Numerical Quantifiers

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1. Introduction

In English, number-words (numerals[1]) *appear* to be used as quantifiers, as in the following examples.

> `**three** students came to the party`
> `there are **four** dogs in the yard`

three `students came to the party`
`there are` **four** `dogs in the yard`

On the other hand, numerals also figure in the following sorts of constructions.

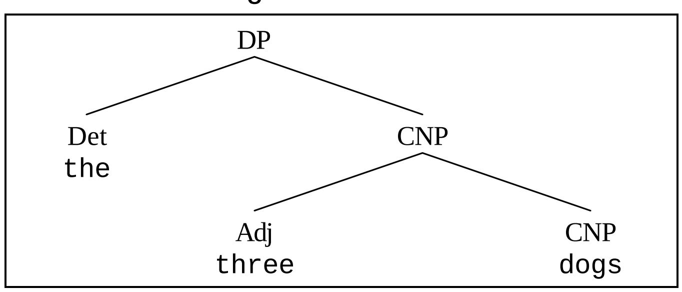
the three `dogs`
my four `dogs`
all five `dogs`
no two `dogs are exactly alike`
`we` **three** `kings …`

If numerals are quantifiers, then the first four phrases violate the prohibition against double-determiners. In the last phrase, the appositive `‘three kings’` cannot be replaced by QPs such as `‘some kings’`, `‘most kings’`, or `‘all kings’`.

2. Our Proposal – Numerals are Plural Adjectives

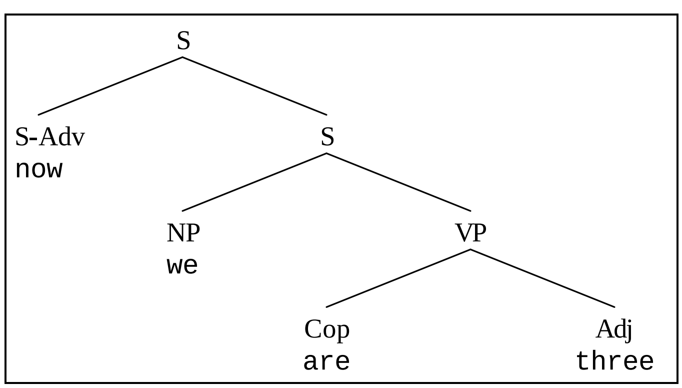
Given the latter data, we propose that numerals are fundamentally, *not* quantifiers, but are rather adjectives,[2] as illustrated in the following tree.

`the three dogs`



Here the numeral `‘three’` serves as a CNP-modifier, which is the principal role of adjectives. Numerals can also be used as bare adjectives, although it is less common, as in the following poetic description of the arrival of one's first child.

`now we are three`



[1] Numerals are sometimes regarded as a species of number words – being logograms like `‘2’` rather than phonograms like `‘two’`. Since this is not a semantically relevant distinction, we will simply use the terms interchangeably.
[2] This is in exact agreement with traditional lexicography.

There is an important difference, however, between numerical adjectives and most other adjectives. In particular, most adjectives are *conjunctive*,[3] but numerical adjectives are not. The latter concept may be defined as follows.

$\mathbb{A}$ is conjunctive $=_{df}$ for any $x_1, \ldots, x_k$
x_1 and … and x_k are $\mathbb{A}$
if and only if
x_1 is $\mathbb{A}$, and … , and x_k is $\mathbb{A}$

So, for example `'happy'` is conjunctive, since for example

Jay and Kay are happy if and only if Jay is happy and Kay is happy.

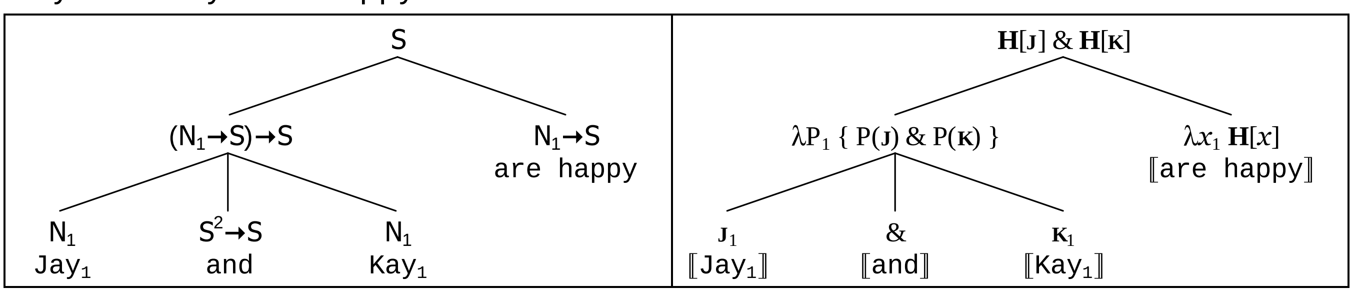
On the other hand, `'two'` is not conjunctive, since for example

Jay and Kay are two, but neither Jay nor Kay is two.

3. Mereological 'and' *versus* Logical 'and'

Notice carefully that the previous sentence introduces a novel use of `'and'`, which we refer to as *mereological* `'and'`, which differs from the more common *logical* `'and'`. Recall that logical `'and'` is a two-place S-operator, which is illustrated in the following.

`Jay and Kay are happy`



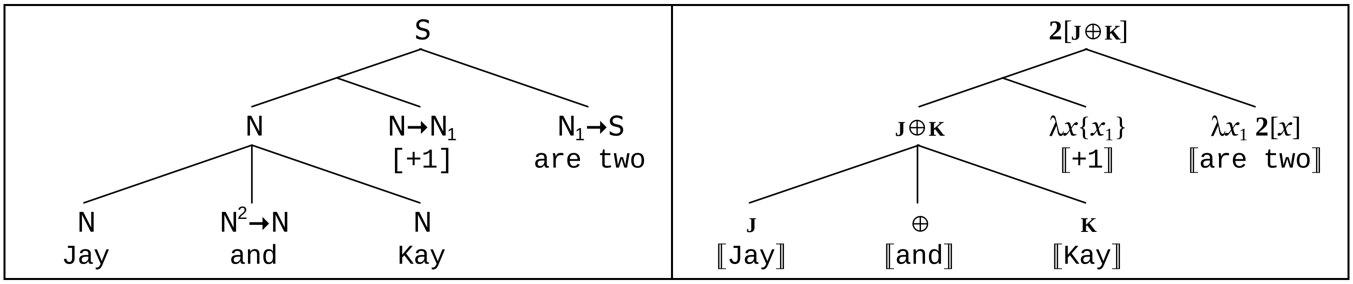
We can apply a completely parallel treatment to `'Jay and Kay are two'`, which accordingly analyzes this sentence as saying that Jay is two and Kay is two. This is an admissible reading of the sentence, to be sure, but it is not the most plausible reading. Rather, the most plausible reading of this sentence treats `'and'`, not as logical, but as mereological, the latter being categorially rendered as follows.

type(and) = $N^2 \rightarrow N$
⟦and⟧ = $\lambda xy\{x \oplus y\}$

Here, $\oplus$ is the *mereological-sum* operator, which is explained in the next section. In the meantime, the following is an example grammatical analysis.

[3] This is closely related to another notion – *distributivity* – which is defined in Section 5.

Jay and Kay are two



Here, **2**[.] is the two-predicate, in the metalanguage, which is explained in the next section.

4. Expansion of the Domain of Entities; Count-Objects

We propose to expand the domain of entities to include the usual *singular-objects* (individuals) as well as *plural-objects* (pluralities), the union of which is the class of *count-objects*.[4] On the one hand, singular-objects are the usual suspects of first-order logic, and include particular individual persons, places, and things. On the other hand, plural-objects are entities that first-order logic studiously sidesteps, and include groups of individuals of various sorts and functions.[5] For example, whereas Nomar Garciapara *is an* individual, the Boston Red Sox infielders *are a* plurality. Even the grammar of plurals sounds strange, as we strain to treat each plurality simultaneously as a "one" and as a "many".

Set theory presents a formal account of collections, which reduces every "many" to a "one", and in particular reduces all plural-talk about collections to singular-talk about sets, which are treated as singular-objects (individuals) completely on a par with "ordinary" singular-objects like numbers and space-time points. Although we don't officially propose to *identify* pluralities with (mathematical) sets, we do propose to *model/explicate* pluralities in terms of sets.[6] In this connection, we offer the following official definitions.

Where $\mathbb{S}$ is a set of *singular-entities* (individuals), the associated set $\mathbb{P}$ of *plural-entities* is defined as follows,

$$\mathbb{P} \quad =_{\text{df}} \quad \{ X : X \subseteq \mathbb{S} \ \& \ X \neq \varnothing \ \& \ \sim\exists y[X=\{y\}] \}$$

and the associated set $\mathbb{C}$ of *count-entities* is defined as follows.

$$\mathbb{C} \quad =_{\text{df}} \quad \mathbb{S} \cup \mathbb{P}$$

In other words, plural-entities correspond to non-empty non-singleton subsets of singular-entities.

We can now define ⊕ as follows.

$$\alpha \oplus \beta \quad =_{\text{df}} \quad \alpha^{+} \cup \beta^{+}$$

$$\text{where:} \quad \varepsilon^{+} \quad =_{\text{df}} \quad \{\varepsilon\} \quad \text{if } \varepsilon \in \mathbb{S}$$
$$=_{\text{df}} \quad \varepsilon \quad \text{if } \varepsilon \in \mathbb{P}$$

[4] Count-objects, the counterparts of *count noun* phrases, are distinguished from mass-objects, the counterparts of *mass noun* phrases.
[5] Some groups are *mere* collections; other groups take on a quasi-autonomous existence, and in particular can serve as agents of actions. For example, a jury can hand down a verdict.
[6] To say that sets *model* pluralities is to say that many properties of pluralities correspond in a specified way to properties of sets.

So, for example, granting that **J** and **K** are individuals (i.e., elements of $\mathbb{S}$),

$$\begin{aligned} \mathbf{J}\oplus\mathbf{K} &= \mathbf{J}^{+} \cup \mathbf{K}^{+} \\ &= \{\mathbf{J}\} \cup \{\mathbf{K}\} \\ &= \{\mathbf{J}, \mathbf{K}\} \end{aligned}$$

This means, in particular, that the interpretation of the compound `'Jay and Kay'` is (modeled by) the set {Jay, Kay}, which consists of Jay and Kay and nothing else.

Granting that pluralities are (modeled by) sets, we can offer the following simple definitions of the numerical adjectives.

$$\begin{aligned} \mathbf{1}[\alpha] &=_{df} \alpha^{+} \text{ has exactly one member} \\ \mathbf{2}[\alpha] &=_{df} \alpha^{+} \text{ has exactly two members} \\ \mathbf{3}[\alpha] &=_{df} \alpha^{+} \text{ has exactly three members} \end{aligned}$$

etc.

These in turn have strictly first-order rewrites, as follows.

$$=_{df} \quad \exists x\, \forall y\{y\in\alpha^{+} \leftrightarrow y=x\}$$
$$=_{df} \quad \exists x_1x_2\, \{\, x_1\neq x_2 \,\&\, \forall y\{y\in\alpha^{+} \leftrightarrow.\ y=x_1 \vee y=x_2\}\, \}$$
$$=_{df} \quad \exists x_1x_2x_3\, \{\, x_1\neq x_2 \,\&\, x_1\neq x_3 \,\&\, x_2\neq x_3 \,\&\, \forall y\{y\in\alpha^{+} \leftrightarrow.\ y=x_1 \vee y=x_2 \vee y=x_3\}\, \}$$

etc.

5. Plural-Predication

We propose that plural-predication is a primitive notion on a par with singular-predication. So, for example, our lexicon might include the following entries.

$$[\![\text{student}]\!] = \lambda x_0\, \mathbf{S}[x]$$
$$[\![\text{meeting}]\!] = \lambda x_0\, \mathbf{M}[x]$$

Here, the variable 'x' ranges over count-entities. So, although we use singular-terms in our mathematical description, we understand them as modeling count-entities, including singular-entities and plural-entities. So if x is a plural-entity, then $\mathbf{S}[x]$ means that x *are* students, and $\mathbf{M}[x]$ means that x *are* meeting.

We further propose that the lexicon will additionally include the following "logical" information.

(L1) distributive[**S**]

(L2) plural[**M**]

These in turn are expanded in accordance with the following definitions.

(d1) distributive[P] $=_{df}$ $\forall x\, \{\, P[x] \leftrightarrow \forall y\{y\sqsubset x \rightarrow P[y]\}\, \}$

(d2) plural[P] $=_{df}$ $\forall x\, \{\, P[x] \rightarrow \mathbb{P}[x]\, \}$

Here, $\sqsubseteq$ is the mereological part-whole relation on the class $\mathbb{C}$ of count-entities, which is officially defined as follows.

$$\alpha\sqsubseteq\beta \quad =_{df} \quad \alpha^{+}\subseteq\beta^{+}$$
$$\alpha\sqsubset\beta \quad =_{df} \quad \alpha\sqsubseteq\beta \,\&\, \beta\not\sqsubseteq\alpha$$

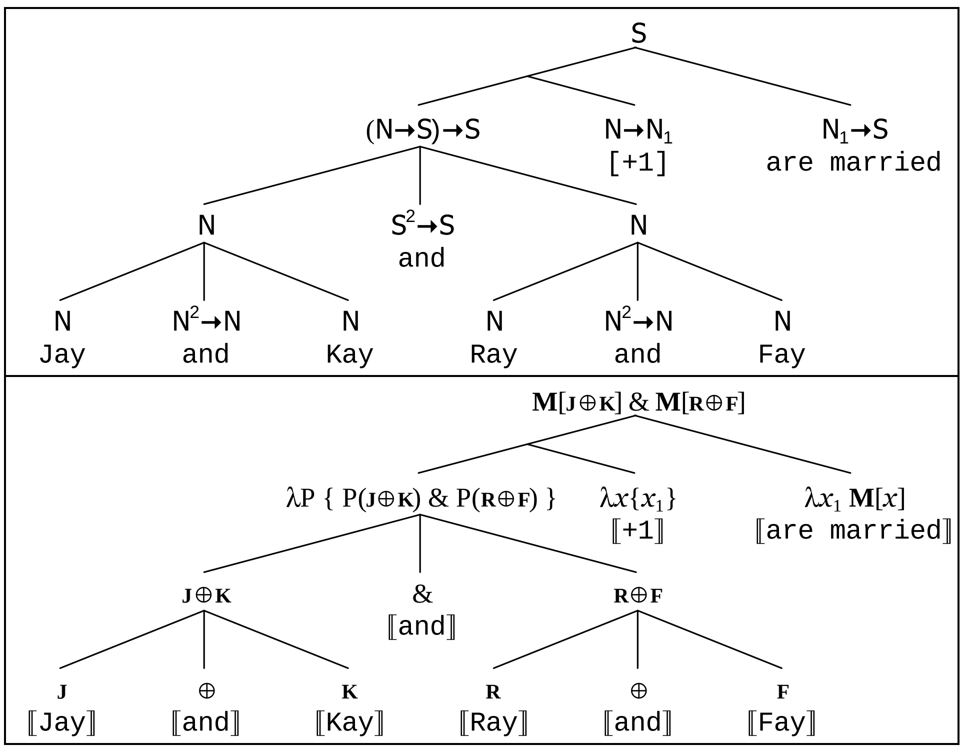
In other words, to say that P is (fully) distributive is to say that a count-entity x is P if and only if every count-entity that x properly contains is P. And to say that P is (inherently) plural is to say that only plural-entities are P. Notice that no (inherently) plural predicate is (fully) distributive, although many plural predicates are *plural-distributive*, which may be defined as follows.

