argument-validity, as follows.

Df
Let \mathcal{L} be a logic semantically specified by (L,V) . Let (Γ/α) be an argument in L .
Then
 (Γ/α) is \mathcal{L} -valid =_{df} $\forall v \{v \in V \rightarrow \forall \gamma (\gamma \in \Gamma \rightarrow v(\gamma)=T) \rightarrow v(\alpha)=T\}$

An alternative account of validity employs an intermediate notion of refutation.

Df

Let L be a formula language, and let v be a valuation on L . Let (Γ/α) be an argument in L . Then

$$v \text{ refutes } (\Gamma/\alpha) =_{df} \forall \gamma \{ \gamma \in \Gamma \rightarrow v(\gamma)=T \} \ \& \ v(\alpha)=F$$

In other words, a valuation refutes an argument if and only if it makes every premise true but makes the conclusion false. This definition allows us to prove the following theorem (exercise).

Th

Let \mathcal{L} be a logic semantically specified by (L, V) . Let (Γ/α) be an argument in L . Then

$$(\Gamma/\alpha) \text{ is } \mathcal{L}\text{-valid} \leftrightarrow \sim \exists v \{ v \in V \ \& \ v \text{ refutes } (\Gamma/\alpha) \}$$

3. Semantically Characterizing Other Standard Logical Notions

In addition to the central concept of logic – argument validity – other logical notions can be defined (even in intro logic!), including tautology, contradiction, consistency, logical implication, and logical equivalence.

Each of these can be given a general definition in the context of logical semantics; some of them are given different names. In the following, we suppose that logic \mathcal{L} is semantically specified by (L, V) . Also, Γ is a subset of formulas of L , and α is a formula of L .

- | | | | |
|-----|--|----------|--|
| (1) | α is \mathcal{L} -valid | $=_{df}$ | $\forall v \in V, v(\alpha)=T$ |
| (2) | α is \mathcal{L} -contra-valid | $=_{df}$ | $\forall v \in V, v(\alpha)=F$ |
| (3) | α is \mathcal{L} -contingent | $=_{df}$ | α is neither \mathcal{L} -valid nor \mathcal{L} -contra-valid |
| (4) | α \mathcal{L} -entails β | $=_{df}$ | $\forall v \in V \{ v(\alpha)=T \rightarrow v(\beta)=T \}$ |
| (5) | α and β are \mathcal{L} -equivalent | $=_{df}$ | $\forall v \in V \{ v(\alpha)=T \leftrightarrow v(\beta)=T \}$ |
| (6) | Γ is \mathcal{L} -consistent | $=_{df}$ | $\exists v \in V, \forall \gamma \in \Gamma, v(\gamma)=T$ |
| (7) | Γ is \mathcal{L} -inconsistent | $=_{df}$ | Γ is not \mathcal{L} -consistent |

4. Validity etc. Revisited

In the present section, we formulate a number of logical notions in terms of a set V of admissible valuations. As usual, it is understood that a valuation on language L is a function that assigns exactly one truth-value, T or F , to every sentence in L . We assume also, of course, that $T \neq F$.

The relevant formal semantic notions are presented in a series of definitions. We begin with matters of terminology. In what follows, we presuppose a formal language L , with sentences S , and we presuppose a set V of admissible valuations. All definitions are relative to L and V .

(D) Let α be a formula (i.e., $\alpha \in S$), let Γ be a subset of formulas (i.e., $\Gamma \subseteq S$), and let W be a subset of valuations (i.e., $W \subseteq V$).

| | | | | |
|-----|------------------|-----|---|----------------------------|
| (1) | $v < \alpha$ | =df | $v(\alpha)=T$ | [v verifies α] |
| (2) | $v < \Gamma$ | =df | $\forall \gamma \in \Gamma, v < \gamma$ | [v verifies Γ] |
| (3) | $W < \Gamma$ | =df | $\forall w \in W, w < \Gamma$ | [W verifies Γ] |
| (4) | $v \perp \alpha$ | =df | $v(\alpha)=F$. | [v falsifies α] |
| (5) | $v \perp \Gamma$ | =df | $\forall \gamma \in \Gamma, v \perp \gamma$ | [v falsifies Γ] |
| (6) | $W \perp \Gamma$ | =df | $\forall w \in W, w \perp \Gamma$ | [W falsifies Γ] |

Having introduced the basic terminology, we next define a variety of logical concepts, including validity and entailment, all defined relative to a set V of admissible valuations.

(D) Let α, β be formulas, and let Γ, Δ be subsets of formulas.

| | | | |
|------|--------------------------------------|-----|---|
| (1) | α is <i>valid</i> | =df | $V < \alpha$ |
| (2) | α is <i>contra-valid</i> | =df | $V \perp \alpha$ |
| (3) | Γ is <i>falsifiable</i> | =df | $\exists v \in V, v \perp \Gamma$ |
| (4) | Γ is <i>unfalsifiable</i> | =df | $\sim \exists v \in V, v \perp \Gamma$ |
| (5) | Γ is <i>verifiable</i> | =df | $\exists v \in V, v < \Gamma$ |
| (6) | Γ is <i>unverifiable</i> | =df | $\sim \exists v \in V, v < \Gamma$ |
| (7) | α <i>entails</i> β | =df | $\forall v \in V \{ v < \alpha \rightarrow v < \beta \}$ |
| (8) | Γ <i>entails</i> α | =df | $\forall v \in V \{ v < \Gamma \rightarrow v < \alpha \}$ |
| (9) | α and β are equivalent | =df | $\forall v \in V \{ v < \alpha \leftrightarrow v < \beta \}$ |
| (10) | Γ and Δ are equivalent | =df | $\forall v \in V \{ v < \Gamma \leftrightarrow v < \Delta \}$ |

Note: ‘entails’ here is short for ‘semantically entails with respect to V ’; the adverbs are omitted in this context. When the omission is troublesome, we will dutifully restore the modifiers.

In order to symbolize some of these predicates, we use a single symbol, the double turnstile ‘ \models ’, ambiguously. As shown later, this ambiguity is harmless; indeed, it is fruitful.

| | | | |
|-----|-------------------------|-----|--------------------------------------|
| (1) | $\models \alpha$ | =df | α is valid |
| (2) | $\alpha \models$ | =df | α is contra-valid |
| (3) | $\models \Gamma$ | =df | Γ is unfalsifiable |
| (4) | $\Gamma \models$ | =df | Γ is unverifiable |
| (5) | $\alpha \models \beta$ | =df | α entails β |
| (6) | $\Gamma \models \alpha$ | =df | Γ entails α |
| (7) | $\alpha \equiv \beta$ | =df | α and β are equivalent |
| (8) | $\Gamma \equiv \Delta$ | =df | Γ and Δ are equivalent |

The notions of validity and contra-validity correspond to the customary logical notions of *logical truth* and *logical falsehood*. The notions of unfalsifiability and unverifiability are generalizations of validity and contra-validity, respectively. Specifically, to say that a set Γ of sentences is "valid" is to say that the sentences of Γ cannot all be made false. Similarly, to say that Γ is "contra-valid" is to say that the sentences of Γ cannot all be made true. Notice that a formula α is valid (resp., contra-valid) iff the singleton $\{\alpha\}$ is unfalsifiable (resp., unverifiable).

In addition to the notions of validity and contra-validity, as well as their generalizations, there are two notions of entailment – binary entailment, and ordinary entailment. Ordinary entailment corresponds to the notion of argument validity.

$$\Gamma \text{ entails } \alpha \quad \leftrightarrow \quad (\Gamma/\alpha) \text{ is valid}$$

Finally, there are two notions of logical equivalence, one pertaining to individual formulas, the other pertaining to sets of formulas. Note carefully that ‘ \equiv ’ is a meta-linguistic two-place predicate, that should not be confused with the symbol employed by some logic authors (but currently no one outside philosophy!) as the object language connective for ‘if and only if’. We will never use ‘ \equiv ’ in this unfortunate manner. Similarly, ‘ \supset ’ is a meta-linguistic two-place predicate, borrowed from set theory, and used to mean ‘properly includes’. Again, ‘ \supset ’ should not be confused with the symbol employed by some logic authors (but currently no one outside philosophy!) as the object language symbol for ‘if...then’. We will never use ‘ \supset ’ in this unfortunate manner.

5. The Mother of All Logical Notions

As used in the previous section, then symbol ‘ \models ’ is used ambiguously; specifically, it represents two different two-place predicates, and four different one-place predicates. It was suggested that this is a harmless, indeed useful, ambiguity. In the present section, we examine a general abstract concept of ‘entails’ [symbolized by ‘ \models ’] from which all the usual logical notions can be derived. The following is the official definition, understood relative to V .

$$\Gamma \models \Delta \quad =_{df} \quad \sim \exists v \in V \{v \langle \Gamma \ \& \ v \perp \Delta \}$$

In other words, set Γ entails set Δ if and only if no admissible valuation verifies Γ and falsifies Δ .

As we see in a later chapter, the symmetry of this notion of entailment plays a crucial role in the general completeness theorem for abstract logics. At the moment, however, we can use this symmetrical entailment to define all previous logical notions. This is the content of the following theorems, whose proofs are left as exercises.

- (1) $\models \alpha \quad \leftrightarrow \quad \emptyset \models \{\alpha\}$
- (2) $\alpha \models \quad \leftrightarrow \quad \{\alpha\} \models \emptyset$
- (3) $\Gamma \models \quad \leftrightarrow \quad \Gamma \models \emptyset$
- (4) $\models \Gamma \quad \leftrightarrow \quad \emptyset \models \Gamma$
- (5) $\alpha \models \beta \quad \leftrightarrow \quad \{\alpha\} \models \{\beta\}$
- (6) $\Gamma \models \alpha \quad \leftrightarrow \quad \Gamma \models \{\alpha\}$

Note: in the above theorems, on the left-hand side, the turnstile is used ambiguously, but on the right-hand side, the turnstile refers exclusively to the symmetric entailment relation. The convention seems clear enough: a single formula can stand in place of its singleton, and the null expression can stand in place of the empty set.

6. Atomic Assignments

Df

Let L be a formal language, and let A be the set of atomic formulas of L . Then

- (1) an *atomic assignment* on L is any function from A into $\{T,F\}$.
- (2) a *finite atomic assignment* on L is any function from A' into $\{T,F\}$, where A' is a finite subset of A

Notice that every valuation on L gives rise to an associated atomic assignment, defined in the obvious manner.

- (T) Let ν be a valuation on L . Then there is a unique atomic assignment α_ν on L , satisfying the following.

$$\text{for any atomic formula } \alpha, \alpha_\nu(\alpha) = \nu(\alpha)$$

In other words, the atomic assignment function α_ν is simply the restriction of the valuation function ν to the atomic formulas of L . Notice that α_ν and ν are not identical, since their respective domains are different – assuming of course that L has at least one non-atomic formula.

Every valuation gives rise to an atomic assignment. Does every atomic assignment give rise to a valuation? In this connection, we propose the following definitions.

Df

Let L be a language with formulas S and atomic formulas A . Let V be a class of valuations on L . Let α be an atomic assignment [resp., finite atomic assignment] on L . Then α is V -consistent if and only if α can be extended to a valuation in V . In other words:

$$\alpha \text{ is } V\text{-consistent} \text{ =}_{df} \exists \nu \in V \forall \alpha \in \text{dom}(\alpha) [\alpha(\alpha) = \nu(\alpha)]$$

Here, $\text{dom}(\alpha)$ is the domain of function α .

V is said to be *atomically-free* if and only if every finite atomic assignment is V -consistent.

In other words, to say that V is atomically-free is to say that every finite atomic assignment gives rise to at least one valuation in V . It is, of course, a fundamental assumption of elementary SL that every finite atomic assignment gives rise to at least one valuation – no assignment of truth-values to finitely-many atomic formulas is logically prohibited. This is related to a central feature of logics – formality – which we discuss in greater detail later. For the moment, however, suppose logic \mathcal{L} is specified by (L,V) . Suppose there is a finite atomic assignment α on L that is not V -consistent. There are a number of ways this could happen, but let us simply suppose $\alpha(P)=T$ and $\alpha(Q)=F$. Since α cannot be extended to a valuation in V , it follows that the argument form (P/Q) is \mathcal{L} -valid. This is a highly undesirable result,

because it entails that we have an \mathcal{L} -valid argument form that has no internal structure in virtue of which it is valid.

So far, we have discussed whether every atomic assignment can be extended to *at least one* valuation. Another interesting question is whether every atomic assignment can be extended to *at most one* valuation. In this connection, we propose the following definition.

Df

Let L be a language with formulas S and atomic formulas A . Let V be a class of valuations on L . Then V is said to be *atomically determined* if and only if:

$$\forall v_1 v_2 \in V \{ \forall \alpha \in A [v_1(\alpha) = v_2(\alpha)] \rightarrow v_1 = v_2 \}$$

Observe that, given that v_1 and v_2 are valuations on L , the sentence ' $v_1 = v_2$ ' amounts to the following.

$$\forall \alpha \in S [v_1(\alpha) = v_2(\alpha)]$$

So, if a semantics is atomically determined, then if two valuations agree on all atomic formulas, then they agree on all formulas, and are therefore identical.