(d3) plural-distributive[P] $=_{df}$ $\forall x \{ P[x] \leftrightarrow. \mathbb{P}[x] \,\&\, \forall y\{\mathbb{P}[y] \,\&\, y \sqsubset x \,.\!\rightarrow P[y]\} \}$

For example, it is fairly plausible to propose that `'meeting'` is plural-distributive; for example, if Jay, Kay, and Elle are meeting, then Jay and Kay are meeting, but we would say that Jay (or Kay, or Elle) is meeting.

The following is an example that uses `'and'` both logically and mereologically.

`Jay and Kay, and Ray and Fay, are married`



6. Plural Quantifiers

First-order logic studiously avoids plural quantifiers, paraphrasing what it can, and ignoring the rest, as illustrated in the following.

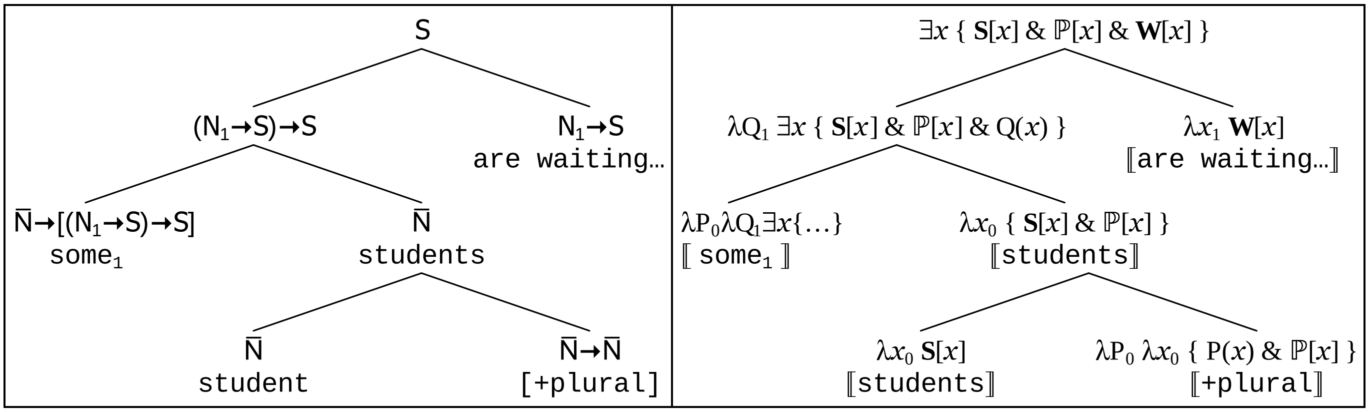
```
all dogs  (are)   ⇒   every dog  (is)
some dogs (are)   ⇒   some dog   (is)
no dogs   (are)   ⇒   no dog     (is)
most dogs (are)   ⇒   ∅
```

Notwithstanding the clarity and precision afforded by singular-only speech, the overall translation method is not entirely successful, as the following examples illustrate.

(1) `some students are waiting in the lounge`
(2) `some students are gathering in the lounge`
(3) `some theatre critics only listen to each other`

For example, given the plural-inflection, (1) conveys the information that the students involved are a plurality, which is grammatically analyzed as follows.

`some students are waiting in the lounge`



This reads the sentence as saying that there is a plural-set of students whose members are waiting in the lounge.

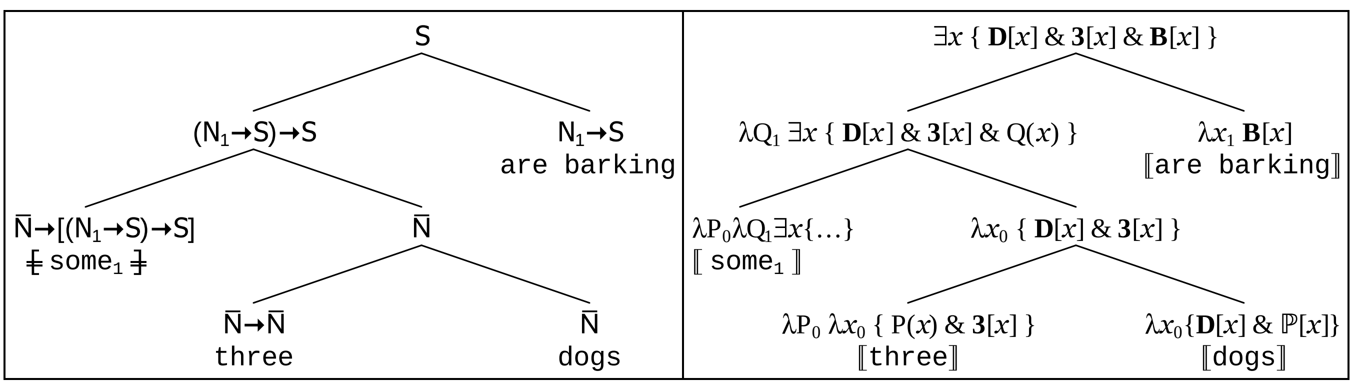
7. Numerical Quantifiers

We are now in position to grammatically analyze numerical quantifiers, as in the following example.

`three dogs are barking`

In particular, we postulate that this sentence contains a covert existential quantifier, as in the following.

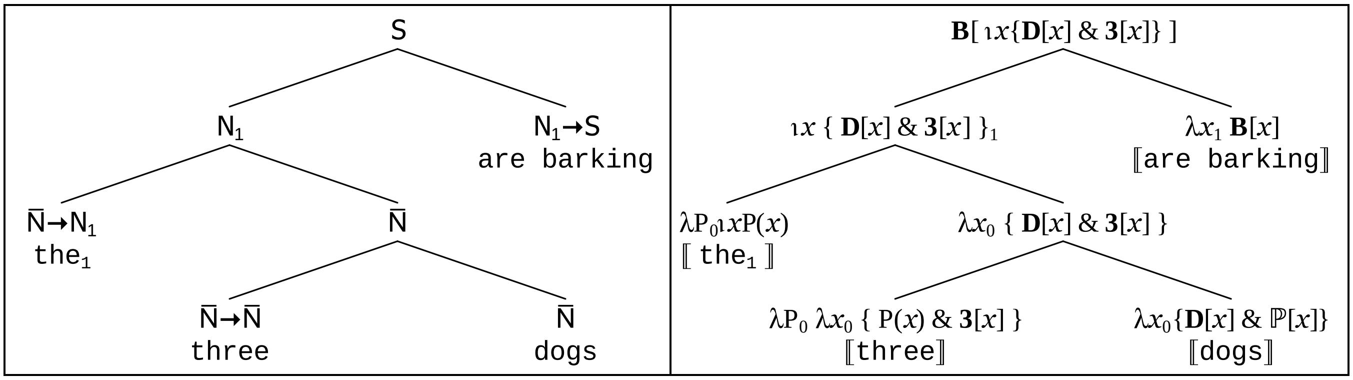
`⫳some⫳ three dogs are barking`



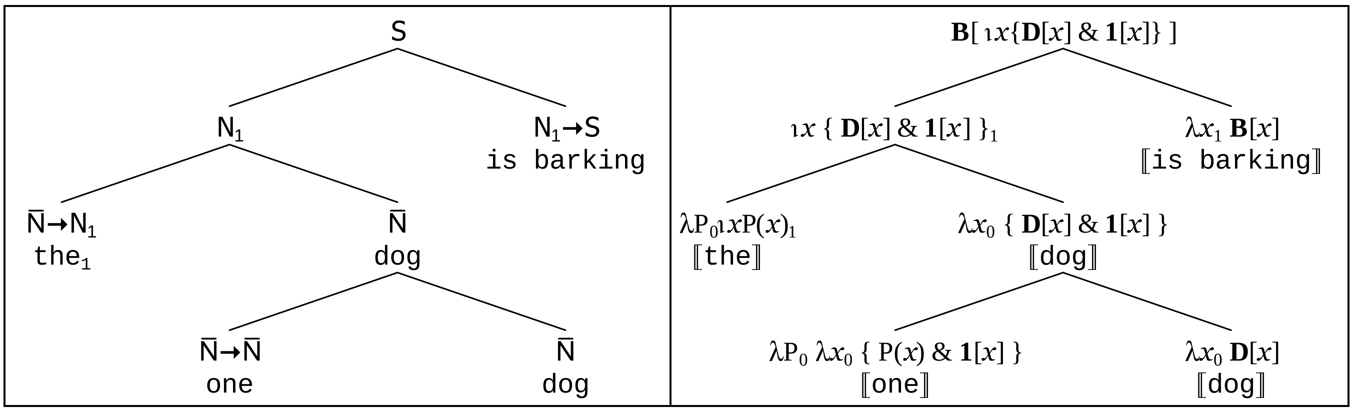
Note that the ℙ-predicate becomes redundant after we add the 3-predicate, so it is dropped. This analysis reads the original sentence as saying that the members of at least one 3-membered set of dogs are barking. Notice that, granting distributivity, this is tantamount to saying that *at least three* dogs are barking. In order to convey that *exactly* three dogs are barking, we need additional semantic information, not postulated in the above analysis. We consider this in a later section (Section 19).

Compare this sentence to the following.

`the three dogs are barking`



`the dog is barking`

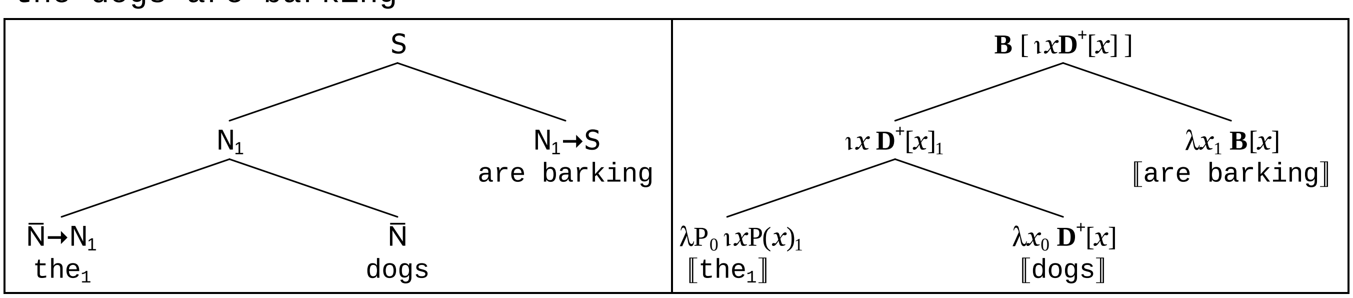


This first sentence is read as saying, in effect, that there is *exactly one* 3-membered set of dogs (in the relevant domain), and its members are barking. The second sentence is read as saying that there is exactly one 1-membered set of dogs (in the relevant domain), and its unique member is barking, which is to say there is exactly one dog, and it is barking. Note that we employ the covert morpheme `'one'`; we can alternatively employ singular-number inflection `[+singular]`.

8. A Problem for our Account of Definite-Determiner Phrases

Traditionally, definite-determiner phrases are logically mapped to definite descriptions involving the iota-operator, as we have done in previous examples. The following example, however, demonstrates the shortcomings of this approach.

`the dogs are barking`



Here, we introduce the following abbreviation.

$$P^{+}[\alpha] \quad =_{df} \quad P[\alpha] \mathbin{\&} \mathbb{P}[\alpha]$$

According to this analysis, the sentence says, in effect, that there is exactly one plural-set of dogs, and its members are barking. The problem is immediate and fatal. Suppose there are in fact three dogs barking; then how many plural-sets of dogs are there whose members are barking? Well, since `‘barking’` is distributive, there are four such sets! But the above analysis claims that there is exactly one such set.

9. Link's Account of Definite-Determiner Phrases

What is needed is an alternative account of `‘the’`. The best-known alternative is due to Godehard Link, who proposes the following.[7]

$$[\![\text{the}]\!] \quad = \quad \lambda P_0\, \mu x P(x)$$

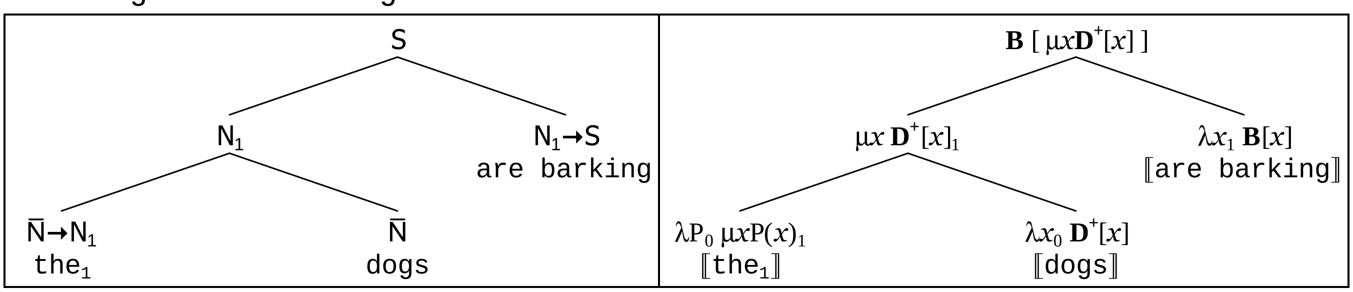
Here, ‘μ’ (mu) is short for ‘maximum’, which is defined relative to the mereological part-whole relation ⊑ among count-objects.[8] In particular,

$$\mu\nu\Phi \quad =_{df} \quad \iota\nu\{\ \Phi\ \&\ \forall\upsilon\{\Phi[\upsilon/\nu] \rightarrow \nu\sqsubseteq\upsilon\}\ \} \qquad [\nu \text{ free for } \upsilon \text{ in } \Phi]$$

In other words, $\mu\nu\Phi$ is the *unique* count-object that is Φ *and furthermore* contains every count-object that is Φ.

Let's see how this works in our earlier case. Given our notation, we don't have to change much – just replace ‘ι’ with ‘μ’.