Combining atomic-freedom and atomic-determination yields the following notion.

Df

Let L be a language with formulas S and atomic formulas A . Let V be a class of valuations on L . Then V is said to be *atomistic* if and only if V is both atomically-free and atomically determined.

Given the definition of 'atomistic', we have the following immediate theorem.

- (T) Let L be a language and let V be a class of valuations on L . Suppose V is atomically determined. Then every atomic assignment on L gives rise to a unique valuation on L .

The following is not automatic, but it is the sort of result we should expect.

- (T) Let L be a ZOL. Suppose V is truth-functional. Then V is atomistic.

A remaining question, in this regard, is whether truth-functionality and atomicity are equivalent. They are not.

- (T) Not every atomistic semantics is truth-functional.

In order to see this, we offer the following example.

- (Ex) Let L be a prefix-formatted ZOL specified by the usual CSL-connectives, plus a one-place connective \odot . Define V so that $v \in V$ iff v satisfies the usual truth-functional conditions, plus the following condition pertaining to the extra connective.

$$\begin{array}{ll} v(\odot\alpha) = T & \text{if } w(\alpha)=T \text{ for every } w \in V \\ = F & \text{otherwise} \end{array}$$

To see that this is not truth-functional, we simply note that $v(\odot\alpha)=F$ for every atomic formula; no atomic formula is an SL-tautology. There is, of course, a valuation v such $v(p)=T$ and $v(q)=F$. This means that, insofar as \odot is truth-functional, its truth-function is given by: $[\odot T=F; \odot F=F]$. So, if \odot is truth-functional, then $v(\odot\alpha)=\odot v(\alpha)=F$, for every formula α . On the other hand, $v(\odot DpNp) = T$ [notice that $DpNp$ ($\approx P \vee \sim P$) is an SL-tautology].

On the other hand, this semantics is atomistic. This is because all valuations are truth-functional with respect to the standard CSL-connectives, and all valuations agree on all formulas whose main connective is \odot . This is the basic idea; a formal proof requires induction.

7. Locality

As we have seen, an atomic assignment is the abstract counterpart of a line in the guide table for a truth-table. Of course, in intro logic, we never assign truth-values to all atomic formulas [why bother?]. Rather, in doing truth-tables, we only assign truth-values to those atomic formulas that are involved in the specific argument under consideration. If we insist on evaluating all atomic formulas, we can simply say that every atomic formula not explicitly mentioned is assigned T (or F , as we wish; it doesn't matter).

This simplification is based on the very plausible assumption that what truth-values we assign to all other atomic formulas is completely irrelevant to the computation of the truth-values of the formulas involved. This is certainly a feature of intro logic, and indeed of all truth-functional logics, but it is not built into the concept of valuation. Noting this assumption, we propose the following definition.

Df

Let L be a language with formulas S and atomic formulas A . Let V be a class of valuations on L . Then V is said to be *local* if and only if:

$$\begin{array}{l} \forall v_1 v_2 \in V \forall \alpha \in S [\forall \beta \{ \beta \in A \ \& \ \beta \preceq \alpha \} \\ \rightarrow v_1(\alpha) = v_2(\alpha)] \end{array}$$

This says that, if two valuations agree on all the atomic subformulas of a formula, then they agree on the formula. Here, we use the symbol ' \preceq ' as the subformula predicate. It is defined inductively as follows, for general ZOL's written in prefix notation.

$\alpha \preceq \alpha$
 $\alpha \preceq \beta \ \& \ \beta \preceq \gamma \rightarrow \alpha \preceq \gamma$
 if χ is a 1-place connective, then $\alpha \preceq \chi\alpha$
 if χ is a 2-place connective, then $\alpha, \beta \preceq \chi\alpha\beta$
 if χ is a 3-place connective, then $\alpha, \beta, \gamma \preceq \chi\alpha\beta\gamma$
 etc.
 that's all!

If we define formulas to be strings of characters, as is customary, then we might define subformula simply to be a sub-string which is a formula. The sub-string predicate is defined using the string concatenation operation as follows.

$$\sigma_1 \lesssim \sigma_2 =_{\text{df}} \exists \sigma_3 \sigma_4 [\sigma_2 = \sigma_3 + \sigma_1 + \sigma_4]$$

[Note: this definition assumes that our theory of strings includes a null-string \emptyset ; then every string is a sub-string of itself.]

[[Defining sub-formula to be a special kind of sub-string works for ZOL's. However, for more complicated languages, including quantified languages, the notion of sub-formula can be construed in a more subtle manner. For example, one can construe subformulas so that ' $\forall xFx$ ' has infinitely many immediate subformulas – 'Fa', 'Fb', 'Fc', etc.]]

Let us consider an extremely simple example. Consider a formal language L with just two formulas, 'p' and ' $\odot p$ '; so the language L is both minuscule and odd. Next, define V so that it contains exactly three valuations, depicted in the following table.

| | p | $\odot p$ |
|-------|---|-----------|
| v_1 | T | T |
| v_2 | T | F |
| v_3 | F | F |

Notice first that this semantics is not atomistic – v_1 and v_2 agree on the sole atomic formula 'p', but do not agree on all formulas; in particular, they do not agree on the "molecular" formula ' $\odot p$ '. Notice also that this semantics is not local either – v_1 and v_2 agree on all the atomic sub-formulas of ' $\odot p$ ', but do not agree on ' $\odot p$ '.

The previous language is not a ZOL; it's way too small! The example can be corrected as follows. Consider a ZOL L with just one connective, \odot , a one-place connective. Next, define V so that it contains exactly three valuations, depicted in the following schematic table.

| | p^* | $\odot p^*$ | $\odot\odot p^*$ | $\odot\odot\odot p^*$ | ... |
|-------|-------|-------------|------------------|-----------------------|-----|
| v_1 | T | T | F | F | ... |
| v_2 | T | F | F | F | ... |
| v_3 | F | F | F | F | ... |

Here, ' p^* ' stands for 'p' followed by some number of sharps; in other words, p^* is an arbitrary atomic formula of L. Notice that this semantics is neither atomistic nor local.

One might naturally wonder whether locality and atomicity are equivalent. The answer is conveyed in the following theorem.

(T) Every atomistic semantics is local, but not every local semantics is atomistic.

Proof???

8. Valuation-Completeness and Super-Valuations

We now consider an interesting question. Once logic \mathcal{L} is semantically specified by a set V of valuations, is there any room for extra valuations? In order to make this idea more precise, we propose the following definitions.

- D1. Let \mathcal{L} be a logic semantically specified by (L, V) . Let v be any valuation on L . Then v is said to be \mathcal{L} -consistent [also, V -consistent] if and only if v does not refute any \mathcal{L} -valid [V -valid] argument.
- D2. Let \mathcal{L} be a logic semantically characterized by (L, V) . Then V is said to be complete if and only if V contains every \mathcal{L} -consistent [V -consistent] valuation.

Here is how these ideas work. We start with a logic \mathcal{L} characterized by a set V of \mathcal{L} -admissible valuations; in particular, V generates the set of \mathcal{L} -valid [V -valid] arguments. This set in turn generates a set of \mathcal{L} -consistent [V -consistent] valuations. Obviously, every valuation in V is V -consistent (exercise). The question is whether every V -consistent valuation is in V . If it is, then V is complete; otherwise, it is incomplete.

1. Example 1

First, we consider a "degenerate" example. Define valuation v_0 as follows.

$$v_0(\alpha) = T, \text{ for every sentence } \alpha \text{ in } S$$

This is a kooky valuation, to be sure, but notice that v_0 is \mathcal{L} -consistent for any logic \mathcal{L} . This is because v_0 refutes no argument whatsoever, because v_0 does not falsify any formula.

2. Example 2

Next, we consider a non-degenerate example. In particular, consider classical sentential logic (CSL), and the valuation v_1 defined as follows.

$$v_1(\alpha) = T \text{ if every CSL-valuation verifies } \alpha; v_1(\alpha) = F, \text{ otherwise.}$$

First, v_1 is not CSL-admissible, because $v_1(P) = v_1(\sim P) = F$, and no CSL-admissible valuation has this property. Although v_1 is not an official CSL valuation, it is nevertheless *CSL-consistent*. To see this, we argue as follows.

Suppose v_1 is not CSL-consistent. Then v_1 refutes at least one CSL-valid argument, call it (Γ/α) . Then v_1 verifies every element of Γ , but falsifies α . Let γ be an element of Γ . Then $v_1(\gamma) = T$, so by the definition of v_1 , every CSL-valuation verifies α ; in other words, α is CSL-valid – it is a logical truth, a tautology. Thus, every element of Γ is CSL-valid. At this point, we can prove the following general lemma (exercise).

$$L1 \quad \Gamma \models \alpha \ \& \ \forall \gamma (\gamma \in \Gamma \rightarrow \models \gamma) \ . \rightarrow \models \alpha$$

In other words, if an argument is valid, and every premise is a logical truth, then its conclusion is also a logical truth. Applying this general result to CSL, we conclude that α is CSL-valid, so every CSL-valuation verifies α , so $v_1(\alpha) = T$. This contradicts our earlier assumption that v_1 falsifies α .

What we have shown is that the standard set of CSL-admissible valuations is not complete. There are at least two valuations (counting the degenerate one) that can be added without changing the logic, at least so far as argument validity is concerned.

3. Super-Valuations

Are there other valuations that are CSL-consistent but not CSL-admissible? As it turns out, there are infinitely-many such valuations. This is in fact a special case of a more general theorem.

In this connection, we offer the following definition.

D3. Let \mathcal{L} be a logic characterized by (L, V) . Let W be any subset of V . Define v_w as follows.

$$v_w(\alpha) = \begin{cases} T & \text{if } \forall w \{ w \in W \rightarrow w(\alpha) = T \} \\ F & \text{otherwise} \end{cases}$$

Terminology: following van Fraassen, we call v_w the *super-valuation* associated with, or determined by, W .

D4. Let V be a set of valuations on language L , and let v be a valuation on L . Then v is said to be a super-valuation over V if and only if there is a $W \subseteq V$ such that $v = v_w$.

Example 1:

if $W = \emptyset$, then obtain Example 1 of the previous subsection.

Example 2:

if $W = V$, then we obtain Example 2 of the previous subsection.

Example 3:

if $W = \{v\}$, then $v_w = v$.

This means that, technically speaking, every ordinary valuation is also a supervaluation.

4. General Theorems

The following theorems sum up the important facts about supervaluations and completeness.

T1. Let \mathcal{L} be a logic characterized by V . Let W be any subset of V , and let v_w be the super-valuation determined by W . Then v_w is \mathcal{L} -consistent.

In other words, every super-valuation over V is V -consistent.

T2. Let \mathcal{L} be a logic characterized by V . Let v be a valuation that is \mathcal{L} -consistent. Then there is a subset W of V , and associated super-valuation v_w , such that $v = v_w$.

In other words, every V -consistent valuation is a super-valuation over V .

T3. A logic (L, V) is complete if and only if every super-valuation over V is an element of V (if other words, V contains all its super-valuations).

9. Elementary Classes and Valuation Spaces

Associated with every formula ϕ , there is a subset of valuations that verify ϕ . This prompts the following official definition.

$$(d) \quad \begin{aligned} V(\phi) &=_{df} \{v: v \ll \phi\} \\ &=_{df} \{v: v(\phi)=T\} \end{aligned}$$

$V(\phi)$ is called the *elementary class* associated with ϕ ; evidently, $V(\phi)$ is the set that consists precisely of those valuations in V that verify ϕ .

This concept can be extended to sets in the obvious manner, as follows.

$$(d) \quad V(\Gamma) =_{df} \{v: v \ll \Gamma\}$$

Once we have the basic notion, we can define the general notion of elementary class as follows.

- (D) A subset W of valuations is said to be an *elementary class* iff there exists a formula ϕ such that $W = V(\phi)$.
- (D) The *valuation space* associated with language L and semantics V is, by definition, the ordered pair (V, E) , where E is the set of all elementary classes; in other words,

$$E = \{V(\phi): \phi \in S\}.$$

Notice that, although every elementary class is a subset of V , not every subset of V is an elementary class. For example, if V is infinite, then $\mathcal{P}(V)$ is uncountable. On the other hand, the number of elementary classes is no larger than the number of formulas, which is denumerable.

By way of concluding this section, we note that the notions of elementary class and valuation space may be used to formulate the various formal semantic notions. We state the basic theorems, leaving the proofs as exercises. We begin with a theorem describing the relation between $V(\phi)$ and $V(\Gamma)$.

- (1) $V(\Gamma) = \bigcap \{V(\gamma): \gamma \in \Gamma\}$
- (2) $\models \alpha \leftrightarrow V(\alpha) = V$
- (3) $\alpha \models \leftrightarrow V(\alpha) = \emptyset$
- (4) $\models \Gamma \leftrightarrow \bigcup \{V(\gamma): \gamma \in \Gamma\} = V$
- (5) $\Gamma \models \leftrightarrow V(\Gamma) = \emptyset$
- (6) $\alpha \models \beta \leftrightarrow V(\alpha) \subseteq V(\beta)$
- (7) $\Gamma \models \alpha \leftrightarrow V(\Gamma) \subseteq V(\alpha)$
- (8) $\Gamma \models \Delta \leftrightarrow V(\Gamma) \subseteq \bigcap \{V(\delta): \delta \in \Delta\}$
- (9) $\alpha \equiv \beta \leftrightarrow V(\alpha) = V(\beta)$
- (10) $\Gamma \equiv \Delta \leftrightarrow V(\Gamma) = V(\Delta)$

In this regard, the following are relevant set-theoretic facts about "big" intersection and union.

- (1) $a \in \bigcap \{V(\gamma): \gamma \in \Gamma\} \leftrightarrow \forall \gamma \{ \gamma \in \Gamma \rightarrow a \in V(\gamma) \}$
- (2) $a \in \bigcup \{V(\gamma): \gamma \in \Gamma\} \leftrightarrow \exists \gamma \{ \gamma \in \Gamma \ \& \ a \in V(\gamma) \}$

10. Exercises

1. Definitions

Presupposing a formal language L and a set V of admissible valuations on L , define the following in primitive notation.

Note 1: For these purposes, set-theoretic notation, including function application notation, is considered part of the primitive vocabulary.

Note 2: There is nothing special about the exact form of the definition. The *definiens* can be any primitive formula that is logically equivalent to the "official" definition. If there is a particular formulation that seems memorable to you, then use it.

- | | | |
|----|--------------------------------------|-------------------------|
| 1. | α is valid | $\models \alpha$ |
| 2. | α is contra-valid | $\alpha \not\models$ |
| 3. | Γ is unfalsifiable | $\models \Gamma$ |
| 4. | Γ is unverifiable | $\Gamma \not\models$ |
| 5. | α entails β | $\alpha \models \beta$ |
| 6. | Γ entails α | $\Gamma \models \alpha$ |
| 7. | Γ entails Δ | $\Gamma \models \Delta$ |
| 8. | α and β are equivalent | $\alpha \equiv \beta$ |
| 9. | Γ and Δ are equivalent | $\Gamma \equiv \Delta$ |

2. General Theorems

In the following a logic \mathcal{L} semantically specified by (L, V) is assumed. It is also assumed that α, β , etc. are formulas of L , and Γ, Δ , etc. are sets of formulas of L .

Note 1: There are three different notions of semantic entailment employed here: binary, ordinary, and symmetric entailment. Similarly, the double turnstile ‘ \equiv ’ is used in seven different ways (see Part 1 above). As usual, the context determines which one is meant.

Note 2: You may use sortal variables and constants – ‘ v ’ (etc.) for valuations in V , ‘ α ’ (etc.) for formulas of L , and ‘ Γ ’ (etc.) for sets of formulas of L .

1. $\Gamma \cap \Delta \neq \emptyset \rightarrow \Gamma \models \Delta$
2. $\alpha \in \Gamma \rightarrow \Gamma \models \alpha$
3. $\Gamma \models \Delta \ \& \ \forall \gamma \{ \gamma \in \Gamma \rightarrow \Sigma \models \gamma \} \rightarrow \Sigma \models \Delta$
4. $\Gamma \models \Delta \ \& \ \Gamma \subseteq \Gamma' \ \& \ \Delta \subseteq \Delta' \rightarrow \Gamma' \models \Delta'$
5. $\Gamma \models \alpha \ \& \ \Gamma \cup \{ \alpha \} \models \beta \rightarrow \Gamma \models \beta$ if Γ entails α , and $\Gamma \cup \{ \alpha \}$ entails β ,
then Γ entails β
6. $\Gamma \models \alpha \leftrightarrow \Gamma \models \{ \alpha \}$ Γ entails α iff Γ entails $\{ \alpha \}$
7. $\alpha \models \beta \leftrightarrow \{ \alpha \} \models \{ \beta \}$ α entails β iff $\{ \alpha \}$ entails $\{ \beta \}$
8. $\models \alpha \leftrightarrow \emptyset \models \alpha$ α is valid iff \emptyset entails α
9. $\alpha \models \leftrightarrow \{ \alpha \} \models \emptyset$ α is contra-valid iff $\{ \alpha \}$ entails \emptyset
10. $\models \Gamma \leftrightarrow \emptyset \models \Gamma$ Γ is unfalsifiable iff \emptyset entails Γ
11. $\Gamma \models \leftrightarrow \Gamma \models \emptyset$ Γ is unverifiable iff Γ entails \emptyset
12. $\alpha \equiv \beta \leftrightarrow \{ \alpha \} \equiv \{ \beta \}$
13. $\alpha \equiv \beta \leftrightarrow \alpha \models \beta \ \& \ \beta \models \alpha$
14. $\Gamma \equiv \Delta \leftrightarrow \forall \alpha \{ \Gamma \models \alpha \leftrightarrow \Delta \models \alpha \}$
15. $\Gamma \equiv \Delta \leftrightarrow \forall \delta \{ \delta \in \Delta \rightarrow \Gamma \models \delta \} \ \& \ \forall \gamma \{ \gamma \in \Gamma \rightarrow \Delta \models \gamma \}$

3. Theorems Pertaining to CSL

In the following, assume that the set of admissible valuations is given by the standard semantics for CSL. Note: in the proposed formalization, the symbols ‘ \sim ’ and ‘ \rightarrow ’ are used in three different ways, as names of object language symbols, as names of the associated truth-functions, and as the metalanguage connectives. The writer and reader of such proofs must be alert to context in reading these symbols.

- | | | |
|-----|--|--|
| 1. | $\Gamma \models \alpha \leftrightarrow \Gamma \cup \{ \sim \alpha \} \models$ | Γ entails α iff $\Gamma \cup \{ \sim \alpha \}$ is unverifiable |
| 2. | $\alpha \models \beta \leftrightarrow \models \alpha \rightarrow \beta$ | α entails β iff $\alpha \rightarrow \beta$ is valid |
| 3. | $\Gamma \cup \{ \alpha \} \models \beta \leftrightarrow \Gamma \models \alpha \rightarrow \beta$ | $\Gamma \cup \{ \alpha \}$ entails β iff Γ entails $\alpha \rightarrow \beta$ |
| 4. | $\Gamma \cup \{ \alpha \} \models \Delta \leftrightarrow \Gamma \models \Delta \cup \{ \sim \alpha \}$ | $\Gamma \cup \{ \alpha \}$ entails Δ iff Γ entails $\Delta \cup \{ \sim \alpha \}$ |
| 5. | $\Gamma \models \Delta \cup \{ \alpha \} \leftrightarrow \Gamma \cup \{ \sim \alpha \} \models \Delta$ | Γ entails $\Delta \cup \{ \alpha \}$ iff $\Gamma \cup \{ \sim \alpha \}$ entails Δ |
| 6. | $\{ \alpha \rightarrow \beta, \alpha \} \models \beta$ | |
| 7. | $\{ \alpha \rightarrow \beta, \sim \beta \} \models \sim \alpha$ | |
| 8. | $\sim \alpha \models \alpha \rightarrow \beta$ | |
| 9. | $\beta \models \alpha \rightarrow \beta$ | |
| 10. | $\{ \alpha, \sim \alpha \} \models$ | |
| 11. | $\models \{ \alpha, \sim \alpha \}$ | |
| 12. | $\exists \alpha \exists \beta \sim [\{ \alpha \rightarrow \beta, \beta \} \models \alpha]$ | |
| 13. | $\exists \alpha \exists \beta \sim [\{ \alpha \rightarrow \beta, \sim \alpha \} \models \sim \beta]$ | |

4. Counterexamples

Give a counterexample to the following:

1. $\Gamma \models \Delta \rightarrow \exists \delta \{ \delta \in \Delta \ \& \ \Gamma \models \delta \}$
2. $\Gamma \equiv \Delta \leftrightarrow \Gamma \models \Delta \ \& \ \Delta \models \Gamma$

11. Answers to Selected Exercises

1. Definitions in Primitive Notation

1. $\models \alpha$ =_{df} $\forall v \{v \in V \rightarrow v(\alpha) = T\}$
2. $\alpha \models$ =_{df} $\forall v \{v \in V \rightarrow v(\alpha) = F\}$
3. $\models \Gamma$ =_{df} $\forall v \{v \in V \rightarrow \exists \gamma \{ \gamma \in \Gamma \ \& \ v(\gamma) = T \} \}$
4. $\Gamma \models$ =_{df} $\sim \exists v \{v \in V \ \& \ \forall \gamma \{ \gamma \in \Gamma \rightarrow v(\gamma) = T \} \}$
5. $\alpha \models \beta$ =_{df} $\forall v \{v \in V \rightarrow v(\alpha) = T \rightarrow v(\beta) = T\}$
6. $\Gamma \models \alpha$ =_{df} $\forall v \{v \in V \rightarrow \forall \gamma \{ \gamma \in \Gamma \rightarrow v(\gamma) = T \} \rightarrow v(\alpha) = T\}$
7. $\Gamma \models \Delta$ =_{df} $\sim \exists v \{v \in V \ \& \ \forall \gamma \{ \gamma \in \Gamma \rightarrow v(\gamma) = T \} \ \& \ \forall \delta \{ \delta \in \Delta \rightarrow v(\delta) = F \} \}$
8. $\alpha \equiv \beta$ =_{df} $\forall v \{v \in V \rightarrow v(\alpha) = v(\beta) \}$
9. $\Gamma \equiv \Delta$ =_{df} $\forall v [v \in V \rightarrow \forall \delta \{ \delta \in \Delta \rightarrow v(\delta) = T \} \leftrightarrow \forall \gamma \{ \gamma \in \Gamma \rightarrow v(\gamma) = T \}]$

2. General Theorems

Note: In what follows, many variables and constants are used sortally, as follows.

| | | |
|-------------------------|---|-------------------------|
| v, w | : | valuations in V |
| α, β, γ | : | formulas in S |
| Γ, Δ | : | sets of formulas in S |

This means that we are presupposing the following shorthand.

| | | |
|--------------------|---|---|
| $\forall v F$ | : | $\forall v \{v \in V \rightarrow F\}$ |
| $\exists v F$ | : | $\exists v \{v \in V \ \& \ F\}$ |
| $\forall \alpha F$ | : | $\forall \alpha \{ \alpha \in S \rightarrow F \}$ |
| $\exists \alpha F$ | : | $\exists \alpha \{ \alpha \in S \ \& \ F \}$ |
| $\forall \Gamma F$ | : | $\forall \Gamma \{ \Gamma \subseteq S \rightarrow F \}$ |
| $\exists \Gamma F$ | : | $\exists \Gamma \{ \Gamma \subseteq S \ \& \ F \}$ |

See later sections for axioms, definitions, and subordinate lemmas.

#1:

| | | |
|------|---|-------------------------------|
| (1) | SHOW: $\Gamma \cap \Delta \neq \emptyset \rightarrow \Gamma \models \Delta$ | CD |
| (2) | $\Gamma \cap \Delta \neq \emptyset$ | As |
| (3) | SHOW: $\Gamma \models \Delta$ | Def $\Gamma \models \Delta$ |
| (4) | SHOW: $\sim \exists v \{v < \Gamma \ \& \ v \perp \Delta\}$ | ID |
| (5) | $\exists v \{v < \Gamma \ \& \ v \perp \Delta\}$ | As |
| (6) | SHOW: \times | 16,17,SL |
| (7) | $w < \Gamma \ \& \ w \perp \Delta$ | 5, $\exists O$ |
| (8) | $\exists x \{x \in \Gamma \ \& \ x \in \Delta\}$ | 2,ST (short for 'Set Theory') |
| (9) | $a \in \Gamma \ \& \ a \in \Delta$ | 8, $\exists O$ |
| (10) | $\forall x \{x \in \Gamma \rightarrow w < x\}$ | 7,Def $v < \Gamma$ |
| (11) | $\forall x \{x \in \Gamma \rightarrow w \perp x\}$ | 7,Def $v \perp \Gamma$ |
| (12) | $w < a$ | 9a,10,QL |
| (13) | $w \perp a$ | 9b,11,QL |
| (14) | $w(a) = T$ | 12,Def $<$ |
| (15) | $w(a) = F$ | 13,Def \perp |
| (16) | $T = F$ | 14,15,IL |
| (17) | $T \neq F$ | Axiom 0 |

#2:

| | | |
|-----|---|-----------------------------|
| (1) | SHOW: $\alpha \in \Gamma \rightarrow \Gamma \models \alpha$ | CD |
| (2) | $\alpha \in \Gamma$ | As |
| (3) | SHOW: $\Gamma \models \alpha$ | Def $\Gamma \models \alpha$ |
| (4) | SHOW: $\forall v \{v < \Gamma \rightarrow v < \alpha\}$ | UCD |
| (5) | $w < \Gamma$ | As |
| (6) | SHOW: $w < \alpha$ | 2,7,QL |
| (7) | $\forall x \{x \in \Gamma \rightarrow w < x\}$ | 5,Def $v < \Gamma$ |