`the dogs are barking`



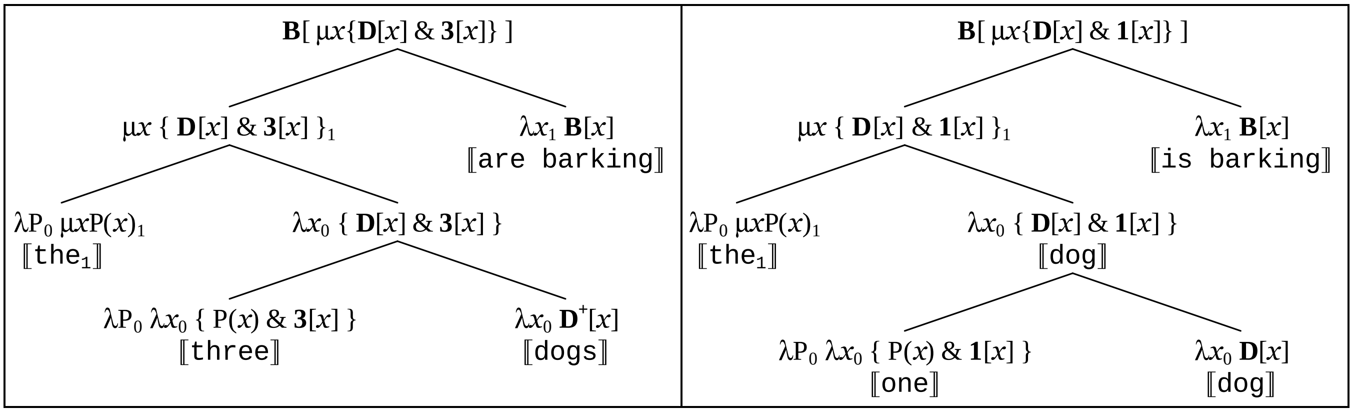
According to the analysis, the sentence says that there is a *maximal* plural-set of dogs (in the relevant domain), and its members are barking.

Link's account of `‘the’` works on problematic examples, but does it work on examples for which we already have a satisfactory solution? If not, then we must postulate two different meanings of `‘the’`, which is theoretically inelegant. Fortunately, our earlier examples work just as well with the new account of `‘the’`, as seen in the following examples.

[7] reference+++

[8] We note that the Link account of `‘the’` also applies to **mass nouns** like `‘water’` as in `‘the water in the bathtub’`, although it requires postulating an expanded domain of entities that includes *mass-entities*, which include all manner of amorphous "quantities of matter". For example, these entities figure in explaining what I mean when I say that the gold in this ring was once scattered across the Galaxy [as currently suggested by cosmologists].

`the three dogs are barking` `the dog is barking`



The first sentence is read as saying that there is a unique maximal 3-membered set of dogs, and its members are barking, and the second sentence is read as saying that there is a unique maximal 1-membered set of dogs, and its members are barking. Note that, as a matter of set theory, there is a maximal 3-membered set of dogs precisely if there is exactly one 3-membered set of dogs, and similarly, there is a maximal 1-membered set of dogs precisely if there is exactly one dog. Accordingly, we can replace 'μ' by 'ι' in the above examples.

10. A Problem for Link's Account – The Cumulative Reading of 'the'

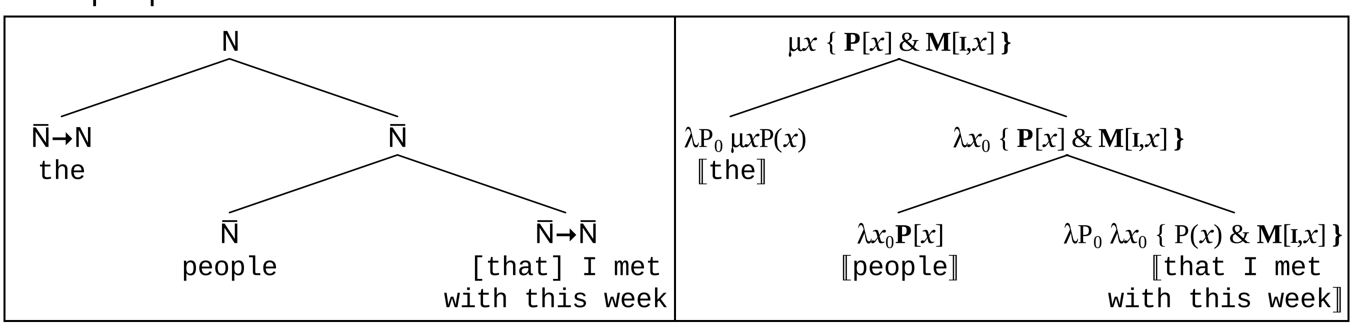
According to Link's account,

$$⟦\texttt{the}⟧ \quad = \quad \lambda P_0\, \mu x P(x)$$

where μxP(x) is the *maximal* set of count-entities that are P.

Consider the following analysis in accordance with this account.

`the people I met-with this week`



Here, we take `'meet with'` as a phrasal verb. Now, suppose that the following summarizes the meet-with events in which I was involved in this week.

I met with Jay on Monday at 10:00 a.m.;
I met with Kay and Elle on Tuesday at 11:00 a.m.;

In particular, on no occasion did I meet with Jay⊕Kay⊕Elle, so there is no maximal set of individuals that I met with this week. Yet it seems reasonable to claim that the people I met with this week *include* Jay, Kay, and Elle (and no one else).

To account for this reading, we propose the following *cumulative reading* of `'the'`.

$[\![\text{the}^+]\!] \quad = \quad \lambda P_0 \bigvee x P(x)$

where:

$\bigvee \nu \Phi$	$=_{df}$	$lub\ \{\nu : \Phi\}$	least upper bound
$ub(S)$	$=_{df}$	$\{x : \forall y(y \in S \rightarrow y \sqsubseteq x\}$	upper bound
$l(S)$	$=_{df}$	$\iota x\{x \in S \ \&\ \forall y\{y \in S \rightarrow x \sqsubseteq y\}\}$	least

For example, the following is the set of all count-entities that I met with.

{Jay, Kay⊕Elle}

So, the least upper bound of this set in the class of count-entities is:

Jay⊕Kay⊕Elle [= {Jay, Kay, Elle}]

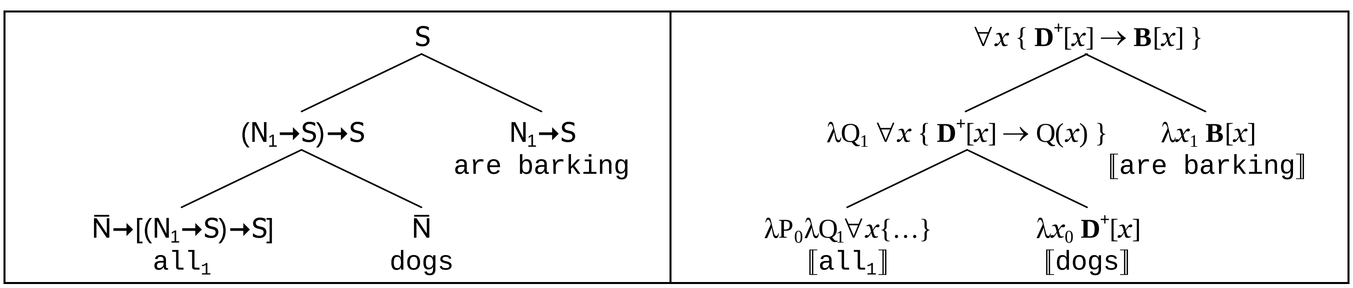
11. Universal Quantifiers and Partitives

We next consider the use of universal quantifiers in plural domains, as illustrated in the following examples.

(1) **`all`** `dogs are barking`
(2) **`all the`** `dogs are barking`
(3) **`all three`** `dogs are barking`

Item (1) is fairly straightforward, being analyzed as follows.

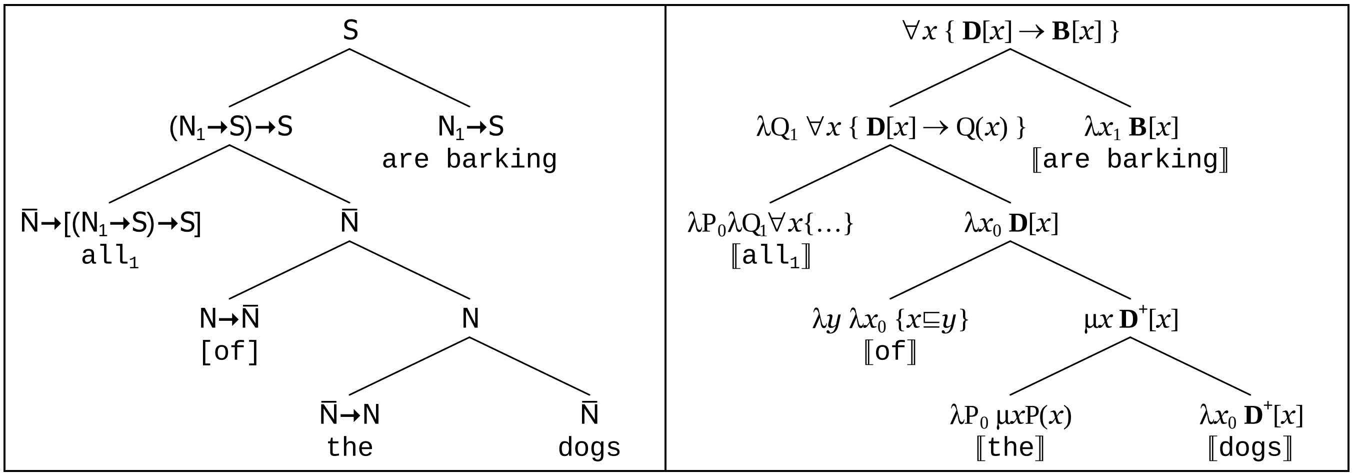
`all dogs are pets`



This reads the sentence as saying that every plural-set of dogs has the following property – its members are barking. Granting that barking is distributive, this amounts to saying that every dog is barking.

Item (2) is not so straightforward, since it appears to have a double-determiner, involving in particular a type-mismatch between `'all'` and `'the'`. In order to resolve this problem, we propose an optionally-pronounced *partitive* `'of'` interposed between `'all'` and `'the'`, as in the following grammatical analysis.

`all [of] the dogs are barking`



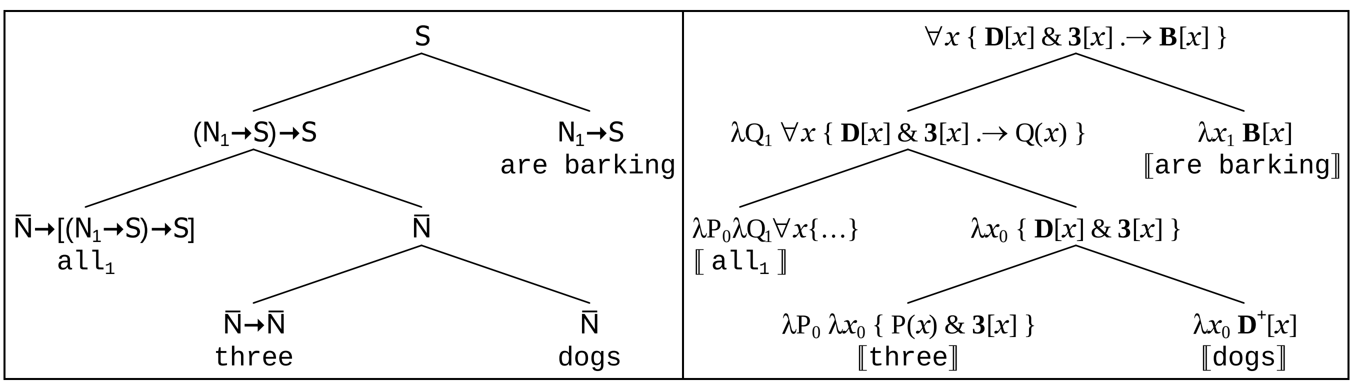
Thus, according to this analysis, the sentence says that every dog-entity "is" barking, which granting distributivity is the same as every individual dog (in the relevant domain) is barking. Note the *partitive use* of `'of'`, which is categorially rendered as follows.

type(`of`)	=	$N \rightarrow \bar{N}$
⟦`of`⟧	=	$\lambda y\, \lambda x_0\, \{x \sqsubseteq y\}$

Thus, `'of'` converts a proper-noun phrase into a common-noun phrase, pretty much reversing the effect of `'the'`.[9]

Item (3) is even less straightforward. First, a naïve analysis goes as follows.

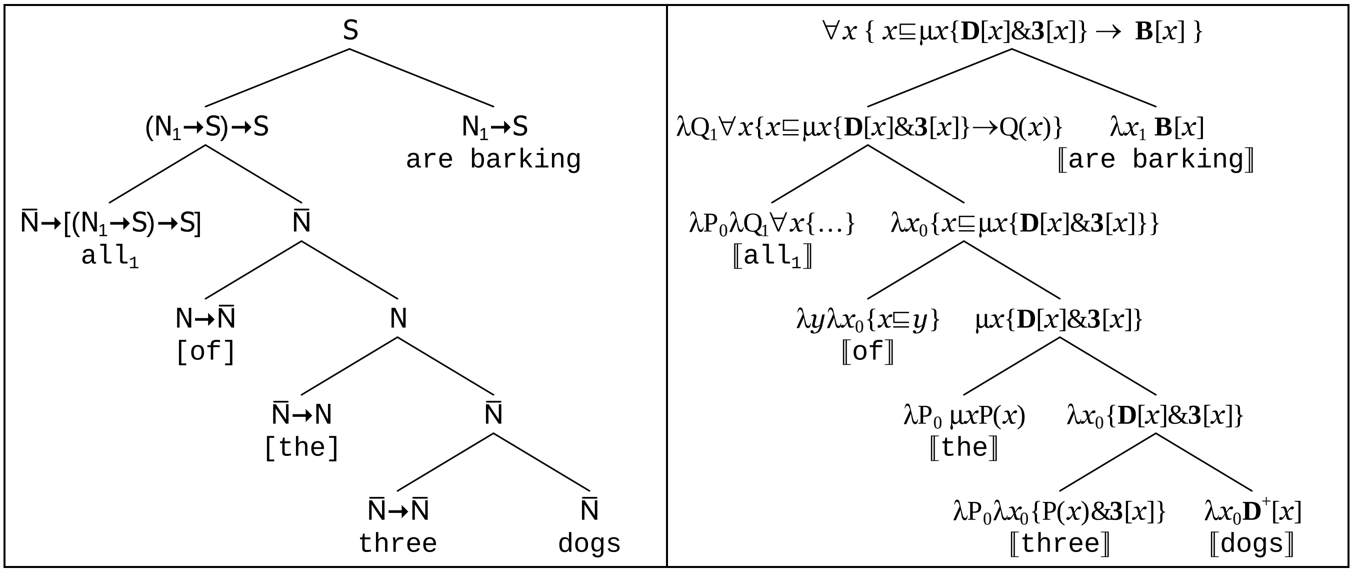
`all three dogs are barking`



This reads the sentence as saying that the members of every 3-membered set of dogs are barking. This is not a very plausible reading! A more plausible reading posits unpronounced material as in the following.

[9] Not perfectly, however, since ⟦`dogs`⟧ includes only pluralities, whereas ⟦`of the dogs`⟧ includes individuals as well as pluralities.