#3:

| | | |
|------|--|---------------------------------|
| (1) | SHOW: $\Gamma \models \Delta \ \& \ \forall x \{x \in \Gamma \rightarrow \Sigma \models x\} \rightarrow \Sigma \models \Delta$ | CD |
| (2) | $\Gamma \models \Delta \ \& \ \forall x \{x \in \Gamma \rightarrow \Sigma \models x\}$ | As |
| (3) | SHOW: $\Sigma \models \Delta$ | Def $\Gamma \models \Delta$ |
| (4) | SHOW: $\sim \exists v \{v < \Sigma \ \& \ v \perp \Delta\}$ | ID |
| (5) | $\exists v \{v < \Sigma \ \& \ v \perp \Delta\}$ | As |
| (6) | SHOW: \times | 12b,15,SL |
| (7) | $w < \Sigma \ \& \ w \perp \Delta$ | 5, $\exists O$ |
| (8) | $\sim \exists v \{v < \Gamma \ \& \ v \perp \Delta\}$ | 2a,Def $\Gamma \models \Delta$ |
| (9) | $\sim [w < \Gamma]$ | 7b,8,QL |
| (10) | $\sim \forall x \{x \in \Gamma \rightarrow w < x\}$ | 9,Def $v < \Gamma$ ($-$) |
| (11) | $\sim (\gamma \in \Gamma \rightarrow w < \gamma)$ | 10, $\sim \forall O$ |
| (12) | $\gamma \in \Gamma \ \& \ \sim [w < \gamma]$ | 11,SL |
| (13) | $\Sigma \models \gamma$ | 2b,12a,QL |
| (14) | $\forall v \{v < \Sigma \rightarrow v < \gamma\}$ | 13, Def $\Gamma \models \alpha$ |
| (15) | $w < \gamma$ | 7a,14,QL |

#4:

| | | |
|------|---|--------------------------------|
| (1) | SHOW: $\Gamma \models \Delta \ \& \ \Gamma \subseteq \Gamma' \ \& \ \Delta \subseteq \Delta' \ . \rightarrow \Gamma' \models \Delta'$ | CD |
| (2) | $\Gamma \models \Delta \ \& \ \Gamma \subseteq \Gamma' \ \& \ \Delta \subseteq \Delta'$ | As |
| (3) | SHOW: $\Gamma' \models \Delta'$ | Def $\Gamma \models \Delta$ |
| (4) | SHOW: $\sim \exists v \{v \langle \Gamma' \ \& \ v \perp \Delta' \}$ | ID |
| (5) | $\exists v \{v \langle \Gamma' \ \& \ v \perp \Delta' \}$ | As |
| (6) | SHOW: \times | 11,15,16,QL |
| (7) | $w \langle \Gamma' \ \& \ w \perp \Delta'$ | 5, $\exists O$ |
| (8) | $\forall x \{x \in \Gamma' \rightarrow w \langle x \}$ | 7a,Def $v \langle \Gamma$ |
| (9) | $\forall x \{x \in \Gamma \rightarrow x \in \Gamma' \}$ | 2b,Def \subseteq |
| (10) | $\forall x \{x \in \Gamma \rightarrow w \langle x \}$ | 8,9,QL |
| (11) | $w \langle \Gamma$ | 10,Def $v \langle \Gamma$ |
| (12) | $\forall x \{x \in \Delta' \rightarrow w \perp x \}$ | 7a,Def $v \perp \Gamma$ |
| (13) | $\forall x \{x \in \Delta \rightarrow x \in \Delta' \}$ | 2c,Def \subseteq |
| (14) | $\forall x \{x \in \Delta \rightarrow w \perp x \}$ | 12,13,QL |
| (15) | $w \perp \Delta$ | 14,Def $v \perp \Gamma$ |
| (16) | $\sim \exists v \{v \langle \Gamma \ \& \ v \perp \Delta \}$ | 2, Def $\Gamma \models \Delta$ |

#5:

| | | |
|------|--|---------------------------------|
| (1) | SHOW: $\Gamma \models \alpha \ \& \ \Gamma \cup \{ \alpha \} \models \beta \ . \rightarrow \Gamma \models \beta$ | CD |
| (2) | $\Gamma \models \alpha \ \& \ \Gamma \cup \{ \alpha \} \models \beta$ | As |
| (3) | SHOW: $\Gamma \models \beta$ | Def $\Gamma \models \alpha$ |
| (4) | SHOW: $\forall v (v \langle \Gamma \rightarrow v \langle \beta)$ | UCD |
| (5) | $w \langle \Gamma$ | As |
| (6) | SHOW: $w \langle \beta$ | DD |
| (7) | $\forall v (v \langle \Gamma \rightarrow v \langle \alpha)$ | 2a, Def $\Gamma \models \alpha$ |
| (8) | $\forall v (v \langle \Gamma \cup \{ \alpha \} \rightarrow v \langle \beta)$ | 2b, Def $\Gamma \models \alpha$ |
| (9) | $w \langle \alpha$ | 5,7,QL |
| (10) | $w \langle \Gamma \cup \{ \alpha \}$ | 5,9, Lemma 2 |
| (11) | $w \langle \beta$ | 8,10,QL |

#6:

| | | |
|------|---|-------------------------------|
| (1) | SHOW: $\Gamma \models \alpha \leftrightarrow \Gamma \models \{ \alpha \}$ | $\leftrightarrow D$ |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\Gamma \models \alpha$ | As |
| (4) | SHOW: $\Gamma \models \{ \alpha \}$ | Def $\Gamma \models \Delta$ |
| (5) | SHOW: $\sim \exists v (v \langle \Gamma \ \& \ v \perp \{ \alpha \})$ | ID |
| (6) | $\exists v (v \langle \Gamma \ \& \ v \perp \{ \alpha \})$ | As |
| (7) | SHOW: \times | 16,17,SL |
| (8) | $w \langle \Gamma \ \& \ w \perp \{ \alpha \}$ | 6, $\exists O$ |
| (9) | $\forall v \{v \langle \Gamma \rightarrow v \langle \alpha \}$ | 3,Def $\Gamma \models \alpha$ |
| (10) | $w \langle \alpha$ | 8a,9,QL |
| (11) | $\forall x \{x \in \{ \alpha \} \rightarrow w \perp x \}$ | 8b,Def $v \perp \Gamma$ |
| (12) | $\alpha \in \{ \alpha \}$ | ST |
| (13) | $w \perp \alpha$ | 11,12,QL |
| (14) | $w(\alpha) = T$ | 10,Def \langle |
| (15) | $w(\alpha) = F$ | 13,Def \perp |
| (16) | $T = F$ | 14,15,IL |
| (17) | $T \neq F$ | Axiom 0 |

#6a:

| | | |
|------|--|-----------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\Gamma = \{\alpha\}$ | As |
| (3) | SHOW: $\Gamma = \alpha$ | Def $\Gamma = \alpha$ |
| (4) | SHOW: $\forall v \{v < \Gamma \rightarrow v < \alpha\}$ | UCD |
| (5) | $w < \Gamma$ | As |
| (6) | SHOW: $w < \alpha$ | 16, Def $<$ |
| (7) | $\sim \exists v (v < \Gamma \ \& \ v \perp \{\alpha\})$ | 2, Def $\Gamma = \Delta$ |
| (8) | $\forall v (v < \Gamma \rightarrow \sim [v \perp \{\alpha\}])$ | 7, QL |
| (9) | $\sim [w \perp \{\alpha\}]$ | 5, 8, QL |
| (10) | $\sim \forall x \{x \in \{\alpha\} \rightarrow w \perp x\}$ | 9, Def $v \perp \Gamma (-)$ |
| (11) | $\exists x \{x \in \{\alpha\} \ \& \ \sim [w \perp x]\}$ | 10, QL |
| (12) | $a \in \{\alpha\} \ \& \ \sim [w \perp a]$ | 11, $\exists O$ |
| (13) | $a = \alpha$ | 12a, ST |
| (14) | $\sim [w \perp \alpha]$ | 12b, 13, IL |
| (15) | $w(\alpha) \neq F$ | 14, Def $\perp (-)$ |
| (16) | $w(\alpha) = T$ | 15, Lemma 0 |

#7:

| | | |
|------|--|--------------------------|
| (1) | SHOW: $\alpha = \beta \leftrightarrow \{\alpha\} = \{\beta\}$ | $\leftrightarrow D$ |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\alpha = \beta$ | As |
| (4) | SHOW: $\{\alpha\} = \{\beta\}$ | Def $\Gamma = \Delta$ |
| (5) | SHOW: $\sim \exists v (v < \{\alpha\} \ \& \ v \perp \{\beta\})$ | ID |
| (6) | $\exists v (v < \{\alpha\} \ \& \ v \perp \{\beta\})$ | As |
| (7) | SHOW: \times | 19, 10, SL |
| (8) | $w < \{\alpha\} \ \& \ w \perp \{\beta\}$ | 6, $\exists O$ |
| (9) | $\forall x \{x \in \{\alpha\} \rightarrow w < x\}$ | 8a, Def $v < \Gamma$ |
| (10) | $\alpha \in \{\alpha\}$ | ST |
| (11) | $w < \alpha$ | 9, 10, QL |
| (12) | $\forall x \{x \in \{\beta\} \rightarrow w \perp x\}$ | 8a, Def $v \perp \Gamma$ |
| (13) | $\beta \in \{\beta\}$ | ST |
| (14) | $w \perp \beta$ | 12, 13, QL |
| (15) | $w(\beta) = F$ | 14, Def $v \perp \alpha$ |
| (16) | $\forall v \{v < \alpha \rightarrow v < \beta\}$ | 3, Def $\alpha = \beta$ |
| (17) | $w < \beta$ | 11, 16, QL |
| (18) | $w(\beta) = T$ | 17, Def $v < \alpha$ |
| (19) | $T = F$ | 15, 18, IL |
| (20) | $T \neq F$ | Axiom 0 |

#7b:

| | | |
|------|---|------------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\{\alpha\} = \{\beta\}$ | As |
| (3) | SHOW: $\alpha = \beta$ | Def $\alpha = \beta$ |
| (4) | SHOW: $\forall v \{v < \alpha \rightarrow v < \beta\}$ | UCD |
| (5) | $w < \alpha$ | As |
| (6) | SHOW: $w < \beta$ | |
| (7) | $\sim \exists v (v < \{\alpha\} \ \& \ v \perp \{\beta\})$ | 2, Def $\Gamma = \Delta$ |
| (8) | $\forall v (v < \{\alpha\} \rightarrow \sim [v \perp \{\beta\}])$ | 7, QL |
| (9) | SHOW: $w < \{\alpha\}$ | Def |
| (10) | SHOW: $\forall x \{x \in \{\alpha\} \rightarrow w < x\}$ | UCD |
| (11) | $e \in \{\alpha\}$ | As |
| (12) | SHOW: $w < e$ | 5, 13, IL |
| (13) | $e = \alpha$ | 11, ST |
| (14) | $\sim [w \perp \{\beta\}]$ | 8, 9, QL |
| (15) | $\sim \forall x \{x \in \{\beta\} \rightarrow w \perp x\}$ | 14, Def $w \perp \Gamma (-)$ |
| (16) | $\exists x \{x \in \{\beta\} \ \& \ \sim [w \perp x]\}$ | 15, QL |
| (17) | $e \in \{\beta\} \ \& \ \sim [w \perp e]$ | 16, $\exists O$ |
| (18) | $e = \beta$ | 17a, ST |
| (19) | $\sim [w \perp \beta]$ | 17b, 18, IL |
| (20) | $\sim [w(\beta) = F]$ | 19, Def $v \perp \alpha (-)$ |
| (21) | $w(\beta) = T$ | 20, Lemma 0 |

#8:

| | | |
|------|---|---------------------------|
| (1) | SHOW: $\models \alpha \leftrightarrow \emptyset = \alpha$ | $\leftrightarrow D$ |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\models \alpha$ | As |
| (4) | SHOW: $\emptyset = \alpha$ | Def $\Gamma = \alpha$ |
| (5) | SHOW: $\forall v (v < \emptyset \rightarrow v < \alpha)$ | UCD |
| (6) | $w < \emptyset$ | As |
| (7) | SHOW: $w < \alpha$ | DD |
| (8) | $\forall v [v < \alpha]$ | 3, Def $\models \alpha$ |
| (9) | $w < \alpha$ | 8, QL |
| (10) | SHOW: \leftarrow | CD |
| (11) | $\emptyset = \alpha$ | As |
| (12) | SHOW: $\models \alpha$ | Def $\models \alpha$ |
| (13) | SHOW: $\forall v [v < \alpha]$ | UD |
| (14) | SHOW: $w < \alpha$ | 15, 16, QL |
| (15) | $\forall v (v < \emptyset \rightarrow v < \alpha)$ | 11, Def $\Gamma = \alpha$ |
| (16) | $w < \emptyset$ | Lemma 5 |