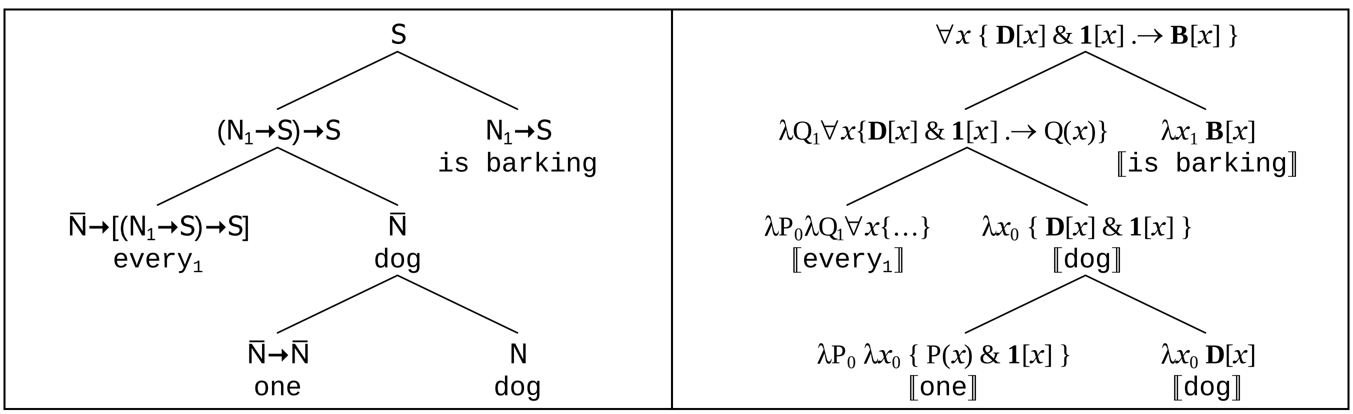
`all [of the] three dogs are barking`



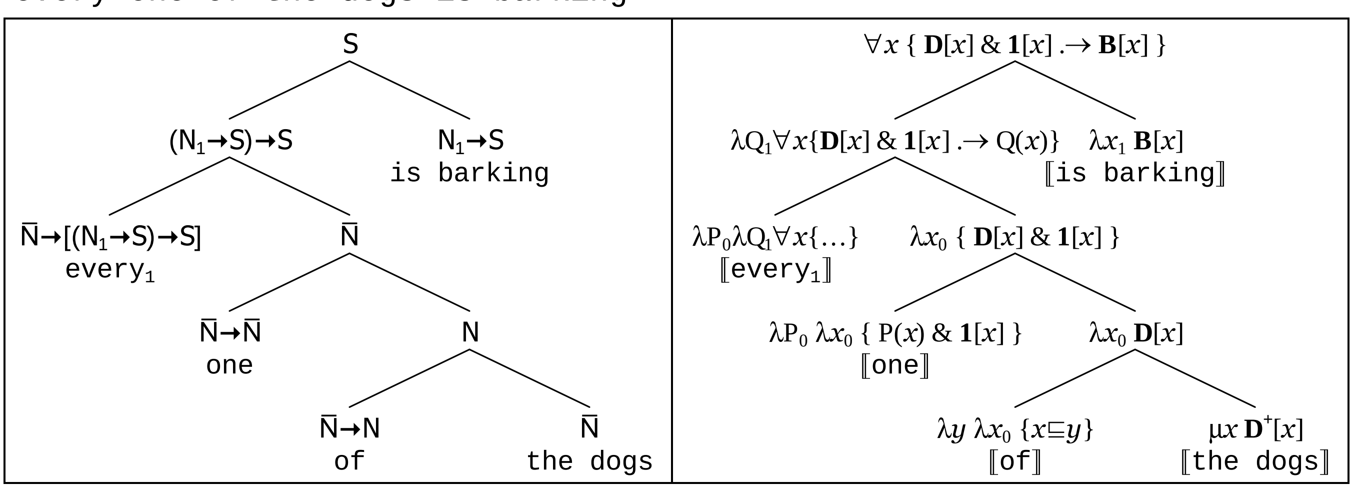
This reads the sentence as saying, in effect, that there are exactly three dogs, and they are *all* barking.

For the sake of comparison, we conclude this section with counterpart examples that employ the singular-quantifier '`every`'.

`every dog is barking`



`every one of the dogs is barking`



12. Exclusive Adverbs

Earlier, we proposed that the fundamental meaning of numerical QPs involves "at least" rather than "exactly". In order to convey "exactly", the sentence either needs appropriate contextual factors, or it needs explicit modifiers such as `‘exactly’`, `‘precisely’`, and `‘only’`. The later are examples of *exclusive adverbs*, which also include the following, among others.

```
just, merely, simply, solely, alone, uniquely, exclusively,
specifically, particularly, barely, scarcely
```

The general idea is that an exclusive adverb *focuses* attention on a phrase, and *conveys* that *other* possibilities are *excluded*.

What makes exclusive adverbs so interesting and perplexing is that the very same surface form can receive many different interpretations. My favorite example comes from a popular song from the 1950's, whose title and key lyric is:

I **only** have eyes for you

I think we all understand the sentiment of this song. But imagine a considerably more gruesome scenario in which the village butcher, who saves body parts for the infamous surgeon Dr. Frankenstein, one day declares:

sorry, Dr. Frankenstein, but today
I only have ***eyes*** for you.

It is also not hard to imagine a less flattering reading of the song in which the speaker (let's say, a boy) tells his girlfriend that *only he* has eyes for her. So, depending upon how it is intonated, the sentence can be paraphrased in the following three different manners.[10]

(1) I have eyes for **you**, but for no one **else**.
(2) I have **eyes** for you, but I have nothing **else** for you.
(3) **I** have eyes for you, but no one **else** does.

Given its focus-sensitivity, and given the variety of phrase types that `‘only’` can modify, the semantics of `‘only’` is subtle and difficult. We propose that `‘only’` is a *multi-categorial* adverb with the following *multi-type*.

type(`only`) = $(K \rightarrow S) \rightarrow (K \rightarrow S)$ [one for each type K]

Here, K is the type of the *focused phrase*, and $K \rightarrow S$ is the type of the *matrix* that contains the focused phrase. For example, in

I only have eyes for **you**

the focused phrase is `‘you’`, which has type N_2, and the matrix is `‘I have-eyes-for…’`, which has type $N_2 \rightarrow S$.

The semantics of `‘only’` is a bit complicated. Our **first approximation** goes as follows.

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[10] We can also concoct a reading in which `‘have’` is focused, and one in which `‘for’` is focused, but these are grammatically far-fetched.

$[\![\texttt{only}]\!] \quad = \quad \lambda\Phi\,\lambda\nu\,\{\,\Phi(\nu)\ \&\ {\sim}\exists\nu'\,\{\,\Phi(\nu')\ \&\ \nu'{\neq}\nu\,\}$
$\qquad\qquad\ \ \equiv \quad \lambda\Phi\,\lambda\nu\,\forall\nu'\,\{\,\Phi(\nu') \leftrightarrow \nu'{=}\nu\,\}$

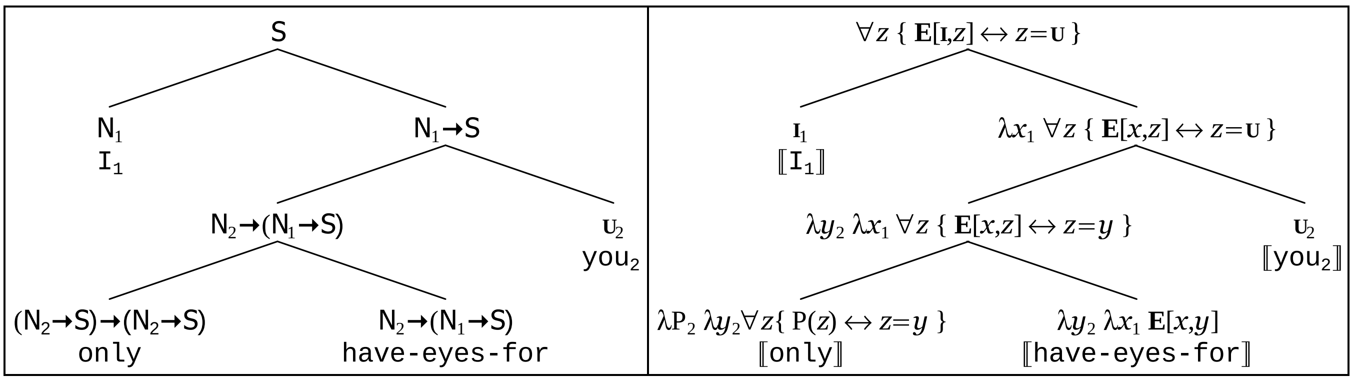
where $\Phi \in [\![\mathsf{K}{\rightarrow}\mathsf{S}]\!]$
$\nu,\nu' \in [\![\mathsf{K}]\!]$

For example, in our current example, $\mathsf{K}{=}\mathsf{N}_2$, and

$[\![\texttt{only}]\!] \quad = \quad \lambda P_2\,\lambda y_2\,\{\,P(y)\ \&\ {\sim}\exists z\{P(z)\ \&\ z{\neq}y\}\,\}$
$\qquad\qquad\ \ \equiv \quad \lambda P_2\,\lambda y_2\,\forall z\,\{\,P(z) \leftrightarrow z{=}y\,\}$

Thus, we have the following grammatical analysis, in which we treat ‘`have eyes for`’ as an idiomatic unit (lexical item).

`I only have-eyes-for` **`you`**



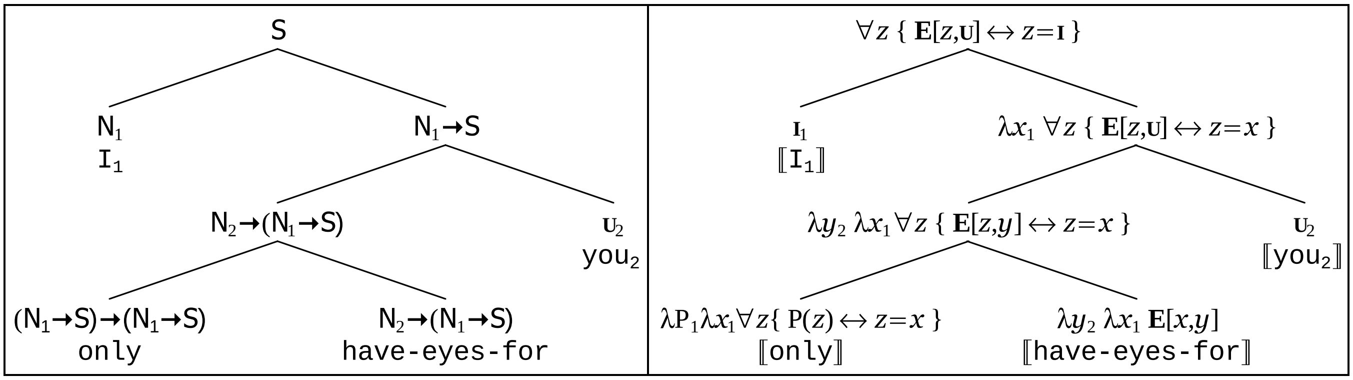
The top node in the semantic tree says that the speaker of the sentence (**I**) has eyes for the audience of the sentence (**U**), but for no one else. The key composition is underwritten by the following derivation.

(1)	$(N_2{\rightarrow}S){\rightarrow}(N_2{\rightarrow}S)$	1	Pr	$\lambda P_2\,\lambda y_2\,\forall z\,\{\,P(z) \leftrightarrow z{=}y\,\}$
(2)	$N_2{\rightarrow}(N_1{\rightarrow}S)$	2	Pr	$\lambda y_2\,\lambda x_1\,\mathbf{E}[x,y]$
(3)	N_2	3	As	y_2
(4)	N_1	4	As	x_1
(5)	$N_2{\rightarrow}S$	24	2,4,MP$_2$	$\lambda y_2\,\mathbf{E}[x,y]$
(6)	$N_2{\rightarrow}S$	124	1,5,→O	$\lambda y_2\,\forall z\,\{\,\mathbf{E}[x,z] \leftrightarrow z{=}y\,\}$
(7)	S	1234	3,6,→O	$\forall z\,\{\,\mathbf{E}[x,z] \leftrightarrow z{=}y\,\}$
(8)	$N_1{\rightarrow}S$	123	4-7,→I	$\lambda x_1\,\forall z\,\{\,\mathbf{E}[x,z] \leftrightarrow z{=}y\,\}$
(9)	$N_2{\rightarrow}(N_1{\rightarrow}S)$	12	3-8,→I	$\lambda y_2\,\lambda x_1\,\forall z\,\{\,\mathbf{E}[x,z] \leftrightarrow z{=}y\,\}$

Compare the above to the following alternative reading in which ‘I’ is focused. In this case, $\mathsf{K}{=}\mathsf{N}_1$, and

$[\![\texttt{only}]\!] \quad = \quad \lambda P_1\,\lambda x_1\,\forall z\,\{\,P(z) \leftrightarrow z{=}x\,\}$

I only have-eyes-for you



The top node in the semantic tree says that the speaker of the sentence (I) has eyes for the audience of the sentence (U), but no one else does. The key composition is underwritten by the following derivation.

(1)	$(N_1 \to S) \to (N_1 \to S)$	1	Pr	$\lambda P_1\, \lambda x_1\, \forall z \,\{\, P(z) \leftrightarrow z{=}x \,\}$
(2)	$N_2 \to (N_1 \to S)$	2	Pr	$\lambda y_2\, \lambda x_1\, \mathbf{E}[x,y]$
(3)	N_2	3	As	y_2
(4)	N_1	4	As	x_1
(5)	$(N_1 \to S) \to S$	14	1,4,MP_2	$\lambda P_1\, \forall z \,\{\, P(z) \leftrightarrow z{=}x \,\}$
(6)	$N_2 \to S$	124	2,5,TR	$\lambda y_2\, \forall z \,\{\, \mathbf{E}[z,y] \leftrightarrow z{=}x \,\}$
(7)	S	1234	3,6,→O	$\forall z \,\{\, \mathbf{E}[z,y] \leftrightarrow z{=}x \,\}$
(8)	$N_1 \to S$	123	4-8,→I	$\lambda x_1\, \forall z\, \forall z \,\{\, \mathbf{E}[z,y] \leftrightarrow z{=}x \,\}$
(9)	$N_2 \to (N_1 \to S)$	12	3-9,→I	$\lambda y_2\, \lambda x_1\, \forall z \,\{\, \mathbf{E}[z,y] \leftrightarrow z{=}x \,\}$

13. A Problem for Our Account, and the Proposed Correction

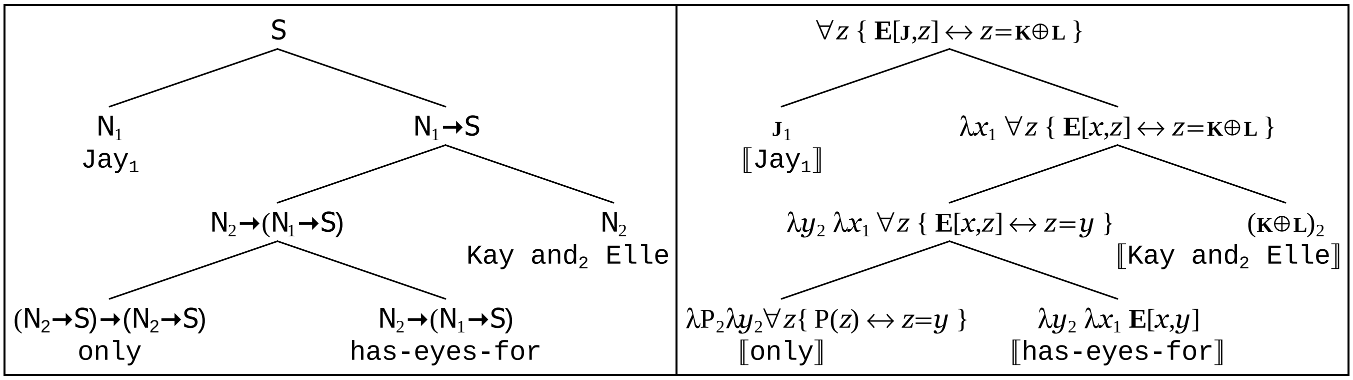
Our account of `'only'` works great for the two previous examples, but consider the following example,

`Jay only has eyes for Kay and Elle`

where we presume that `'Kay and Elle'` is the focused phrase.