#9:

| | | |
|------|--|----------------------------------|
| (1) | SHOW: $\alpha \models \leftrightarrow \{\alpha\} \models \emptyset$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\alpha \models$ | As |
| (4) | SHOW: $\{\alpha\} \models \emptyset$ | Def $\Gamma \models \Delta$ |
| (5) | SHOW: $\forall v (v < \{\alpha\} \rightarrow \exists \delta \{\delta \in \emptyset \ \& \ v < \delta\})$ | UCD |
| (6) | $w < \{\alpha\}$ | As |
| (7) | SHOW: $\exists \delta \{\delta \in \emptyset \ \& \ w < \delta\}$ | 15,16,SL |
| (8) | $\forall v [v \perp \alpha]$ | 3, Def $\alpha \models$ |
| (9) | $\forall v [v(\alpha) = F]$ | 8, Def \perp |
| (10) | $w(\alpha) = F$ | 9,QL |
| (11) | $\forall x \{x \in \{\alpha\} \rightarrow v < x\}$ | 6, Def $<$ |
| (12) | $\alpha \in \{\alpha\}$ | ST |
| (13) | $w < \alpha$ | 11,12,QL |
| (14) | $w(\alpha) = T$ | 13, Def $<$ |
| (15) | $T = F$ | 10,14,IL |
| (16) | $T \neq F$ | Axiom 0 |
| (17) | SHOW: \leftarrow | CD |
| (18) | $\{\alpha\} \models \emptyset$ | As |
| (19) | SHOW: $\alpha \models$ | Def $\alpha \models$ |
| (20) | SHOW: $\sim \exists v [v < \alpha]$ | $\sim \exists$ D |
| (21) | $w < \alpha$ | As |
| (22) | SHOW: \times | 27a,28,QL |
| (23) | $w < \{\alpha\}$ | 21, Lemma 6 |
| (24) | $\sim \exists v \{v < \{\alpha\} \ \& \ v \perp \emptyset\}$ | 18, Def $\Gamma \vdash \Delta$ |
| (25) | $\sim [w \perp \emptyset]$ | 21,24,QL |
| (26) | $\sim \forall x \{x \in \emptyset \rightarrow w \perp x\}$ | 25, Def $v \perp \Gamma (-)$ |
| (27) | $a \in \emptyset \ \& \ \sim [w \perp a]$ | 26, $\sim \forall \rightarrow$ O |
| (28) | $\sim \exists x [x \in \emptyset]$ | ST |

#10:

| | | |
|------|---|---------------------------------|
| (1) | SHOW: $\models \Gamma \leftrightarrow \emptyset \models \Gamma$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\models \Gamma$ | As |
| (4) | SHOW: $\emptyset \models \Gamma$ | Def $\Gamma \models \Delta$ |
| (5) | SHOW: $\sim \exists v \{v < \emptyset \ \& \ v \perp \Gamma\}$ | $\sim \exists$ D |
| (6) | $w < \emptyset \ \& \ w \perp \Gamma$ | As |
| (7) | SHOW: \times | 6b,8,QL |
| (8) | $\sim \exists v [v \perp \Gamma]$ | 3, Def $\models \Gamma$ |
| (9) | SHOW: \leftarrow | CD |
| (10) | $\emptyset \models \Gamma$ | As |
| (11) | SHOW: $\models \Gamma$ | Def $\models \Gamma$ |
| (12) | SHOW: $\sim \exists v [v \perp \Gamma]$ | $\sim \exists$ D |
| (13) | $w \perp \Gamma$ | As |
| (14) | SHOW: \times | 21b,22,SL |
| (15) | $\forall x (x \in \Gamma \rightarrow w \perp x)$ | 13, Def $v \perp \Gamma$ |
| (16) | $\sim \exists v \{v < \emptyset \ \& \ v \perp \Gamma\}$ | 10, Def $\Gamma \models \Delta$ |
| (17) | $w < \emptyset$ | Lemma 5 |
| (18) | $\sim [w \perp \Gamma]$ | 16,17,QL |
| (19) | $\sim \forall x \{x \in \Gamma \rightarrow w \perp x\}$ | 18, Def $v \perp \Gamma$ (-) |
| (20) | $\exists x \{x \in \Gamma \ \& \ \sim [w \perp x]\}$ | 19,QL |
| (21) | $a \in \Gamma \ \& \ \sim [w \perp a]$ | 26, \exists O |
| (22) | $w \perp a$ | 15,21a,QL |

11:

| | | |
|------|--|---------------------------------|
| (1) | SHOW: $\Gamma \models \leftrightarrow \Gamma \models \emptyset$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\Gamma \models$ | As |
| (4) | SHOW: $\Gamma \models \emptyset$ | Def $\Gamma \models \Delta$ |
| (5) | SHOW: $\forall v \{v < \Gamma \rightarrow \exists x(x \in \emptyset \ \& \ v < x)\}$ | UCD |
| (6) | $w < \Gamma$ | As |
| (7) | SHOW: $\exists x(x \in \emptyset \ \& \ v < x)$ | ID |
| (8) | $\sim \exists v[v < \Gamma]$ | 2, Def $\Gamma \models$ |
| (9) | $\sim [w < \Gamma]$ | 7,QL |
| (10) | x | 5,8 |
| (11) | SHOW: \leftarrow | CD |
| (12) | $\Gamma \models \emptyset$ | As |
| (13) | SHOW: $\Gamma \models$ | Def $\Gamma \models$ |
| (14) | SHOW: $\sim \exists v[v < \Gamma]$ | ID |
| (15) | $\exists v[v < \Gamma]$ | As |
| (16) | SHOW: x | 22,23,SL |
| (17) | $w < \Gamma$ | 15, \exists O |
| (18) | $\sim \exists v \{v < \Gamma \ \& \ v \perp \emptyset\}$ | 12, Def $\Gamma \models \Delta$ |
| (19) | $\sim [w \perp \emptyset]$ | 17,18,QL |
| (20) | $\sim \forall x \{x \in \emptyset \rightarrow w \perp x\}$ | 19, Def $v \perp \Gamma$ |
| (21) | $\exists x(x \in \emptyset \ \& \ \sim [w \perp x])$ | 20,QL |
| (22) | $\exists x[x \in \emptyset]$ | 21,QL |
| (23) | $\sim \exists x[x \in \emptyset]$ | ST |

#12:

| | | |
|-----|---|------------------------------|
| (1) | SHOW: $\alpha \equiv \beta \leftrightarrow \{\alpha\} \equiv \{\beta\}$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\alpha \equiv \beta$ | As |
| (4) | SHOW: $\{\alpha\} \equiv \{\beta\}$ | Def $\Gamma \equiv \Delta$ |
| (5) | SHOW: $\forall v(v < \{\alpha\} \leftrightarrow v < \{\beta\})$ | UD |
| (6) | SHOW: $w < \{\alpha\} \leftrightarrow w < \{\beta\}$ | 8, Lemma 6,SL |
| (7) | $\forall v \{v < \alpha \leftrightarrow v < \beta\}$ | 3, Def $\alpha \equiv \beta$ |
| (8) | $w < \alpha \leftrightarrow w < \beta$ | 7,QL |

#13:

| | | |
|------|--|---------------------------------|
| (1) | SHOW: $\alpha \equiv \beta \leftrightarrow \alpha \models \beta \ \& \ \beta \models \alpha$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\alpha \equiv \beta$ | As |
| (4) | $\forall v \{v < \alpha \leftrightarrow v < \beta\}$ | 3, Def $\alpha \equiv \beta$ |
| (5) | SHOW: $\alpha \models \beta \ \& \ \beta \models \alpha$ | &D |
| (6) | SHOW: $\alpha \models \beta$ | Def $\alpha \models \beta$ |
| (7) | SHOW: $\forall v \{v < \alpha \rightarrow v < \beta\}$ | 4,QL |
| (8) | SHOW: $\beta \models \alpha$ | Def $\alpha \models \beta$ |
| (9) | SHOW: $\forall v \{v < \beta \rightarrow v < \alpha\}$ | 4,QL |
| (10) | SHOW: \leftarrow | CD |
| (11) | $\alpha \models \beta \ \& \ \beta \models \alpha$ | As |
| (12) | $\forall v \{v < \alpha \rightarrow v < \beta\}$ | 11a, Def $\alpha \models \beta$ |
| (13) | $\forall v \{v < \beta \rightarrow v < \alpha\}$ | 11b, Def $\alpha \models \beta$ |
| (14) | SHOW: $\alpha \equiv \beta$ | Def $\alpha \equiv \beta$ |
| (15) | SHOW: $\forall v \{v < \alpha \leftrightarrow v < \beta\}$ | 12,13,QL |

#14:

| | | |
|------|---|--------------------------------|
| (1) | SHOW: $\Gamma \equiv \Delta \leftrightarrow \forall x\{\Gamma \models x \leftrightarrow \Delta \models x\}$ | $\leftrightarrow D$ |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\Gamma \equiv \Delta$ | As |
| (4) | $\forall v\{v < \Gamma \leftrightarrow v < \Delta\}$ | 3, Def $\Gamma \equiv \Delta$ |
| (5) | SHOW: $\forall x\{\Gamma \models x \leftrightarrow \Delta \models x\}$ | UD |
| (6) | SHOW: $\Gamma \models \alpha \leftrightarrow \Delta \models \alpha$ | $\leftrightarrow D$ |
| (7) | SHOW: \rightarrow | CD |
| (8) | $\Gamma \models \alpha$ | As |
| (9) | $\forall v\{v < \Gamma \rightarrow v < \alpha\}$ | 8, Def $\Gamma \models \alpha$ |
| (10) | SHOW: $\Delta \models \alpha$ | Def $\Gamma \models \alpha$ |
| (11) | SHOW: $\forall v\{v < \Delta \rightarrow v < \alpha\}$ | UCD |
| (12) | $w < \Delta$ | As |
| (13) | SHOW: $v < \alpha$ | 9, 14, QL |
| (14) | $w < \Gamma$ | 4, 11, QL |
| (15) | SHOW: \leftarrow | CD |
| (16) | 8-14 <i>mutatis mutandis</i> | |

#14b:

| | | |
|------|--|----------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\forall x\{\Gamma \models x \leftrightarrow \Delta \models x\}$ | As |
| (3) | SHOW: $\Gamma \equiv \Delta$ | Def $\Gamma \equiv \Delta$ |
| (4) | SHOW: $\forall v\{v < \Gamma \leftrightarrow v < \Delta\}$ | UD |
| (5) | SHOW: $w < \Gamma \leftrightarrow w < \Delta$ | $\leftrightarrow D$ |
| (6) | SHOW: \rightarrow | CD |
| (7) | $w < \Gamma$ | As |
| (8) | $\forall x\{x \in \Gamma \rightarrow w < x\}$ | 7, Def $v < \Gamma$ |
| (9) | SHOW: $w < \Delta$ | Def $v < \Gamma$ |
| (10) | SHOW: $\forall x\{x \in \Delta \rightarrow w < x\}$ | UCD |
| (11) | $\delta \in \Delta$ | As |
| (12) | SHOW: $w < \delta$ | 8, 14, QL |
| (13) | $\Delta \models \delta$ | 11, Th#2, QL |
| (14) | $\Gamma \models \delta$ | 2, 13, QL |
| (15) | SHOW: \leftarrow | |
| (16) | 7-14 <i>mutatis mutandis</i> | |

#15:

| | | |
|------|---|-------------------------------|
| (1) | SHOW: $\Gamma \equiv \Delta \leftrightarrow \forall x\{x \in \Delta \rightarrow \Gamma \models x\} \ \& \ \forall x\{x \in \Gamma \rightarrow \Delta \models x\}$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\Gamma \equiv \Delta$ | As |
| (4) | $\forall v\{v < \Gamma \leftrightarrow v < \Delta\}$ | 3, Def $\Gamma \equiv \Delta$ |
| (5) | SHOW: $\forall x\{x \in \Delta \rightarrow \Gamma \models x\} \ \& \ \forall x\{x \in \Gamma \rightarrow \Delta \models x\}$ | &D |
| (6) | SHOW: $\forall x\{x \in \Delta \rightarrow \Gamma \models x\}$ | UCD |
| (7) | $\delta \in \Delta$ | As |
| (8) | SHOW: $\Gamma \models \delta$ | Def $\Gamma \models \alpha$ |
| (9) | SHOW: $\forall v\{v < \Gamma \rightarrow v < \delta\}$ | UCD |
| (10) | $w < \Gamma$ | As |
| (11) | SHOW: $w < \delta$ | 12,13,QL |
| (12) | $w < \Delta$ | 4,10,QL |
| (13) | $\forall x\{x \in \Delta \rightarrow w < x\}$ | 12, Def $v < \Gamma$ |
| (14) | SHOW: $\forall \gamma\{\gamma \in \Gamma \rightarrow \Delta \models \gamma\}$ | CD |
| (15) | 6-13 <i>mutatis mutandis</i> | |