First, we must face the issue of whether `'and'` is logical-conjunction or mereological-conjunction. If `'and'` is mereological, then `'Kay and Elle'` is a proper-noun phrase (N), and Kay-and-Elle is a plural entity, in which case we obtain the following analysis.

`Jay only has eyes for Kay and Elle`



According to this analysis, the sentence says that Jay has eyes for an entity if and only if that entity is (identical to) the plurality Kay⊕Elle. Therefore, since Kay ≠ Kay⊕Elle, Jay does not have eyes for Kay, and similarly he does not have eyes for Elle; he only has eyes for Kay-and-Elle (as a unit so to speak). Perhaps the envisaged circumstances are odd (or even kinky!), but it is nevertheless an admissible reading of the sentence.

On the other hand, if '`and`' is logical-conjunction, then '`Kay and Elle`' is a quantifier phrase [(N→S)→S], and Kay-and-Elle is a QP-object, in which case we categorially render '`only`' as follows.

type(`only`) = $[(N_2 \to S) \to S] \to [(N_2 \to S) \to S]$

⟦`only`⟧ = $\lambda \mathbb{Q}_2\, \lambda P_2\, \forall Q_2 \{\, \mathbb{Q}(Q) \leftrightarrow Q{=}P \,\}$

$\equiv \lambda \mathbb{Q}_2\, \lambda P_2 \{\, \mathbb{Q}(P)\ \&\ {\sim}\exists Q \{\, \mathbb{Q}(Q)\ \&\ Q{\neq}P \,\} \,\}$

where $\mathbb{Q}_2 \in [\![(N_2 \to S) \to S]\!]$

$P_2, Q_2 \in [\![N_2 \to S]\!]$

Unfortunately, this does not yield appropriate truth-conditions for the above sentence, since it implies that the above sentence says that Jay has eyes for neither Kay nor Elle (exercise).

In light of the difficulties faced by our original account of ⟦`only`⟧, we now consider the following **second approximation** account of ⟦`only`⟧.

⟦`only`⟧ = $\lambda \Phi\, \lambda \nu \{\, \Phi(\nu)\ \&\ {\sim}\exists \nu' \{\, \Phi(\nu')\ \&\ \nu' \perp \nu \,\} \,\}$

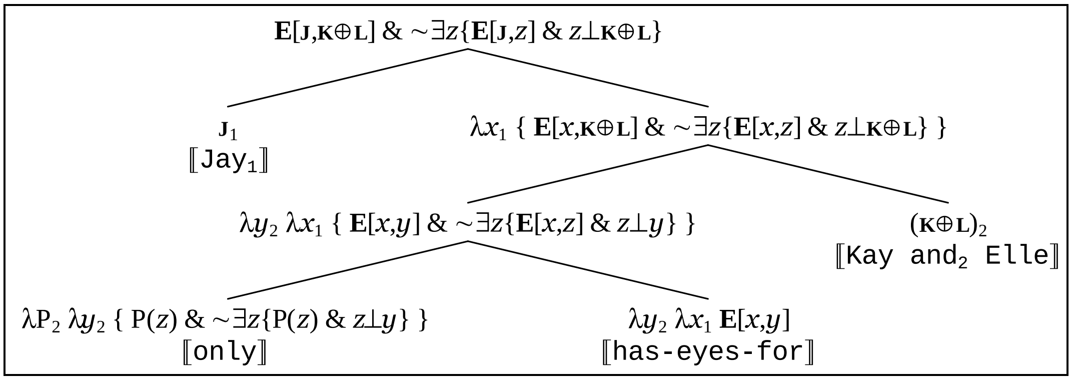
where $\Phi \in [\![K \to S]\!]$

$\nu, \nu' \in [\![K]\!]$

Note that this account is obtained from the first approximation by replacing '≠' by '⊥', where '⊥' refers to the **disjointness relation**, the exact definition of which varies from sort/type to sort/type.

For example, when applied exclusively to singular-entities, disjointness (⊥) coincides with non-identity (≠), and the revised account subsumes our earlier account. This is to be expected, since the earlier account makes correct predictions when applied to singular-entities. On the other hand, the revised account does not coincide with the original account when applied to plural-entities. For example, when applied to our standing example, we obtain the following analysis.

`Jay only has eyes for Kay and Elle`



This analysis reads the sentence as saying that Jay has eyes for the plural-entity Kay-and-Elle, but for no entity disjoint from Kay-and-Elle. So it does not say that Jay does not have eyes for Kay, or for Elle. Unfortunately, it does not say that Jay *does* have eyes for Kay or Elle, *unless* we further hypothesize that the **E**-relation is distributive.

14. Applying the New Definition to QPs

According to the most plausible reading , the sentence under scrutiny says that

Jay has eyes for Kay, *and*
Jay has eyes for Elle, *but*
Jay has eyes for *no one else*.

This suggests that the appropriate reading of `'and'` is the logical reading, in which case the focus of `'only'` is a QP, but the alternatives considered are not all QP-objects, but only those QP-objects that are "like" Kay, Elle, and Kay-and-Elle. Putting all this together, we obtain the following categorial rendering of `'only'`.

$$[\![\texttt{only}]\!] \quad = \quad \lambda\mathcal{Q}_2\,\lambda\mathbb{Q}_2\,\{\mathcal{U}(\mathbb{Q})\}\ \{\ \mathcal{Q}(\mathbb{Q})\ \&\ {\sim}\exists\mathbb{P}\{\mathcal{U}(\mathbb{P})\ \&\ \mathcal{Q}(\mathbb{P})\ \&\ \mathbb{P}\perp\mathbb{Q}\}\ \}$$

where

$$\mathcal{Q} \in [\![[(\mathsf{N}\rightarrow\mathsf{S})\rightarrow\mathsf{S}]\rightarrow\mathsf{S}]\!]$$

$$\mathbb{P},\mathbb{Q} \in [\![(\mathsf{N}\rightarrow\mathsf{S})\rightarrow\mathsf{S}]\!]$$

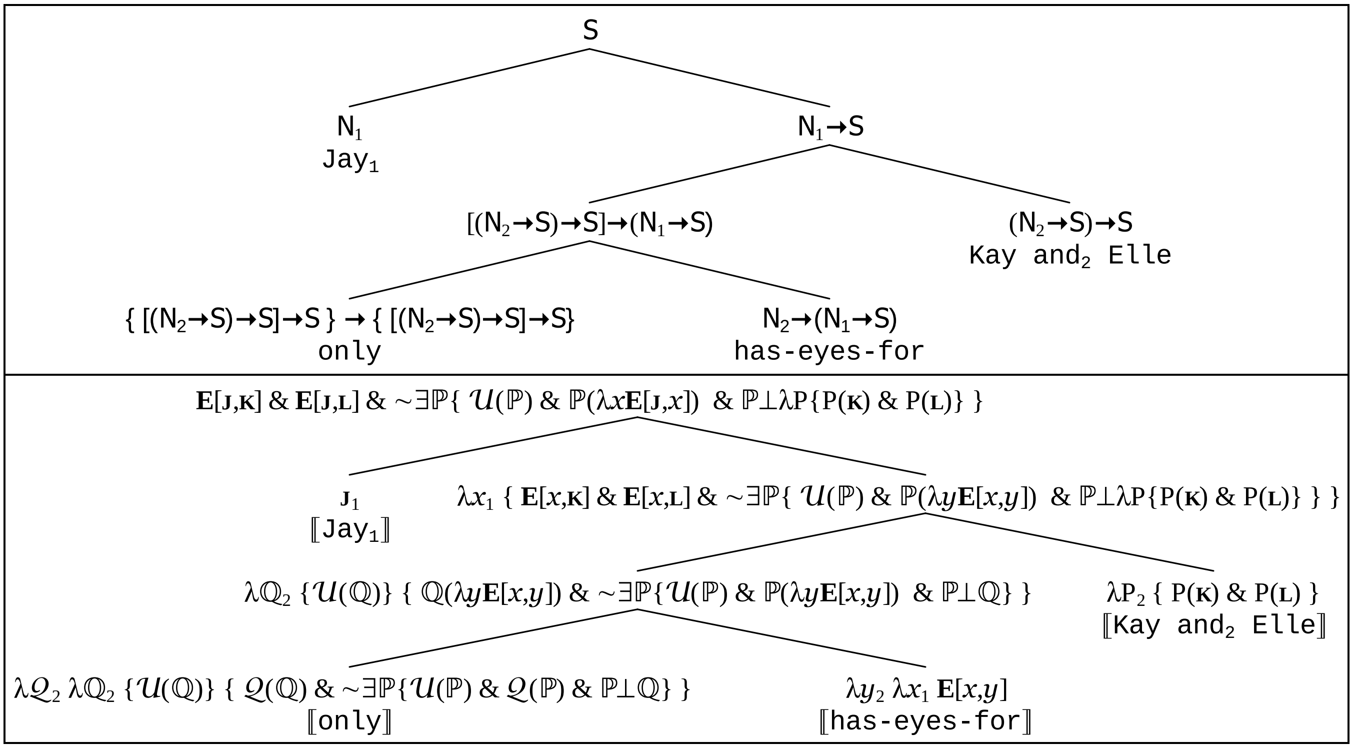
$$\mathcal{U}(\mathbb{P}) \quad =_{df} \quad \exists\mathrm{P}\ [\ \exists x\mathrm{P}(x)\ \&\ \mathbb{P} = \lambda\mathrm{Q}\forall x\{\mathrm{P}(x)\rightarrow\mathrm{Q}(x)\}\ ]$$

$$\mathbb{P}\perp\mathbb{Q} \quad =_{df} \quad {\sim}\exists\mathrm{P}\{\ \mathbb{P}(\mathrm{P})\ \&\ \mathbb{Q}(\mathrm{P})\ \}$$

Note the introduction of the additional restriction on the domain of QP-objects, which in particular restricts the domain to universal-QP-objects.

The following is the associated analysis, where $\mathbf{E}(\alpha) =_{df} \lambda\nu\mathbf{E}[\alpha,\nu]$ [α free for ν].

`Jay only has eyes for Kay and Elle`



First, note that, in the second composition, the input is of the appropriate sort for the functor. Next, since the top node quantifies over second-order predicates, it is not obvious what it says. As it turns out, it is equivalent to the following, which is exactly what we want.

$$\forall x \{ \mathbf{E}[\mathbf{J},x] \leftrightarrow.\ x{=}\mathbf{K} \vee x{=}\mathbf{L} \}$$

The equivalence is demonstrated in the following type-theory derivations, where '**E**' is short for '$\lambda x\mathbf{E}[\mathbf{J},x]$'.

(1)	$\forall x \{ \mathbf{E}(x) \leftrightarrow.\ x{=}\mathbf{K} \vee x{=}\mathbf{L} \}$	Pr
(2)	~~SHOW~~: $\mathbf{E}(\mathbf{K})$ & $\mathbf{E}(\mathbf{L})$ & $\sim\exists\mathbb{P}\{ \mathcal{U}(\mathbb{P})$ & $\mathbb{P}(\mathbf{E})$ & $\mathbb{P}\perp\lambda P\{P(\mathbf{K})$ & $P(\mathbf{L})\} \}$	3,4,SL
(3)	$\mathbf{E}(\mathbf{K})$ & $\mathbf{E}(\mathbf{L})$	1,IL
(4)	~~SHOW~~: $\sim\exists\mathbb{P}\{ \mathcal{U}(\mathbb{P})$ & $\mathbb{P}(\mathbf{E})$ & $\mathbb{P}\perp\lambda P\{P(\mathbf{K})$ & $P(\mathbf{L})\} \}$	ID
(5)	$\exists\mathbb{P}\{ \mathcal{U}(\mathbb{P})$ & $\mathbb{P}(\mathbf{E})$ & $\mathbb{P}\perp\lambda P\{P(\mathbf{K})$ & $P(\mathbf{L})\} \}$	As
(6)	$\mathcal{U}(\mathbb{P})$ & $\mathbb{P}(\mathbf{E})$ & $\mathbb{P}\perp\lambda P\{P(\mathbf{K})$ & $P(\mathbf{L})\} \}$	∃O
(7)	~~SHOW~~: ※	16,19,SL
(8)	$\exists P \{ \exists xP(x)$ & $\mathbb{P} = \lambda Q\forall x\{P(x)\to Q(x)\} \}$	6a,Def $\mathcal{U}$
(9)	$\exists xP(x)$	∃&O
(10)	$P(a)$	∃O
(11)	$\mathbb{P} = \lambda Q\, \forall x \{ P(x) \to Q(x) \}$	10, ∃&O
(12)	$\lambda Q\, \forall x \{ P(x) \to Q(x) \} \perp \lambda P\{P(\mathbf{K})$ & $P(\mathbf{L})\} \}$	6c,11,IL
(13)	$\lambda P\{P(\mathbf{K})$ & $P(\mathbf{L})\} = \lambda Q\forall x \{ x{=}\mathbf{K} \vee x{=}\mathbf{L} .\!\to Q(x) \}$	λC
(14)	$\lambda Q\, \forall x \{ P(x) \to Q(x) \} \perp \lambda Q\, \forall x \{ x{=}\mathbf{K} \vee x{=}\mathbf{L} .\!\to Q(x) \}$	12,13,IL
(15)	$\sim\exists x \{ P(x)$ &. $x{=}\mathbf{K} \vee x{=}\mathbf{L} \}$	14,Def ⊥
(16)	$a{\neq}\mathbf{K}$ & $a{\neq}\mathbf{L}$	10,15,QL
(17)	$\forall x \{ P(x) \to \mathbf{E}(x) \}$	6b,8b,IL,λC
(18)	$\mathbf{E}(a)$	10,17,QL
(19)	$a{=}\mathbf{K} \vee a{=}\mathbf{L}$	1,18,QL

(1)	$\mathbf{E}(\mathbf{K}) \,\&\, \mathbf{E}(\mathbf{L}) \,\&\, \sim\exists\mathbb{P}\{\, \mathcal{U}(\mathbb{P}) \,\&\, \mathbb{P}(\mathbf{E}) \,\&\, \mathbb{P}\perp\lambda P\{P(\mathbf{K}) \,\&\, P(\mathbf{L})\}\,\}$	Pr
(2)	~~SHOW~~: $\forall x\,\{\, \mathbf{E}(x) \leftrightarrow.\; x{=}\mathbf{K} \vee x{=}\mathbf{L} \,\}$	1a,1b,3,QL
(3)	~~SHOW~~: $\forall x\,\{\, \mathbf{E}(x) \rightarrow.\; x{=}\mathbf{K} \vee x{=}\mathbf{L} \,\}$	UCD
(4)	$\mathbf{E}(a)$	As
(5)	~~SHOW~~: $a{=}\mathbf{K} \vee a{=}\mathbf{L}$	DD
(6)	$[\lambda P\{P(a)\}](\mathbf{E})$	4,λC
(7)	$P(a) \leftrightarrow \forall x\{x{=}a \rightarrow P(x)\}$	IL
(8)	$\lambda P\{P(a)\} = \lambda P\forall x\{\, [\lambda x\{x{=}a\}](x) \rightarrow P(x) \,\}$	7,λC
(9)	$\mathcal{U}(\, \lambda P\{P(a)\}\,)$	8,QL,Def $\mathcal{U}$
(10)	$\lambda P\{P(\mathbf{K}) \,\&\, P(\mathbf{L})\} \not\perp \lambda P\{P(a)\}$	1c,6,9,QL
(11)	$\lambda P\{P(\mathbf{K}) \,\&\, P(\mathbf{L})\} \;=\; \lambda Q\forall x\,\{\, x{=}\mathbf{K} \vee x{=}\mathbf{L} \,.\!\rightarrow Q(x) \,\}$	λC
(12)	$\lambda P\{P(a)\} \;=\; \lambda Q\forall x\,\{\, x{=}a \rightarrow Q(x) \,\}$	λC
(13)	$\lambda Q\forall x\,\{\, x{=}\mathbf{K} \vee x{=}\mathbf{L} \,.\!\rightarrow Q(x) \,\} \;\not\perp\; \lambda Q\forall x\,\{\, x{=}a \rightarrow Q(x) \,\}$	10-12,IL
(14)	$\exists x\,\{\, x{=}\mathbf{K} \vee x{=}\mathbf{L} \,\&\, x{=}a \,\}$	13,Def ⊥
(15)	$a{=}\mathbf{K} \vee a{=}\mathbf{L}$	14,IL

15. What happens when the focus is a CNP?

Consider the following example.

```
only poisonous snakes are dangerous
```

Depending on the focus of ‘`only`’, this is ambiguous among the following.

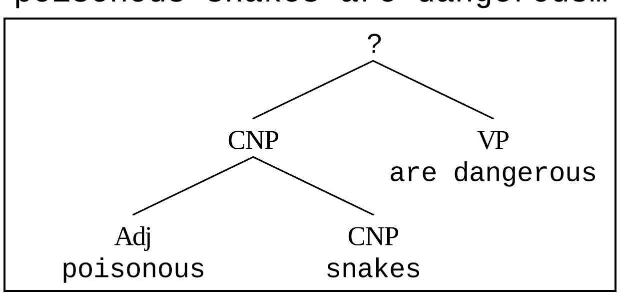
`poisonous` `snakes are dangerous,`
`but non-poisonous snakes are not dangerous`

`poisonous` **`snakes`** `are dangerous,`
`but other poisonous things are not dangerous`

`poisonous-snakes` `are dangerous,`
`but other things are not dangerous`

We first consider the last one, since its overall structure is the simplest. We consider the other two in later sections. The first thing to notice is that the paraphrase is not well-formed by simple categorial formation rules.

`poisonous snakes are dangerous…`



We need an NP to serve as the subject of the VP, but what we have is a CNP. There seems to be a missing determiner – but which one? When we say that poisonous snakes are dangerous, do we mean *all* poisonous snakes, *most* poisonous snakes, *some* poisonous snakes, or what?

There does not seem to be a generally agreed upon answer to this question. However, we propose that, at least in the presence of `'only'`, the missing determiner is `'some'`. So, for example, the sentence

`only men play NFL football`

says that *some* men play NFL football,[11] but *no non*-men play NFL football. It most certainly does not say that *all*, or *most*, or *generally*, men play NFL football.

How does this fit into our account of `'only'`? The focus phrase is `'men'` which is a CNP, so the applicable sub-category of `'only'` is:

$(\bar{N}\to S)\to(\bar{N}\to S)$

and the associated interpretation of `'only'` is:

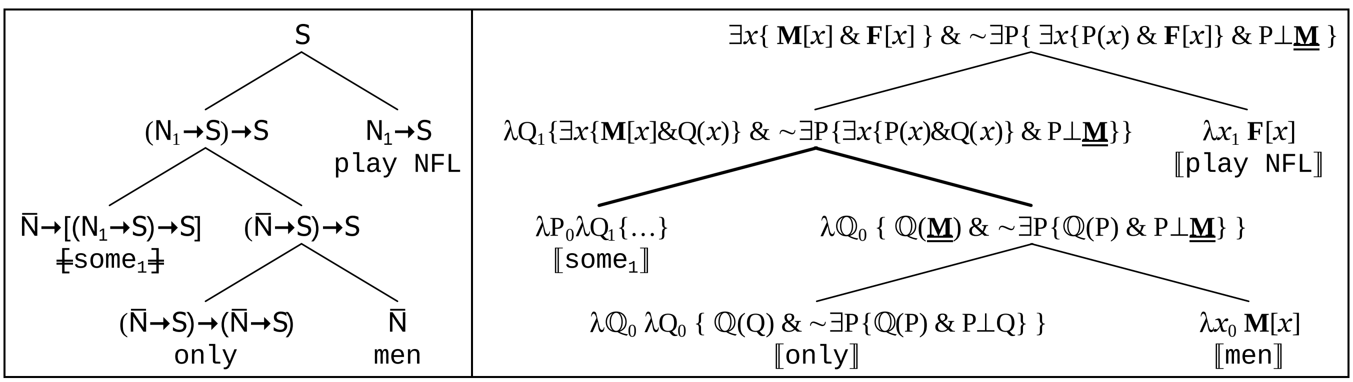
$\lambda\mathbb{Q}_0\,\lambda Q_0\,\{\ \mathbb{Q}(Q)\ \ \&\ \ {\sim}\exists P\{\mathbb{Q}(P)\ \&\ P\perp Q\}\ \}$

where

$P\perp Q \quad =_{df} \quad {\sim}\exists x\ \{\ P(x)\ \&\ Q(x)\ \}$

With this account of ⟦only⟧ in hand, we offer the following analysis. Note the covert determiner `'some'`. Also note that, for the sake of brevity, we occasionally write '$\underline{\underline{\mathbf{M}}}$' in place of '$\lambda x\mathbf{M}[x]$'.

`only men play NFL football`



The key composition is underwritten by the following derivation.

(1)	$(\bar{N}\to S)\to S$	1	Pr	$\lambda\mathbb{Q}_0\ \{\ \mathbb{Q}(\underline{\underline{\mathbf{M}}})\ \&\ {\sim}\exists P\{\ \mathbb{Q}(P)\ \&\ P\perp\underline{\underline{\mathbf{M}}}\}\ \}$
(2)	$\bar{N}\to[(N_1\to S)\to S]$	2	Pr	$\lambda P_0\,\lambda Q_1\,\exists x\{\ P(x)\ \&\ Q(x)\ \}$
(3)	$N_1\to S$	3	As	Q_1
(4)	$\bar{N}\to S$	23	2,3,MP$_2$	$\lambda P_0\,\exists x\{\ P(x)\ \&\ Q(x)\ \}$
(5)	S	123	1,4,→O	$\exists x\{\mathbf{M}[x]\&Q(x)\}\ \&\ {\sim}\exists P\{\ \exists x\{P(x)\&Q(x)\}\ \&\ P\perp\underline{\underline{\mathbf{M}}}\ \}$
(6)	$(N_1\to S)\to S$	12	3-5,→I	$\lambda Q_1\,\exists x\{\mathbf{M}[x]\&Q(x)\}\ \&\ {\sim}\exists P\{\ \exists x\{P(x)\&Q(x)\}\ \&\ P\perp\underline{\underline{\mathbf{M}}}\ \}$

Now, let us examine the top node more carefully. It clearly contains the information that some men play NFL football. The question is whether it also says that no one else does. In other words, is the top node equivalent to the following.

[11] Note carefully, however, that there is also a weak sense of `'only'`; see Section 18.

∃x{ **M**[x] & **F**[x] } & ~∃x { ~**M**[a] & **F**[a] }

By way of answering this question, we offer the following type-theory derivations.

(1)	~∃P{ ∃x{P(x) & **F**[x]} & P⊥**M** }	Pr
(2)	~~SHOW~~: ~∃x { ~**M**[a] & **F**[a] }	ID
(3)	∃x { ~**M**[a] & **F**[a] }	As
(4)	~**M**[a] & **F**[a]	2,∃O
(5)	~~SHOW~~: ※	4a,12,SL
(6)	[λx(x=a)](a)	IL,λC
(7)	∃x{ [λx(x=a)](x) & **F**[x] }	4b,6,QL
(8)	λx(x=a) ⊥̸ **M**	1b,7,QL
(9)	∃x{ [λx(x=a)](x) & **M**[x] }	8,Def ⊥
(10)	[λx(x=a)](b) & **M**[b]	9,∃O
(11)	b=a	9a,λC
(12)	**M**[a]	9b,10,IL

(1)	~∃x { ~**M**[x] & **F**[x] }	Pr
(2)	~~SHOW~~: ~∃P{ ∃x{P(x) & **F**[x]} & P⊥**M** }	ID
(3)	∃P{ ∃x{P(x) & **F**[x]} & P⊥**M** }	As
(4)	~~SHOW~~: ※	6b,8,SL
(5)	∃x{P(x) & **F**[x]} & P⊥**M**	3,∃O
(6)	P(a) & **F**[a]	4a,∃O
(7)	~∃x { P(x) & **M**[x] }	3b,Def ⊥
(8)	~**M**[a]	6a,7,QL
(8)	~**F**[a]	1,7,QL

16. What happens when the focus is an Adjective?

We next consider an example in which the focus is an adjective, as in the following example.

`only` **`poisonous`** `snakes are dangerous`

In this case the focus-type is N̄→N̄, and the matrix-type is (N̄→N̄)→S, and the associated interpretation of '`only`' is as follows.

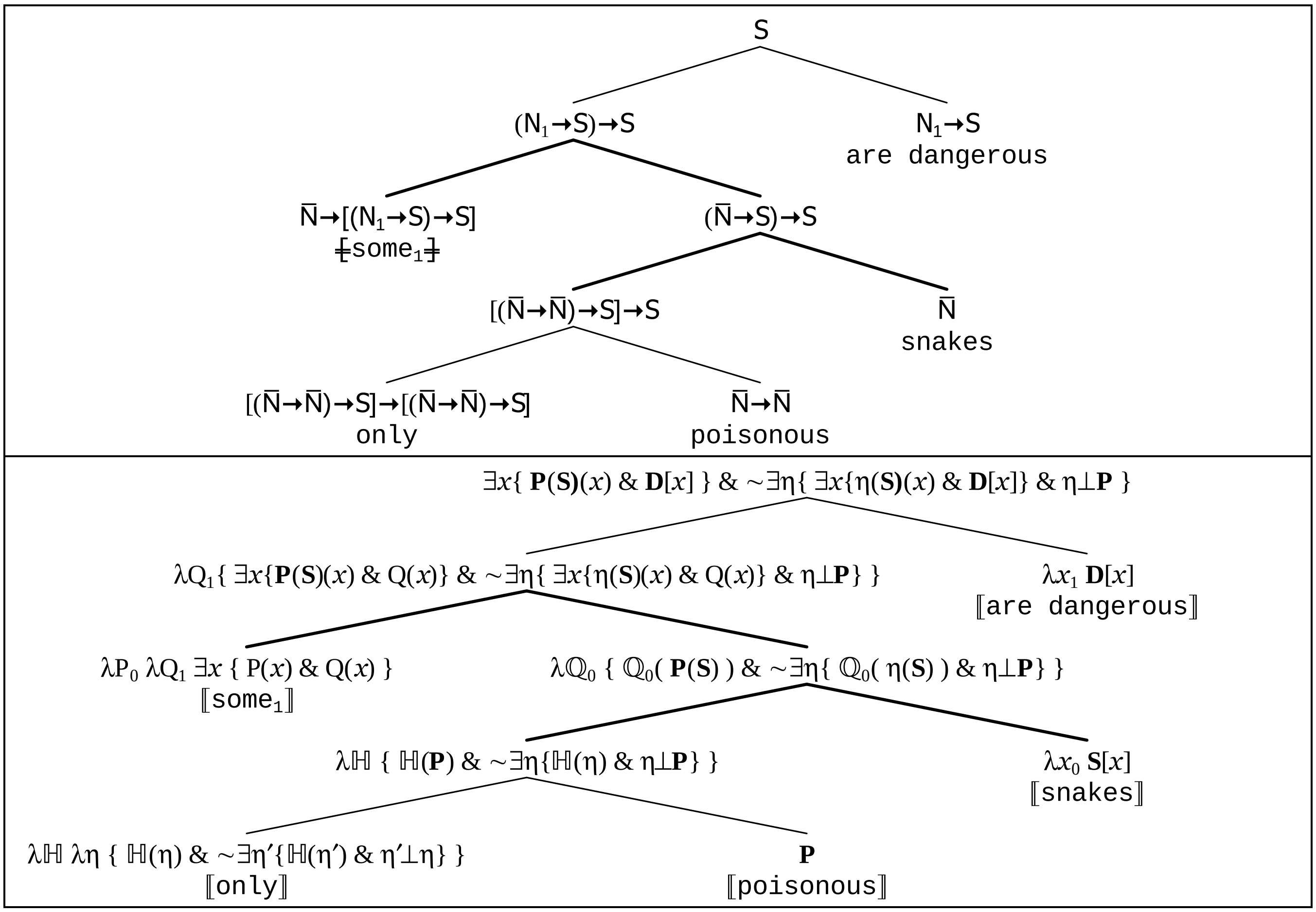
⟦only⟧	=	λℍ λη { ℍ(η) & ~∃η′{ℍ(η′) & η′⊥η} }
where		ℍ ∈ ⟦(N̄→N̄)→S⟧
		η ∈ N̄→N̄
		η′⊥η $=_{df}$ ~ ∃P_0 ∃x_0 { η′(P_0)(x_0) & η(P_0)(x_0) }

Note carefully that the construction only makes sense for *subsective* adjectives,[12] so in particular, the η-variables range over subsective adjectives. Compare the following pieces of nonsense involving non-subsective adjectives.[13]

only **alleged** snakes are dangerous
only **former** snakes are dangerous

The following is the associated grammatical analysis.

only **poisonous** snakes are dangerous



The key compositions are underwritten by the following derivations.

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[12] Basically, η is subsective if $\eta(S) \subseteq S$, for every set S of entities. For example, `'poisonous'` is subsective since every poisonous N is an N.

[13] Note that these make sense if `'alleged snakes'` and `'former snakes'` are focused, although they describe rather odd worlds.