#15b:

| | | |
|------|--|---------------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\forall x\{x \in \Delta \rightarrow \Gamma \models x\} \ \& \ \forall x\{x \in \Gamma \rightarrow \Delta \models x\}$ | As |
| (3) | SHOW: $\Gamma \equiv \Delta$ | Def $\Gamma \equiv \Delta$ |
| (4) | SHOW: $\forall v\{v < \Gamma \leftrightarrow v < \Delta\}$ | UBD |
| (5) | SHOW: \rightarrow | CD |
| (6) | $w < \Gamma$ | As |
| (7) | SHOW: $w < \Delta$ | Def $v < \Gamma$ |
| (8) | SHOW: $\forall x\{x \in \Delta \rightarrow w < x\}$ | UCD |
| (9) | $\delta \in \Delta$ | As |
| (10) | SHOW: $w < \delta$ | 6,12,QL |
| (11) | $\Gamma \models \delta$ | 2a,9,QL |
| (12) | $\forall v\{v < \Gamma \rightarrow v < \delta\}$ | 11, Def $\Gamma \models \alpha$ |
| (13) | SHOW: \leftarrow | CD |
| (14) | 6-12, <i>mutatis mutandis</i> (2b/2a in line 11) | |

3. Theorems Pertaining to CSL

#1:

| | | |
|------|---|--------------------------------|
| (1) | SHOW: $\Gamma \models \alpha \leftrightarrow \Gamma \cup \{\sim \alpha\} \models$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\Gamma \models \alpha$ | As |
| (4) | SHOW: $\Gamma \cup \{\sim \alpha\} \models$ | Def $\Gamma \models$ |
| (5) | SHOW: $\sim \exists v[v < \Gamma \cup \{\sim \alpha\}]$ | ID |
| (6) | $\exists v[v < \Gamma \cup \{\sim \alpha\}]$ | As |
| (7) | SHOW: \times | 19,20,SL |
| (8) | $w < \Gamma \cup \{\sim \alpha\}$ | 6, \exists O |
| (9) | $\forall x(x \in \Gamma \cup \{\sim \alpha\} \rightarrow w < x)$ | 8, Def $<$ |
| (10) | $\forall v(v < \Gamma \rightarrow v < \alpha)$ | 3, Def $\Gamma \models \alpha$ |
| (11) | $w < \Gamma$ | 8, Lemma 3, ST |
| (12) | $w < \alpha$ | 10,11,QL |
| (13) | $w(\alpha) = T$ | 10, Def $<$ |
| (14) | $\sim \alpha \in \Gamma \cup \{\sim \alpha\}$ | ST |
| (15) | $w < \sim \alpha$ | 9,14,QL |
| (16) | $w(\sim \alpha) = T$ | 15, Def $<$ |
| (17) | $w(\sim \alpha) = \sim w(\alpha)$ | Def CSL-val |
| (18) | $T = \sim T$ | 13,16,17,IL |
| (19) | $T = F$ | 18, Def $-$ |
| (20) | $T \neq F$ | Axiom 0 |

#1b:

| | | |
|------|---|-----------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\Gamma \cup \{\sim \alpha\} \models$ | CD |
| (3) | SHOW: $\Gamma \models \alpha$ | Def $\Gamma \models \alpha$ |
| (4) | SHOW: $\forall v(v < \Gamma \rightarrow v < \alpha)$ | UCD |
| (5) | $w < \Gamma$ | As |
| (6) | SHOW: $w < \alpha$ | 11,12-16,17-25,SC |
| (7) | $\sim \exists v[v < \Gamma \cup \{\sim \alpha\}]$ | 2,Def $\Gamma \models$ |
| (8) | $\sim [w < \Gamma \cup \{\sim \alpha\}]$ | 7,QL |
| (9) | $\sim \forall x(x \in \Gamma \cup \{\sim \alpha\} \rightarrow w < x)$ | 8, Def $v < \Gamma (-)$ |
| (10) | $a \in \Gamma \cup \{\sim \alpha\} \ \& \ \sim [w < a]$ | 9, $\sim \forall$ O |
| (11) | $a \in \Gamma \vee a \in \{\sim \alpha\}$ | 10a,Def \cup |
| (12) | case 1: $a \in \Gamma$ | As |
| (13) | $\forall x(x \in \Gamma \rightarrow w < x)$ | 5, Def $<$ |
| (14) | $w < a$ | 12,13,QL |
| (15) | \times | 10b,14,SL |
| (16) | $w < \alpha$ | 15,SL |
| (17) | case 2: $a \in \{\sim \alpha\}$ | As |
| (18) | $a = \sim \alpha$ | 17, ST |
| (19) | $\sim [w < \sim \alpha]$ | 10b,17,IL |
| (20) | $w(\sim \alpha) \neq T$ | Def $< (-)$ |
| (21) | $w(\sim \alpha) = F$ | 20, Lemma 0 |
| (22) | $w(\sim \alpha) = \sim w(\alpha)$ | Def CSL-val |
| (23) | $\sim w(\alpha) = F$ | 21,22,IL |
| (24) | $w(\alpha) = T$ | 23, Def $-$ |
| (25) | $w < \alpha$ | 24, Def $<$ |

#2:

| | | |
|------|---|-------------------------|
| (1) | SHOW: $\alpha = \beta \leftrightarrow \models \alpha \rightarrow \beta$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\alpha = \beta$ | As |
| (4) | SHOW: $\models \alpha \rightarrow \beta$ | Def $\models \alpha$ |
| (5) | SHOW: $\forall v[v < \alpha \rightarrow \beta]$ | UD |
| (6) | SHOW: $w < \alpha \rightarrow \beta$ | Def $<$ |
| (7) | SHOW: $w(\alpha \rightarrow \beta) = T$ | ID |
| (8) | $w(\alpha \rightarrow \beta) \neq T$ | As |
| (9) | $w(\alpha \rightarrow \beta) = F$ | 8, Lemma 0 |
| (10) | $w(\alpha \rightarrow \beta) = w(\alpha) \rightarrow w(\beta)$ | Def CSL-val |
| (11) | $w(\alpha) \rightarrow w(\beta) = F$ | 9,10,IL |
| (12) | $w(\alpha) = T \ \& \ w(\beta) = F$ | 11, Def \rightarrow |
| (13) | $\forall v(v < \alpha \rightarrow v < \beta)$ | 3, Def $\alpha = \beta$ |
| (14) | $w < \alpha$ | 12a, Def $<$ |
| (15) | $w < \beta$ | 13,14,QL |
| (16) | $w(\beta) = T$ | 15, Def $<$ |
| (17) | $T = F$ | 12b,16,IL |
| (18) | $T \neq F$ | Axiom 0 |

#2b:

| | | |
|------|--|--------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\models \alpha \rightarrow \beta$ | As |
| (3) | SHOW: $\alpha = \beta$ | Def $\alpha = \beta$ |
| (4) | SHOW: $\forall v(v < \alpha \rightarrow v < \beta)$ | UCD |
| (5) | $w < \alpha$ | As |
| (6) | SHOW: $w < \beta$ | Def $<$ |
| (7) | SHOW: $w(\beta) = T$ | ID |
| (8) | $w(\beta) \neq T$ | As |
| (9) | SHOW: \times | 18,19,SL |
| (10) | $w(\beta) = F$ | 8, Lemma 0 |
| (11) | $\forall v[v < \alpha \rightarrow \beta]$ | 2, Def $\models \alpha$ |
| (12) | $w < \alpha \rightarrow \beta$ | 11,QL |
| (13) | $w(\alpha \rightarrow \beta) = T$ | 12, Def $<$ |
| (14) | $w(\alpha \rightarrow \beta) = w(\alpha) \rightarrow w(\beta)$ | Def CSL-val |
| (15) | $w(\alpha) \rightarrow w(\beta) = T$ | 13,14,IL |
| (16) | $w(\alpha) = F$ | 10,15, Def \rightarrow |
| (17) | $w(\alpha) = T$ | 5, Def $<$ |
| (18) | $T = F$ | 31,32,IL |
| (19) | $T \neq F$ | Axiom 0 |

#3:

| | | |
|------|--|--------------------------------|
| (1) | SHOW: $\Gamma \cup \{\alpha\} \models \beta \leftrightarrow \Gamma \models \alpha \rightarrow \beta$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | SHOW: $\Gamma \cup \{\alpha\} \models \beta \rightarrow \Gamma \models \alpha \rightarrow \beta$ | CD |
| (4) | $\Gamma \cup \{\alpha\} \models \beta$ | As |
| (5) | SHOW: $\Gamma \models \alpha \rightarrow \beta$ | Def $\Gamma \models \alpha$ |
| (6) | SHOW: $\forall v (v < \Gamma \rightarrow v < \alpha \rightarrow \beta)$ | UCD |
| (7) | $w < \Gamma$ | As |
| (8) | SHOW: $w < \alpha \rightarrow \beta$ | Def $<$ |
| (9) | SHOW: $w(\alpha \rightarrow \beta) = T$ | ID |
| (10) | $w(\alpha \rightarrow \beta) \neq T$ | As |
| (11) | SHOW: \times | 23,24-31,SC |
| (12) | $w(\alpha \rightarrow \beta) = F$ | 10, Lemma 0 |
| (13) | $w(\alpha \rightarrow \beta) = w(\alpha) \rightarrow w(\beta)$ | Def CSL-val |
| (14) | $w(\alpha) \rightarrow w(\beta) = F$ | 12,13,IL |
| (15) | $w(\alpha) = T \ \& \ w(\beta) = F$ | 14, Def \rightarrow |
| (16) | $w < \alpha$ | 15a, Def $<$ |
| (17) | $w(\beta) \neq T$ | 15b, Lemma 1 |
| (18) | $\sim [w < \beta]$ | 17, Def $<$ |
| (19) | $\forall v (v < \Gamma \cup \{\alpha\} \rightarrow v < \beta)$ | 4, Def $\Gamma \models \alpha$ |
| (20) | $\sim [w < \Gamma \cup \{\alpha\}]$ | 18,19,QL |
| (21) | $\sim \forall x (x \in \Gamma \cup \{\alpha\} \rightarrow w < x)$ | 20, Def $< (-)$ |
| (22) | $a \in \Gamma \cup \{\alpha\} \ \& \ \sim [w < a]$ | 21, $\sim \forall$ O |
| (23) | $a \in \Gamma \vee a \in \{\alpha\}$ | 22a,ST |
| (24) | case 1: $a \in \Gamma$ | As |
| (25) | $\forall x (x \in \Gamma \rightarrow w < x)$ | 7, Def $<$ |
| (26) | $w < a$ | 24,25,QL |
| (27) | \times | 22b,26 |
| (28) | case 2: $a \in \{\alpha\}$ | As |
| (29) | $a = \alpha$ | 28, Def $\{\}$ |
| (30) | $\sim [w < \alpha]$ | 22b,29,IL |
| (31) | \times | 16,30 |

#3b:

| | | |
|------|---|--------------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\Gamma \models \alpha \rightarrow \beta$ | As |
| (3) | SHOW: $\Gamma \cup \{\alpha\} \models \beta$ | Def $\Gamma \models \alpha$ |
| (4) | SHOW: $\forall v(v < \Gamma \cup \{\alpha\} \rightarrow v < \beta)$ | UCD |
| (5) | $w < \Gamma \cup \{\alpha\}$ | As |
| (6) | SHOW: $w < \beta$ | DD |
| (7) | $\forall v(v < \Gamma \rightarrow v < \alpha \rightarrow \beta)$ | 2, Def $\Gamma \models \alpha$ |
| (8) | $w < \Gamma$ | 5, Lemma 3, ST |
| (9) | $w < \alpha \rightarrow \beta$ | 7,8, QL |
| (10) | $w(\alpha \rightarrow \beta) = T$ | 9, Def $<$ |
| (11) | $\forall x(x \in \Gamma \cup \{\alpha\} \rightarrow w < x)$ | 5, Def $<$ |
| (12) | $\alpha \in \Gamma \cup \{\alpha\}$ | ST |
| (13) | $w < \alpha$ | 11,12, SL |
| (14) | $w(\alpha) = T$ | 13, Def $<$ |
| (15) | $w(\alpha \rightarrow \beta) = w(\alpha) \rightarrow w(\beta)$ | Def CSL-val |
| (16) | $T = T \rightarrow w(\beta)$ | 10,14,15, IL |
| (17) | $w(\beta) = T$ | 16, Def \rightarrow |
| (18) | $w < \beta$ | 17, Def $<$ |

#4:

| | | |
|------|--|--------------------------------|
| (1) | SHOW: $\Gamma \cup \{\alpha\} \models \Delta \leftrightarrow \Gamma \models \Delta \cup \{\sim \alpha\}$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $\Gamma \cup \{\alpha\} \models \Delta$ | As |
| (4) | SHOW: $\Gamma \models \Delta \cup \{\sim \alpha\}$ | Def $\Gamma \models \Delta$ |
| (5) | SHOW: $\sim \exists v[v < \Gamma \ \& \ v \perp \Delta \cup \{\sim \alpha\}]$ | $\sim \exists$ D |
| (6) | $w < \Gamma \ \& \ w \perp \Delta \cup \{\sim \alpha\}$ | As |
| (7) | SHOW: \times | 18,20, SL |
| (8) | $\forall x\{x \in \Delta \cup \{\sim \alpha\} \rightarrow w \perp x\}$ | 6b, Def $v \perp \Gamma$ |
| (9) | $\sim \alpha \in \Delta \cup \{\sim \alpha\}$ | ST |
| (10) | $w \perp \sim \alpha$ | 8,9, QL |
| (11) | $w(\sim \alpha) = F$ | 10, Def $v \perp \alpha$ |
| (12) | $w(\sim \alpha) = \sim w(\alpha)$ | Def CSL-valuation |
| (13) | $\sim w(\alpha) = F$ | 11,12, IL |
| (14) | $w(\alpha) = T$ | 13, Def \sim |
| (15) | $w < \alpha$ | 14, Def $<$ |
| (16) | $w < \Gamma \cup \{\alpha\}$ | 6a,15, Lemma 2 |
| (17) | $\sim \exists v\{v < \Gamma \cup \{\alpha\} \ \& \ v \perp \Delta\}$ | 3, Def $\Gamma \models \Delta$ |
| (18) | $\sim [w \perp \Delta]$ | 16,17, QL |
| (19) | $\Delta \subseteq \Delta \cup \{\sim \alpha\}$ | ST |
| (20) | $w \perp \Delta$ | 6b,19, Lemma 4 |