(1)	$[(\bar{N}\to\bar{N})\to S]\to S$	1	Pr	$\lambda\mathbb{H}\ \{\ \mathbb{H}(\mathbf{P})\ \&\ {\sim}\exists\eta\{\mathbb{H}(\eta)\ \&\ \eta\perp\mathbf{P}\}\ \}$
(2)	$\bar{N}$	2	Pr	$\lambda x_0\ \mathbf{S}[x]$
(3)	$\bar{N}\to S$	3	As	$\mathbb{Q}_0$
(4)	$\bar{N}\to\bar{N}$	4	As	η
(5)	$\bar{N}$	24	2,4,→O	$\eta(\mathbf{S})$
(6)	S	234	3,5,→O	$\mathbb{Q}_0(\ \eta(\mathbf{S})\ )$
(7)	$(\bar{N}\to\bar{N})\to S$	23	4-6,→I	$\lambda\eta\ \mathbb{Q}_0(\ \eta(\mathbf{S})\ )$
(8)	S	123	1,7,→O	$\mathbb{Q}_0(\ \mathbf{P}(\mathbf{S})\ )\ \&\ {\sim}\exists\eta\{\ \mathbb{Q}_0(\ \eta(\mathbf{S})\ )\ \&\ \eta\perp\mathbf{P}\}$
(9)	$(\bar{N}\to S)\to S$	12	3-8,→I	$\lambda\mathbb{Q}_0\ \{\ \mathbb{Q}_0(\ \mathbf{P}(\mathbf{S})\ )\ \&\ {\sim}\exists\eta\{\ \mathbb{Q}_0(\ \eta(\mathbf{S})\ )\ \&\ \eta\perp\mathbf{P}\}\ \}$

(1)	$(\bar{N}\to S)\to S$	1	Pr	$\lambda\mathbb{Q}_0\ \{\ \mathbb{Q}_0(\ \mathbf{P}(\mathbf{S})\ )\ \&\ {\sim}\exists\eta\{\ \mathbb{Q}_0(\ \eta(\mathbf{S})\ )\ \&\ \eta\perp\mathbf{P}\}\ \}$
(2)	$\bar{N}\to[(N_1\to S)\to S]$	2	Pr	$\lambda P_0\ \lambda Q_1\ \exists x\ \{\ P(x)\ \&\ Q(x)\ \}$
(3)	$N_1\to S$	3	As	Q_1
(4)	$\bar{N}\to S$	23	2,3,MP_2	$\lambda P_0\ \exists x\ \{\ P(x)\ \&\ Q(x)\ \}$
(5)	S	123	1,4,→O	$\exists x\{\mathbf{P}(\mathbf{S})(x)\ \&\ Q(x)\}\ \&\ {\sim}\exists\eta\{\exists x\{\eta(\mathbf{S})(x)\ \&\ Q(x)\}\ \&\ \eta\perp\mathbf{P}\}$
(6)	$(N_1\to S)\to S$	12	3-5,→I	$\lambda Q_1\ \exists x\{\mathbf{P}(\mathbf{S})(x)\ \&\ Q(x)\}\ \&\ {\sim}\exists\eta\{\exists x\{\eta(\mathbf{S})(x)\ \&\ Q(x)\}\ \&\ \eta\perp\mathbf{P}\}$

Let us now examine the top node which is

$$\exists x\{\ \mathbf{P}(\mathbf{S})(x)\ \&\ \mathbf{D}[x]\ \}\ \&\ {\sim}\exists\eta\{\ \exists x\{\eta(\mathbf{S})(x)\ \&\ \mathbf{D}[x]\}\ \&\ \eta\perp\mathbf{P}\ \}$$

This clearly says that some poisonous snakes are dangerous. The question is whether it denies that any non-poisonous snakes are dangerous,[14] which is to say whether it is equivalent to the following.

$$\exists x\{\ \mathbf{P}(\mathbf{S})(x)\ \&\ \mathbf{D}[x]\ \}\ \&\ {\sim}\exists x\ \{\ \mathbf{S}[x]\ \&\ {\sim}\mathbf{P}(\mathbf{S})(x)\ \&\ \mathbf{D}[x]\ \}$$

This is settled in the following type-theory derivations.

(1)	${\sim}\exists\eta\ \{\ \exists x\{\eta(\mathbf{S})(x)\ \&\ \mathbf{D}[x]\}\ \&\ \eta\perp\mathbf{P}\ \}$	Pr
(2)	~~SHOW~~: ${\sim}\exists x\ \{\ \mathbf{S}[x]\ \&\ {\sim}\mathbf{P}(\mathbf{S})(x)\ \&\ \mathbf{D}[x]\ \}$	ID
(3)	$\exists x\ \{\ \mathbf{S}[x]\ \&\ {\sim}\mathbf{P}(\mathbf{S})(x)\ \&\ \mathbf{D}[x]\ \}$	As
(4)	~~SHOW~~: ※	13b,13c,SL
(5)	$\mathbf{S}[a]\ \&\ {\sim}\mathbf{P}(\mathbf{S})(a)\ \&\ \mathbf{D}[a]$	3,∃O
(6)	[let] $\eta' = \lambda Q_0\ \lambda x_0\ \{\ Q(x)\ \&\ {\sim}\mathbf{P}(Q_0)(x_0)\ \}$	λC
(7)	$\eta'(\mathbf{S}) = \lambda x_0\ \{\ \mathbf{S}[x]\ \&\ {\sim}\mathbf{P}(\mathbf{S})(x_0)\ \}$	6,IL
(8)	$\eta'(\mathbf{S})(a)$	5a,b,7,IL,λC
(9)	$\exists x\{\eta'(\mathbf{S})(x)\ \&\ \mathbf{D}[x]$	5c,8,QL
(10)	$\eta\not\perp\mathbf{P}$	1,9,QL
(11)	$\exists Q_0\ \exists x_0\ \{\ \eta'(Q_0)(x_0)\ \&\ \mathbf{P}(Q_0)(x_0)\ \}$	10,Def ⊥
(12)	$\eta'(Q_0)(b_0)\ \&\ \mathbf{P}(Q_0)(b_0)$	11,∃O
(13)	$Q(b)\ \&\ {\sim}P(Q_0)(b_0)\ \&\ \mathbf{P}(Q_0)(b_0)$	6,12,IL,λC

[14] Note carefully the difference between a non-poisonous snake and a non(poisonous snake). Here, it is critical to the semantics that the η-variables range over subsective adjectives.

(1)	$\sim\exists x \{ \mathbf{S}[x] \mathbin{\&} \sim\mathbf{P}(\mathbf{S})(x) \mathbin{\&} \mathbf{D}[x] \}$	Pr
(2)	~~SHOW~~: $\sim\exists\eta \{ \exists x\{\eta(\mathbf{S})(x) \mathbin{\&} \mathbf{D}[x]\} \mathbin{\&} \eta\perp\mathbf{P} \}$	ID
(3)	$\exists\eta\{ \exists x\{\eta(\mathbf{S})(x) \mathbin{\&} \mathbf{D}[x]\} \mathbin{\&} \eta\perp\mathbf{P} \}$	As
(4)	~~SHOW~~: ⨳	5b,9,SL
(5)	$\exists x\{\eta(\mathbf{S})(x) \mathbin{\&} \mathbf{D}[x]\} \mathbin{\&} \eta\perp\mathbf{P}$	3,∃O
(6)	$\eta(\mathbf{S})(a) \mathbin{\&} \mathbf{D}[a]$	5a,∃O
(7)	$\sim\mathbf{P}(\mathbf{S})(a)$	6b,5a,Def ⊥
(8)	$\mathbf{S}[a]$	6a,η is subsective
(9)	$\sim\mathrm{D}[a]$	1,7,8,QL

17. 'The Only'

One of the more perplexing combinations involves `'the'` and `'only'`. First, note that the following are not equivalent.

> **`the only`** `people I respect are kind`
> **`only the`** `people I respect are kind`

So, clearly `'the only'` ≠ `'only the'`, so we have to be careful in constructing our categorial analysis.

We propose that, when it appears in the phrase `'the only'`, the role of `'only'` is largely emphatic, similar to how `'unique'` operates inside `'the unique'`. In other words, `'the'` does the real work, and `'only'` simply provides emphasis.

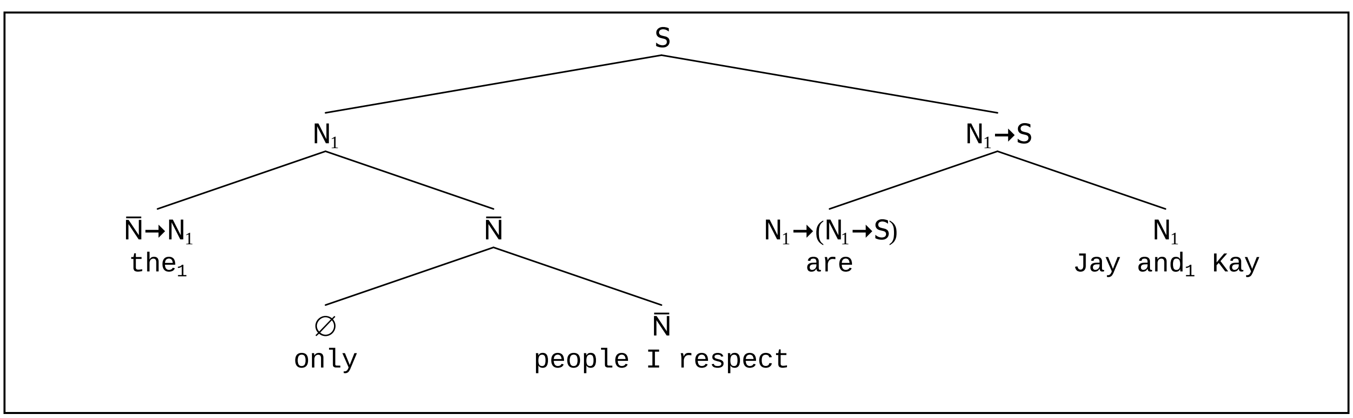
Let us apply this proposal to a few examples. First consider the following

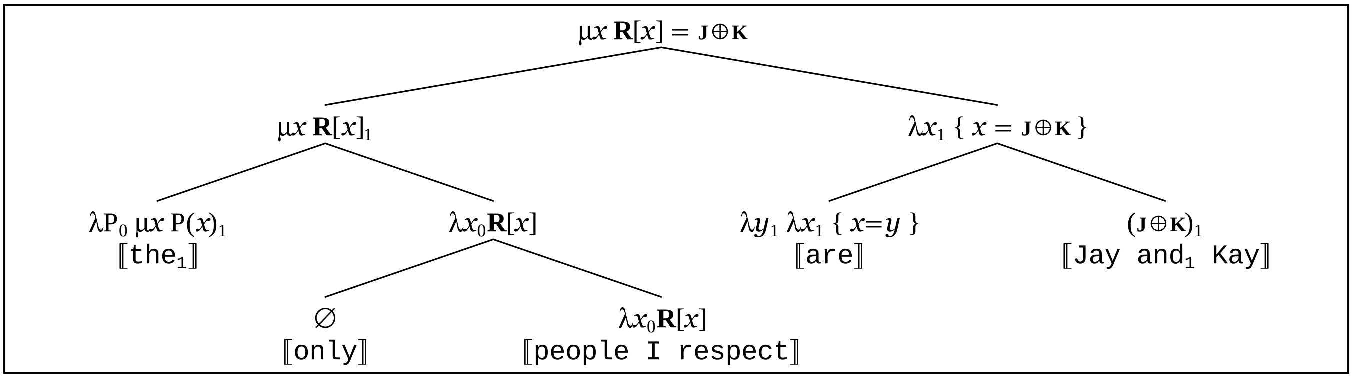
> `the only` **`people I respect`** `are Jay and Kay`

which is equivalent to

> `Jay and Kay are the only` **`people I respect`**

which suggests that `'be'` is transitive. This is further clarified in the following.





According to this analysis, the sentence says that the maximal count entity containing all the people I respect is identical to the count-entity Jay⊕Kay. By standard mereological reasoning, granting that *respect* is distributive, this is equivalent to saying that I respect an individual if and only if that individual is Jay or Kay.

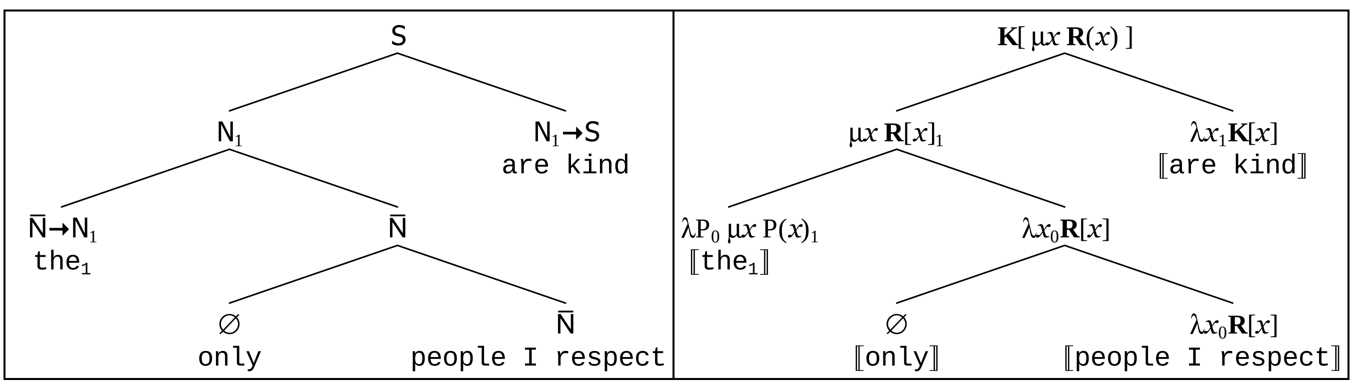
Now, back to our original example.

`the only` **`people I respect`** `are kind`

Note that this is not equivalent to

`kind are the only` **`people I respect`**

since the latter is ill-formed. This suggests that `'be'` is a copula. This is further clarified in the following.



According to this analysis, the sentence says that the maximal count-entity containing all the people I respect is counted among the entities that are kind. This entails that everyone I respect is kind, provided we read the predicate `'is kind'` as distributive. This is strongly encouraged by the presence of the word `'only'` in the original phrase.