#4b:

| | | |
|------|---|---------------------------------|
| (1) | SHOW: \leftarrow | CD |
| (2) | $\Gamma = \Delta \cup \{\sim\alpha\}$ | As |
| (3) | SHOW: $\Gamma \cup \{\alpha\} = \Delta$ | Def $\Gamma = \Delta$ |
| (4) | SHOW: $\sim\exists v\{v < \Gamma \cup \{\alpha\} \& v \perp \Delta\}$ | $\sim\exists D$ |
| (5) | $w < \Gamma \cup \{\alpha\} \& w \perp \Delta$ | As |
| (6) | SHOW: \times | 15,16-31,SC |
| (7) | $\forall x\{x \in \Gamma \cup \{\alpha\} \rightarrow w < x\}$ | 5a, Def $v < \Gamma$ |
| (8) | $\forall x(x \in \Gamma \rightarrow x \in \Gamma \cup \{\alpha\})$ | ST |
| (9) | $\forall x(x \in \Gamma \rightarrow w < x)$ | 7,8,QL |
| (10) | $w < \Gamma$ | 9, Def $v < \Gamma$ |
| (11) | $\sim\exists v(v < \Gamma \& v \perp \Delta \cup \{\sim\alpha\})$ | 2, Def $\Gamma = \Delta$ |
| (12) | $\sim[w \perp \Delta \cup \{\sim\alpha\}]$ | 10,11,QL |
| (13) | $\sim\forall x(x \in \Delta \cup \{\sim\alpha\} \rightarrow w \perp x)$ | 12, Def $w \perp \Gamma (-)$ |
| (14) | $e \in \Delta \cup \{\sim\alpha\} \& \sim[w \perp e]$ | 13, $\sim\forall \rightarrow O$ |
| (15) | $e \in \Delta \vee e = \sim\alpha$ | 14a, ST |
| (16) | c1: $e \in \Delta$ | As |
| (17) | $\forall x\{x \in \Delta \rightarrow w \perp x\}$ | 5b, Def $v \perp \Delta$ |
| (18) | $w \perp e$ | 16,17,QL |
| (19) | \times | 14b,18,SL |
| (20) | c2: $e = \sim\alpha$ | As |
| (21) | $\sim[w \perp \sim\alpha]$ | 14b,20,IL |
| (22) | $w(\sim\alpha) \neq F$ | 21, Def $v \perp \alpha (-)$ |
| (23) | $w(\sim\alpha) = \sim w(\alpha)$ | Def CSL-valuation |
| (24) | $\sim w(\alpha) \neq F$ | 22,23,IL |
| (25) | $\sim w(\alpha) = T$ | 24, Lemma 0 |
| (26) | $w(\alpha) = F$ | 25, Def $-$ |
| (27) | $\alpha \in \Gamma \cup \{\alpha\}$ | ST |
| (28) | $w < \alpha$ | 7,27,QL |
| (29) | $w(\alpha) = T$ | 28, Def $v < \alpha$ |
| (30) | $T = F$ | 26,29,IL |
| (31) | \times | 30, Axiom 0 |

#5:

- (32) $\Gamma = \Delta \cup \{\alpha\} \leftrightarrow \Gamma \cup \{\sim\alpha\} = \Delta$
Very similar to #4.

#6:

| | | |
|------|---|-----------------------|
| (1) | SHOW: $\{\alpha \rightarrow \beta, \alpha\} = \beta$ | Def $\Gamma = \alpha$ |
| (2) | SHOW: $\forall v(v < \{\alpha \rightarrow \beta, \alpha\} \rightarrow v < \beta)$ | UCD |
| (3) | $w < \{\alpha \rightarrow \beta, \alpha\}$ | As |
| (4) | SHOW: $w < \beta$ | 13, Def $<$ |
| (5) | $\forall x(x \in \{\alpha \rightarrow \beta, \alpha\} \rightarrow w < x)$ | 3, Def $v < \Gamma$ |
| (6) | $\alpha \rightarrow \beta, \alpha \in \{\alpha \rightarrow \beta, \alpha\}$ | ST |
| (7) | $w < \alpha \rightarrow \beta$ | 5,6a,QL |
| (8) | $w < \alpha$ | 5,6b,QL |
| (9) | $w(\alpha \rightarrow \beta) = T$ | 7, Def $<$ |
| (10) | $w(\alpha) = T$ | 8, Def $<$ |
| (11) | $w(\alpha \rightarrow \beta) = w(\alpha) \rightarrow w(\beta)$ | Def CSL-val |
| (12) | $T = T \rightarrow w(\beta)$ | 9,10,11,IL |
| (13) | $w(\beta) = T$ | 12, Def \rightarrow |

#7:

| | | |
|------|--|-----------------------------|
| (1) | SHOW: $\{\alpha \rightarrow \beta, \sim \beta\} \models \sim \alpha$ | Def $\Gamma \models \alpha$ |
| (2) | SHOW: $\forall v (v < \{\alpha \rightarrow \beta, \sim \beta\} \rightarrow v < \sim \alpha)$ | UCD |
| (3) | $w < \{\alpha \rightarrow \beta, \sim \beta\}$ | As |
| (4) | SHOW: $w < \sim \alpha$ | 19, Def < |
| (5) | $\forall x (x \in \{\alpha \rightarrow \beta, \sim \beta\} \rightarrow w < x)$ | 3, Def $v < \Gamma$ |
| (6) | $\alpha \rightarrow \beta, \sim \beta \in \{\alpha \rightarrow \beta, \sim \beta\}$ | ST |
| (7) | $w < \alpha \rightarrow \beta$ | 5, 6a, QL |
| (8) | $w < \sim \beta$ | 5, 6b, QL |
| (9) | $w(\alpha \rightarrow \beta) = T$ | 7, Def < |
| (10) | $w(\sim \beta) = T$ | 8, Def < |
| (11) | $w(\sim \beta) = \sim w(\beta)$ | Def CSL-val |
| (12) | $\sim w(\beta) = T$ | 10, 11, IL |
| (13) | $w(\beta) = F$ | 12, Def - |
| (14) | $w(\alpha \rightarrow \beta) = w(\alpha) \rightarrow w(\beta)$ | Def CSL-valuation |
| (15) | $T = w(\alpha) \rightarrow F$ | 9, 13, 14, IL |
| (16) | $w(\alpha) = F$ | 15, Def \rightarrow |
| (17) | $w(\sim \alpha) = \sim w(\alpha)$ | Def CSL-val |
| (18) | $w(\sim \alpha) = \sim F$ | 16, 17, IL |
| (19) | $w(\sim \alpha) = T$ | 18, Def - |

#8:

| | | |
|------|--|----------------------------|
| (1) | SHOW: $\sim \alpha \models \alpha \rightarrow \beta$ | Def $\alpha \models \beta$ |
| (2) | SHOW: $\forall v \{v < \sim \alpha \rightarrow v < \alpha \rightarrow \beta\}$ | UCD |
| (3) | $w < \sim \alpha$ | As |
| (4) | SHOW: $w < \alpha \rightarrow \beta$ | Def < |
| (5) | SHOW: $w(\alpha \rightarrow \beta) = T$ | Def CSL-val |
| (6) | SHOW: $w(\alpha) \rightarrow w(\beta) = T$ | 11, 12, IL |
| (7) | $w(\sim \alpha) = T$ | 3, Def < |
| (8) | $w(\sim \alpha) = \sim w(\alpha)$ | Def CSL-val |
| (9) | $\sim w(\alpha) = T$ | 7, 8, IL |
| (10) | $w(\alpha) = F$ | 9, Def - |
| (11) | $w(\alpha) \rightarrow w(\beta) = F \rightarrow w(\beta)$ | 10, IL |
| (12) | $F \rightarrow w(\beta) = T$ | Def \rightarrow |

#9:

| | | |
|------|--|----------------------------|
| (1) | $\beta \models \alpha \rightarrow \beta$ | |
| (2) | SHOW: $\beta \models \alpha \rightarrow \beta$ | Def $\alpha \models \beta$ |
| (3) | SHOW: $\forall v \{v < \beta \rightarrow v < \alpha \rightarrow \beta\}$ | UCD |
| (4) | $w < \beta$ | As |
| (5) | SHOW: $w < \alpha \rightarrow \beta$ | Def < |
| (6) | SHOW: $w(\alpha \rightarrow \beta) = T$ | Def CSL-val |
| (7) | SHOW: $w(\alpha) \rightarrow w(\beta) = T$ | 9, 10, IL |
| (8) | $w(\beta) = T$ | 3, Def < |
| (9) | $w(\alpha) \rightarrow w(\beta) = w(\alpha) \rightarrow T$ | 10, IL |
| (10) | $w(\alpha) \rightarrow T = T$ | Def \rightarrow |

#10:

| | | |
|------|---|----------------------|
| (1) | SHOW: $\{\alpha, \sim\alpha\} \models$ | Def $\Gamma \models$ |
| (2) | SHOW: $\sim \exists v \forall x \{x \in \{\alpha, \sim\alpha\} \rightarrow v < x\}$ | $\sim \exists D$ |
| (3) | $\forall x \{x \in \{\alpha, \sim\alpha\} \rightarrow w < x\}$ | As |
| (4) | SHOW: \times | 13, Axiom 0 |
| (5) | $\alpha, \sim\alpha \in \{\alpha, \sim\alpha\}$ | ST |
| (6) | $w < \alpha$ | 3,5a,QL |
| (7) | $w < \sim\alpha$ | 3,5b,QL |
| (8) | $w(\alpha) = T$ | 6, Def < |
| (9) | $w(\sim\alpha) = T$ | 7, Def < |
| (10) | $w(\sim\alpha) = \sim w(\alpha)$ | Def CSL-val |
| (11) | $\sim w(\alpha) = T$ | 9,10,IL |
| (12) | $w(\alpha) = F$ | 11, Def - |
| (13) | $T = F$ | 8,12,IL |

#11:

| | | |
|------|---|----------------------|
| (1) | SHOW: $\models \{\alpha, \sim\alpha\}$ | Def $\models \Gamma$ |
| (2) | SHOW: $\sim \exists v \forall x \{x \in \{\alpha, \sim\alpha\} \rightarrow v \perp x\}$ | $\sim \exists D$ |
| (3) | $\forall x \{x \in \{\alpha, \sim\alpha\} \rightarrow w \perp x\}$ | As |
| (4) | SHOW: \times | 13, Axiom 0, SL |
| (5) | $\alpha, \sim\alpha \in \{\alpha, \sim\alpha\}$ | ST |
| (6) | $w \perp \alpha$ | 3,5a,QL |
| (7) | $w \perp \sim\alpha$ | 3,5b,QL |
| (8) | $w(\alpha) = F$ | 6, Def < |
| (9) | $w(\sim\alpha) = F$ | 7, Def < |
| (10) | $w(\sim\alpha) = \sim w(\alpha)$ | Def CSL-val |
| (11) | $\sim w(\alpha) = F$ | 9,10,IL |
| (12) | $w(\alpha) = T$ | 11, Def - |
| (13) | $T = F$ | 8,12,IL |

#12:

| | | |
|------|--|--------------------------------|
| (1) | SHOW: $\exists \alpha \exists \beta \sim [\{\alpha \rightarrow \beta, \beta\} \models \alpha]$ | 2, QL |
| (2) | SHOW: $\sim [\{P \rightarrow Q, Q\} \models P]$ | ID |
| (3) | $\{P \rightarrow Q, Q\} \models P$ | As |
| (4) | SHOW: \times | 18, Axiom 0, SL |
| (5) | $\forall v \{v < \{P \rightarrow Q, Q\} \rightarrow v < P\}$ | 2, Def $\Gamma \models \alpha$ |
| (6) | $\exists v \{v(P) = F \ \& \ v(Q) = T\}$ | Big Lemma |
| (7) | $w(P) = F \ \& \ w(Q) = T$ | 6, $\exists O$ |
| (8) | $w(P \rightarrow Q) = w(P) \rightarrow w(Q)$ | Def CSL-val |
| (9) | $w(P \rightarrow Q) = F \rightarrow T$ | 7,8,IL |
| (10) | $w(P \rightarrow Q) = T$ | 9, Def \rightarrow |
| (11) | $w < P \rightarrow Q$ | 10, Def < |
| (12) | $w < Q$ | 7b, Def < |
| (13) | $\forall x \{x \in \{P \rightarrow Q, Q\} \leftrightarrow x = P \rightarrow Q \vee x = Q\}$ | ST |
| (14) | $\forall x \{x \in \{P \rightarrow Q, Q\} \rightarrow w < x\}$ | 11,12,13,QL |
| (15) | $w < \{P \rightarrow Q, Q\}$ | 14, Def $v < \Gamma$ |
| (16) | $w < P$ | 5,15,QL |
| (17) | $w(P) = T$ | 16, Def < |
| (18) | $T = F$ | 7,17,IL |

#13:

| | | |
|------|--|--------------------------------|
| (1) | $\exists\alpha\exists\beta\sim[\{\alpha\rightarrow\beta, \sim\alpha\} \models \sim\beta]$ | 2,QL |
| (2) | SHOW: $\sim[\{P\rightarrow Q, \sim P\} \models \sim Q]$ | ID |
| (3) | $\{P\rightarrow Q, \sim P\} \models \sim Q$ | As |
| (4) | SHOW: \times | 25,Axiom 0,SL |
| (5) | $\forall v\{v\langle\{P\rightarrow Q, \sim P\}\rangle \rightarrow v\langle\sim Q\rangle\}$ | 2,Def $\Gamma\models\alpha$ |
| (6) | $\exists v\{v(P)=F \ \& \ v(Q)=T\}$ | Big Lemma |
| (7) | $w(P)=F \ \& \ w(Q)=T$ | 6, $\exists O$ |
| (8) | $w(P\rightarrow Q) = w(P)\rightarrow w(Q)$ | Def CSL-val |
| (9) | $w(P\rightarrow Q) = F\rightarrow T$ | 7,8,IL |
| (10) | $w(P\rightarrow Q) = T$ | 9,Def \rightarrow |
| (11) | $w\langle P\rightarrow Q\rangle$ | 10, Def \langle |
| (12) | $w\langle Q\rangle$ | 7b,Def \langle |
| (13) | $w(\sim P) = \sim w(P)$ | Def CSL-val |
| (14) | $w(\sim P) = \sim F$ | 7a,13,IL |
| (15) | $w(\sim P) = T$ | 14,Def \sim |
| (16) | $w\langle\sim P\rangle$ | 15,Def \langle |
| (17) | $\forall x\{x\in\{P\rightarrow Q, \sim P\} \leftrightarrow x=P\rightarrow Q \vee x=\sim P\}$ | ST |
| (18) | $\forall x\{x\in\{P\rightarrow Q, \sim P\} \rightarrow w\langle x\rangle\}$ | 11,16,17,IL |
| (19) | $w\langle\{P\rightarrow Q, \sim P\}\rangle$ | 18,Def $v\langle\Gamma\rangle$ |
| (20) | $w\langle\sim Q\rangle$ | 5,19,QL |
| (21) | $w(\sim Q)=T$ | 20, Def \langle |
| (22) | $w(\sim Q) = \sim w(Q)$ | Def CSL-val |
| (23) | $\sim w(Q) = T$ | 21,22,IL |
| (24) | $w(Q) = F$ | 23,Def \sim |
| (25) | $T=F$ | 7b,24,IL |

4. Official Definitions for Purposes of Doing Proofs of Theorems:

- (1) $v < \alpha \quad =_{df} \quad v(\alpha) = T$
- (2) $v < \Gamma \quad =_{df} \quad \forall x \{ x \in \Gamma \rightarrow v < x \}$
- (3) $v \perp \alpha \quad =_{df} \quad v(\alpha) = F$
- (4) $v \perp \Gamma \quad =_{df} \quad \forall x \{ x \in \Gamma \rightarrow v \perp x \}$
- (5) $\Gamma \models \alpha \quad =_{df} \quad \forall v (v < \Gamma \rightarrow v < \alpha)$
- (6) $\Gamma \models \Delta \quad =_{df} \quad \sim \exists v (v < \Gamma \ \& \ v \perp \Delta)$
- (7) $\models \alpha \quad =_{df} \quad \forall v [v < \alpha]$
- (8) $\alpha \models \quad =_{df} \quad \forall v [v \perp \alpha]$
- (9) $\models \Gamma \quad =_{df} \quad \sim \exists v [v \perp \Gamma]$
- (10) $\Gamma \models \quad =_{df} \quad \sim \exists v [v < \Gamma]$
- (11) $\alpha \equiv \beta \quad =_{df} \quad \forall v \in V \{ v < \alpha \leftrightarrow v < \beta \}$
- (12) $\Gamma \equiv \Delta \quad =_{df} \quad \forall v \in V \{ v < \Gamma \leftrightarrow v < \Delta \}$
- (13) Def \sim
 $\sim T \quad =_{df} \quad F$
 $\sim F \quad =_{df} \quad T$
- (14) Def \rightarrow
 $T \rightarrow T \quad =_{df} \quad T$
 $T \rightarrow F \quad =_{df} \quad F$
 $F \rightarrow T \quad =_{df} \quad T$
 $F \rightarrow F \quad =_{df} \quad T$
- (15) Def CSL-val
 v is a CSL-valuation $=_{df}$
 $\forall \alpha [v(\sim \alpha) = \sim v(\alpha)] \ \& \ \forall \alpha \beta [v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)] \ \& \dots$

5. Supporting Lemmas

Axiom 0:

$$T \neq F$$

This is taken for granted in formal semantics. Logically speaking, it is an axiom (primitive thesis, fundamental postulate) of formal semantics.

Lemma 0:

| | | |
|------|--|---------------------------------|
| (26) | SHOW: $v(\alpha) = T \vee v(\alpha) = F$ | |
| (27) | v is a function from S into $\{T, F\}$ | Def of valuation on L |
| (28) | $\forall x(x \in S \rightarrow v(x) \in \{T, F\})$ | 2, Def function from A to B |
| (29) | $\alpha \in S$ | sortal assumption |
| (30) | $v(\alpha) \in \{T, F\}$ | 3,4,QL |
| (31) | $v(\alpha) = T \vee v(\alpha) = F$ | 5, Def $\{ \}$ |

Lemma 1:

$$v(\alpha) = T \rightarrow v(\alpha) \neq F$$

$$v(\alpha) = F \rightarrow v(\alpha) \neq T$$

Both are immediate corollaries to Axiom 0.

Lemma 2:

| | | |
|------|---|-----------------|
| (1) | SHOW: $v < \Gamma \ \& \ v < \alpha \ . \rightarrow v < \Gamma \cup \{ \alpha \}$ | CD |
| (2) | $v < \Gamma \ \& \ v < \alpha$ | As |
| (3) | SHOW: $v < \Gamma \cup \{ \alpha \}$ | Def $<$ |
| (4) | SHOW: $\forall x(x \in \Gamma \cup \{ \alpha \} \rightarrow v < x)$ | UCD |
| (5) | $a \in \Gamma \cup \{ \alpha \}$ | As |
| (6) | SHOW: $v < a$ | SC |
| (7) | $a \in \Gamma \vee a \in \{ \alpha \}$ | 5, Def \cup |
| (8) | case 1: $a \in \Gamma$ | As |
| (9) | $\forall x(x \in \Gamma \rightarrow v < x)$ | 2a, Def $<$ |
| (10) | $v < a$ | 8,9,QL |
| (11) | case 2: $a \in \{ \alpha \}$ | As |
| (12) | $a = \alpha$ | 11, Def $\{ \}$ |
| (13) | $v < a$ | 2b,12,IL |

Lemma 3:

| | | |
|-----|---|--------------------|
| (1) | SHOW: $\Gamma \subseteq \Delta \rightarrow v < \Delta \rightarrow v < \Gamma$ | CD2 |
| (2) | $\Gamma \subseteq \Delta$ | As |
| (3) | $v < \Delta$ | As |
| (4) | SHOW: $v < \Gamma$ | Def $<$ |
| (5) | SHOW: $\forall x(x \in \Gamma \rightarrow v < x)$ | DD |
| (6) | $\forall x(x \in \Gamma \rightarrow x \in \Delta)$ | 2, Def \subseteq |
| (7) | $\forall x(x \in \Delta \rightarrow v < x)$ | 3, Def $<$ |
| (8) | $\forall x(x \in \Gamma \rightarrow v < x)$ | 6,7,QL |

Lemma 4:

| | | |
|-----|---|--------------------|
| (1) | SHOW: $\Gamma \subseteq \Delta \rightarrow v \perp \Delta \rightarrow v \perp \Gamma$ | CD2 |
| (2) | $\Gamma \subseteq \Delta$ | As |
| (3) | $v \perp \Delta$ | As |
| (4) | SHOW: $v \perp \Gamma$ | Def \perp |
| (5) | SHOW: $\forall x(x \in \Gamma \rightarrow v \perp x)$ | DD |
| (6) | $\forall x(x \in \Gamma \rightarrow x \in \Delta)$ | 2, Def \subseteq |
| (7) | $\forall x(x \in \Delta \rightarrow v \perp x)$ | 3, Def \perp |
| (8) | $\forall x(x \in \Gamma \rightarrow v \perp x)$ | 6,7,QL |

Lemma 5:

| | | |
|-----|--|------------------|
| (1) | SHOW: $v < \emptyset$ | Def $v < \Gamma$ |
| (2) | SHOW: $\forall x\{x \in \emptyset \rightarrow v < x\}$ | 3,QL |
| (3) | $\sim \exists x\{x \in \emptyset\}$ | ST |

Lemma 6:

| | | |
|------|---|---------------------|
| (1) | SHOW: $w < \{\alpha\} \leftrightarrow w < \alpha$ | \leftrightarrow D |
| (2) | SHOW: \rightarrow | CD |
| (3) | $w < \{\alpha\}$ | As |
| (4) | SHOW: $w < \alpha$ | 5,6,QL |
| (5) | $\forall x\{x \in \{\alpha\} \rightarrow w < x\}$ | 3, Def $v < \Gamma$ |
| (6) | $\alpha \in \{\alpha\}$ | ST |
| (7) | SHOW: \leftarrow | CD |
| (8) | $w < \alpha$ | As |
| (9) | SHOW: $w < \{\alpha\}$ | Def $v < \Gamma$ |
| (10) | SHOW: $\forall x\{x \in \{\alpha\} \rightarrow w < x\}$ | UCD |
| (11) | $e \in \{\alpha\}$ | As |
| (12) | SHOW: $w < e$ | 8,13,IL |
| (13) | $e = \alpha$ | 11,ST |

Big Lemma:

In CSL, the (syntactically) atomic formulas are also semantically atomic. In other words, any assignment of truth values to any subset of atomic formulas can be extended to an admissible valuation.

Proof: by induction!

6. Theorems of Set Theory

#1:

| | | |
|-----|---------------------------------------|----------------|
| (1) | SHOW: $a \in \Gamma \cup \{a\}$ | Def \cup |
| (2) | SHOW: $a \in \Gamma \vee a \in \{a\}$ | DD |
| (3) | $a=a$ | IL |
| (4) | $a \in \{a\}$ | 3, Def $\{ \}$ |
| (5) | $a \in \Gamma \vee a \in \{a\}$ | 4, SL |

#2:

| | | |
|-----|------------------------|----------------|
| (1) | SHOW: $a \in \{a\}$ | DD |
| (2) | $a=a$ | IL |
| (3) | $a \in \{a\}$ | 2, Def $\{ \}$ |
| (1) | SHOW: $a \in \{a, b\}$ | DD |
| (2) | $a=a$ | IL |
| (3) | $a=a \vee a=b$ | 2, SL |
| (4) | $a \in \{a, b\}$ | 2, Def $\{ \}$ |

#3:

| | | |
|-----|--|-----------------|
| (1) | SHOW: $\Gamma \subseteq \Gamma \cup \Delta$ | Def \subseteq |
| (2) | SHOW: $\forall x(x \in \Gamma \rightarrow x \in \Gamma \cup \Delta)$ | UCD |
| (3) | $a \in \Gamma$ | As |
| (4) | SHOW: $a \in \Gamma \cup \Delta$ | Def \cup |
| (5) | SHOW: $a \in \Gamma \vee a \in \Delta$ | DD |
| (6) | $a \in \Gamma \vee a \in \Delta$ | 3, SL |

#4:

| | | |
|-----|---|-----------------------|
| (1) | SHOW: $\sim \exists x[x \in \emptyset]$ | ID |
| (2) | $\exists x[x \in \emptyset]$ | As |
| (3) | SHOW: \times | 5, 6, SL |
| (4) | $a \in \emptyset$ | 2, $\exists \text{O}$ |
| (5) | $a \neq a$ | 3, Def \emptyset |
| (6) | $a=a$ | IL |

#4:

$$e \in \Gamma \cup \Delta \leftrightarrow e \in \Gamma \vee e \in \Delta$$

#5:

$$e \in \Gamma \cup \{\alpha\} \leftrightarrow e \in \Gamma \vee e = \alpha$$

#6:

$$\Gamma \cap \Delta \neq \emptyset \rightarrow \exists x\{x \in \Gamma \ \& \ x \in \Delta\}$$