18. ‘Only if’ and the Weak Sense of ‘Only’

We next consider how ‘`only`’ interacts with ‘`if`’ in phrases such as:

```
I will get into Cal Tech only if I ace all my exams
a number is even only if it is divisible by two
```

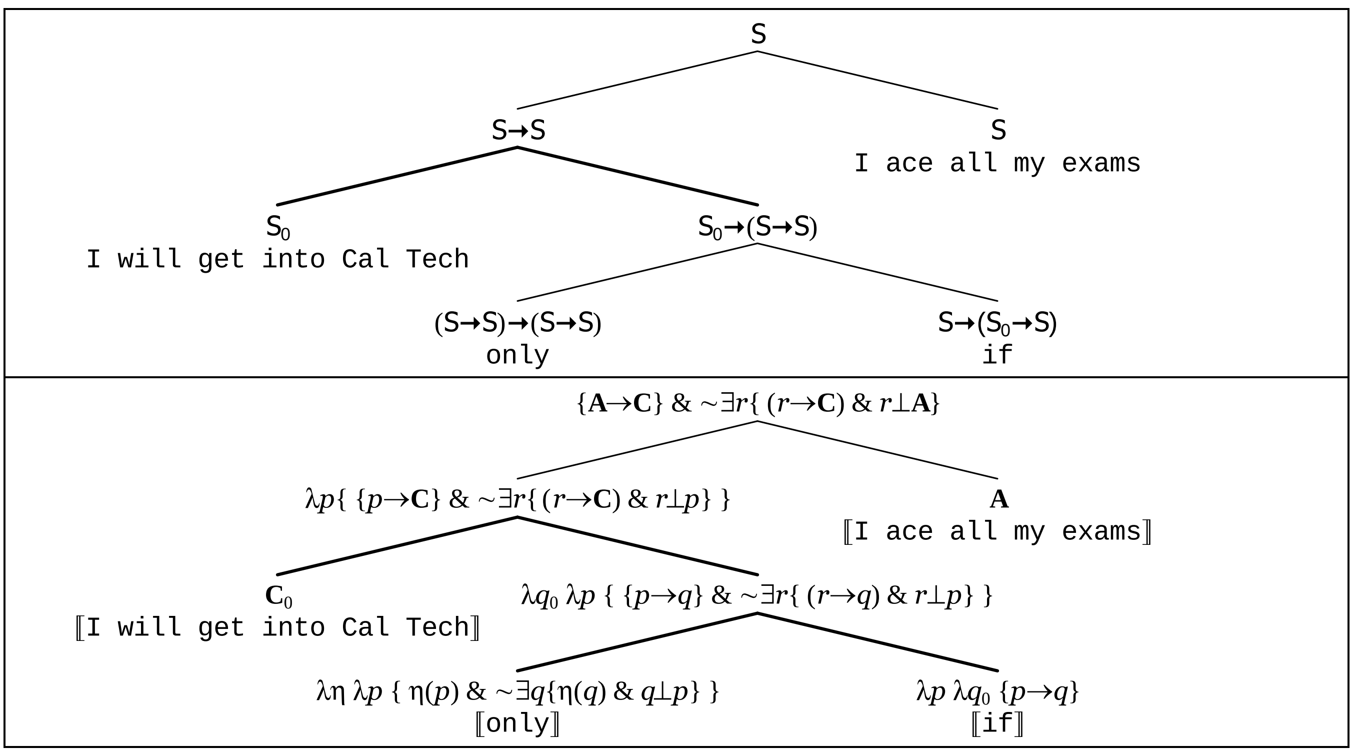
We propose that, in ‘`only if`’ clauses, the focus of ‘`only`’ is the antecedent (i.e., the complement of ‘`if`’). Thus, applying our general analysis of ‘`only`’, we have the following.

type(`only`)	=	$(S\rightarrow S)\rightarrow(S\rightarrow S)$
⟦`only`⟧	=	$\lambda\eta\, \lambda p\ \{\ \eta(p)\ \&\ {\sim}\exists q\{\eta(q)\ \&\ q\perp p\}\ \}$
where		$\eta \in [\![S\rightarrow S]\!]$
		$q\perp p\ =_{df}\ q\&{\sim}p$

In other words,

The following is an example analysis.

`I will get into Cal Tech only if I ace all my exams`



(1)	$(S\rightarrow S)\rightarrow(S\rightarrow S)$	1	Pr	$\lambda\eta\ \lambda p\ \{\ \eta(p)\ \&\ {\sim}\exists r\{\eta(r)\ \&\ r\perp p\}\ \}$
(2)	$S\rightarrow(S_0\rightarrow S)$	2	Pr	$\lambda p\ \lambda q_0\ \{p\rightarrow q\}$
(3)	S_0	3	As	q_0
(4)	$S\rightarrow S$	23	2,3,MP_2	$\lambda p\ \{p\rightarrow q\}$
(5)	$S\rightarrow S$	123	1,4,→O	$\lambda p\ \{\ \{p\rightarrow q\}\ \&\ {\sim}\exists r\{\ (r\rightarrow q)\ \&\ r\perp p\}\ \}$
(6)	$S_0\rightarrow(S\rightarrow S)$	12	3-5,→I	$\lambda q_0\ \lambda p\ \{\ \{p\rightarrow q\}\ \&\ {\sim}\exists r\{\ (r\rightarrow q)\ \&\ r\perp p\}\ \}$

The following type-theory derivations demonstrate that the top node is equivalent to the following.

A↔**C**

(1)	{**A**→**C**} & ~∃*r*{ (*r*→**C**) & *r*⊥**A**}	Pr
(2)	~~SHOW~~: **A**↔**C**	1a,3,SL
(3)	~~SHOW~~: **C**→**A**	6,SL
(4)	**C**→**C**	SL
(5)	**C** ∠̸ **A**	1b,4,QL
(6)	~(**C** & ~**A**)	5, Def ⊥

(1)	**A**↔**C**	Pr
(2)	~~SHOW~~: {**A**→**C**} & ~∃*r*{ (*r*→**C**) & *r*⊥**A**}	1,SL,3,SL
(3)	~~SHOW~~: ~∃*r*{ (*r*→**C**) & *r*⊥**A**}	ID
(4)	∃*r*{ (*r*→**C**) & *r*⊥**A**}	As
(5)	~~SHOW~~: ※	6a,7,SL
(6)	B→**C** & B⊥**A**	4,∃O
(7)	B & ~**A**	6b,Def ⊥

Now, here is the problem. It does not seem that `‘only if’` is equivalent to `‘if and only if’`, so we propose that `‘only’` has both a strong sense and a weak sense, the latter being the negative half of the former.

$[\![\text{only}_w]\!] \quad = \quad \lambda\Phi\ \lambda\nu\ {\sim}\exists\nu'\ \{\ \Phi(\nu')\ \&\ \nu'\perp\nu\ \}$

where $\Phi \in [\![\mathsf{K}\rightarrow\mathsf{S}]\!]$
$\nu,\nu' \in [\![\mathsf{K}]\!]$

This gives us the following truth conditions for `‘C only if A’`.[15]

C only if A ≡ if C then A

The weak sense of `‘only’` also applies to CNP applications, according to which

`only A's are B's`

does not logically entail

`some A's are B's`

but only (the weak half):

`no non-A's are B's`

This is left as an exercise.

[15] Note carefully that this equivalence, which is counterintuitive, is largely a product of the oddity of our truth conditions for `‘if’`, according to which `‘if’` is truth-functional. We should not automatically expect this equivalence to obtain for other (more robust) versions of `‘if…then’`.

19. Applying these Ideas to Numerical Adjectives

Finally, we consider how `'exactly'` works in phrases such as

`exactly three dogs are barking`

We propose that, in this context, `'exactly'` behaves semantically like `'only'` where the focus is the numerical adjective `'three'`. In particular, for numerical adjectives, we propose the following type-analysis of `'only'`.

$$\text{type}(\texttt{only}) \;=\; [(\bar{N}\to\bar{N})\to S]\to[(\bar{N}\to\bar{N})\to S]$$

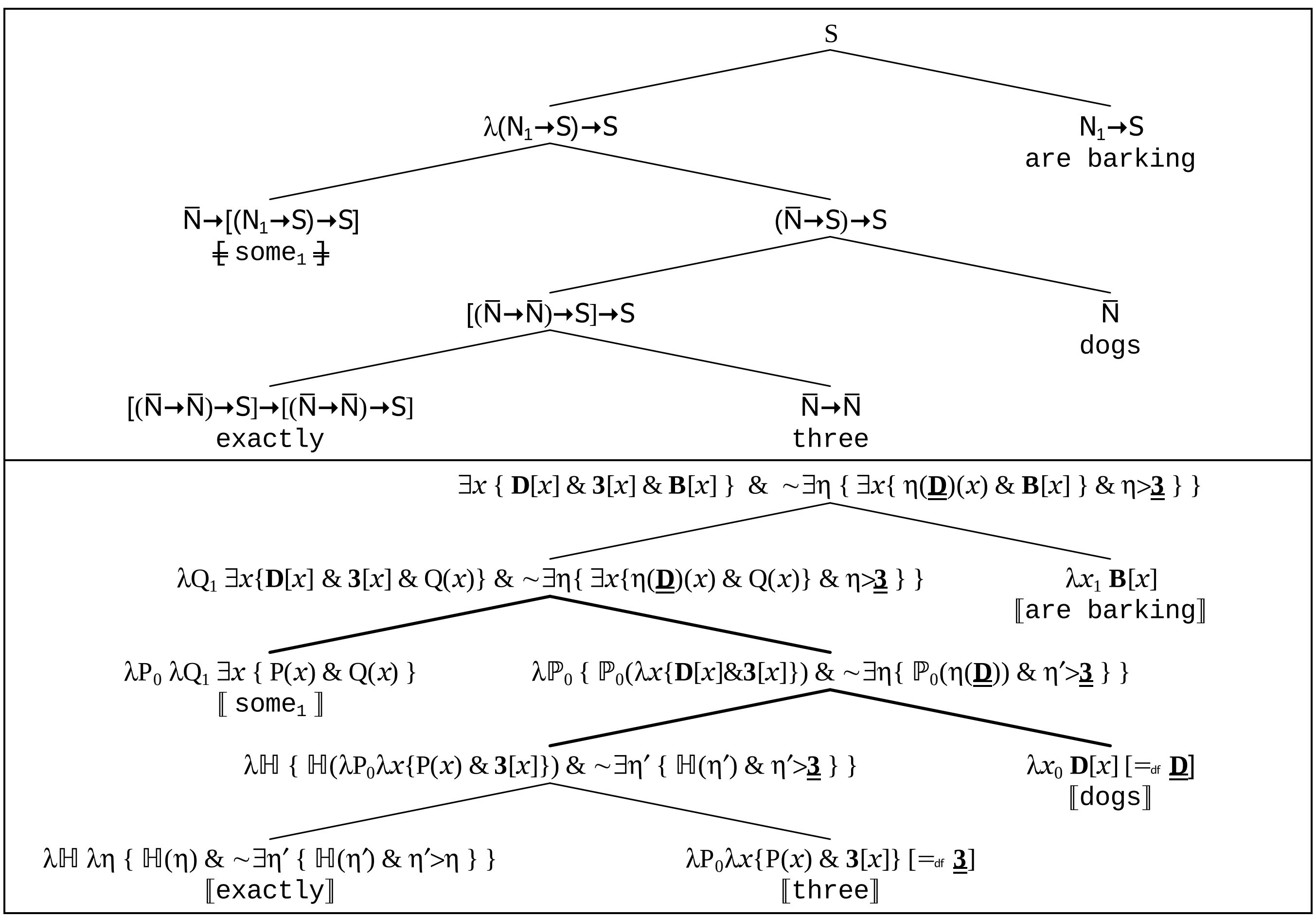
$$[\![\texttt{only}]\!] \;=\; \lambda\mathbb{H}\,\lambda\eta\,\{\,\mathbb{H}(\eta)\;\&\;\sim\exists\eta'\,\{\,\mathbb{H}(\eta')\;\&\;\eta'\perp\eta\,\}\,\}$$

where $\mathbb{H} \in [\![(\bar{N}\to\bar{N})\to S]\!]$
$\eta,\eta' \in \mathbb{N}$ [the class of numeral objects, a subclass of $[\![\bar{N}\to\bar{N}]\!]$]

where $\eta'\perp\eta \;=_{df}\; \eta' > \eta$

With this proposal in hand, we now offer the following analysis.

`exactly three dogs are barking`



According to this analysis, the sentence says that there is a three-membered set of dogs whose members are barking and there is no *larger* set of dogs whose members are barking. The following derivations underwrite the key compositions. Note the abbreviation: $\mathbf{3D}[\alpha] =_{df} \mathbf{D}[\alpha]\&\mathbf{3}[\alpha]$.

(1)	$[(\bar{N}\to\bar{N})\to S]\to S$	1	Pr	$\lambda\mathbb{H}\ \{\ \mathbb{H}(\lambda P_0\lambda x\{P(x)\ \&\ \mathbf{3}[x]\})\ \&\ {\sim}\exists\eta'\ \{\ \mathbb{H}(\eta')\ \&\ \eta'>\underline{\underline{\mathbf{3}}}\ \}\ \}$
(2)	$\bar{N}$	2	Pr	$\lambda x_0\ \mathbf{D}[x]$
(3)	$\bar{N}\to S$	3	As	$\mathbb{P}_0$
(4)	$\bar{N}\to\bar{N}$	4	As	η
(5)	$\bar{N}$	24	$2,4,\to O$	$\eta\ (\ \lambda x_0\ \mathbf{D}[x]\ )$
(6)	S	234	$3,5,\to O$	$\mathbb{P}_0\ (\ \eta\ (\ \lambda x_0\ \mathbf{D}[x]\ )\ )$
(7)	$(\bar{N}\to\bar{N})\to S$	23	$4\text{-}6,\to I$	$\lambda\eta\ \{\ \mathbb{P}_0\ (\ \eta\ (\ \lambda x_0\ \mathbf{D}[x]\ )\ )\ \}$
(8)	S	123	$1,7,\to O$	$\mathbb{P}_0(\lambda x\{\mathbf{3D}[x]\})\ \&\ {\sim}\exists\eta'\{\mathbb{P}_0(\eta'(\lambda x_0\ \mathbf{D}[x]))\ \&\ \eta'>\mathbf{3}\ \}$
(9)	$(\bar{N}\to S)\to S$	12	$3\text{-}8,\to I$	$\lambda\mathbb{P}_0\{\mathbb{P}_0(\lambda x\{\mathbf{3D}[x]\})\ \&\ {\sim}\exists\eta\{\mathbb{P}_0(\eta(\lambda x_0\mathbf{D}[x]))\ \&\ \eta'>\underline{\underline{\mathbf{3}}}\}\}$

(1)	$(\bar{N}\to S)\to S$	1	Pr	$\lambda\mathbb{P}_0\{\mathbb{P}_0(\lambda x\{\mathbf{3D}[x]\})\ \&\ {\sim}\exists\eta\{\mathbb{P}_0(\eta(\lambda x_0\mathbf{D}[x]))\ \&\ \eta>\underline{\underline{\mathbf{3}}}\}\}$
(2)	$\bar{N}\to[(N_1\to S)\to S]$	2	Pr	$\lambda P_0\ \lambda Q_1\ \exists x\ \{\ P(x)\ \&\ Q(x)\ \}$
(3)	$N_1\to S$	3	As	Q_1
(4)	$\bar{N}\to S$	23	$2,3,MP_2$	$\lambda P_0\ \exists x\ \{\ P(x)\ \&\ Q(x)\ \}$
(5)	S	123	$1,4,\to O$	$\exists x\{\mathbf{3D}[x]\&Q(x)\}\ \&\ {\sim}\exists\eta\{\exists x\{\eta(\underline{\underline{\mathbf{D}}})(x)\ \&\ Q(x)\}\ \&\ \eta>\underline{\underline{\mathbf{3}}}\ \}\}$
(6)	$(N_1\to S)\to S$	12	$3\text{-}5,\to O$	$\lambda Q_1\ \exists x\{\mathbf{3D}[x]\&Q(x)\}\ \&\ {\sim}\exists\eta\{\exists x\{\eta(\mathbf{D})(x)\ \&\ Q(x)\}\ \&\ \eta>\underline{\underline{\mathbf{3}}}\ \}\}$