

# 9

## First-Order Modal Logic with Actuality

---

1.	Introduction.....	2
2.	Russell's Yacht Reconsidered .....	2
3.	Syntactic Analysis of 'Actual' .....	3
4.	Direct Comparison versus Mediated Comparison .....	4
5.	Semantic Analysis of 'Actual' – First Attempt.....	4
6.	Flexible Actuality.....	5
7.	Rigid Actuality Revisited.....	7
8.	The Equivalence of $\bigcirc$ -Modified Terms and Scoped-Terms .....	8
9.	'Actually' – Actuality As Applied to Predicates and Sentences .....	9
10.	Examples of 'Actually' from Ordinary Language.....	9
11.	De Re Plurals and Actuality.....	11
12.	Another Example of 'Actually' .....	12
13.	Official Rules of Derivation for the Actuality Adverbs.....	15
14.	Appendix – Pitfalls in the Logic of Actuality .....	15
1.	Introduction.....	15
2.	A Problem with Adding ' $\bigcirc$ ' to Modal Systems that Extend K.....	15
3.	Modal Depth .....	17
4.	The New T-Rules .....	17
5.	What About Other Rules?.....	18
6.	Summary of Adjusted Rules for K+ Systems with Actuality .....	20
15.	Derivation Exercises .....	21
16.	Answers to Selected Exercises.....	22

---

## 1. Introduction

In an earlier chapter, we dealt with one notion of actuality, which is used to distinguish between items that actually exist and items that don't. There is another notion of actuality, which is motivated by a number of examples from ordinary language. First we consider actuality as it applies to singular-terms and function signs. Then we consider actuality as it applies to formulas and predicates.

## 2. Russell's Yacht Reconsidered

Recall Bertrand Russell's yacht example, which we formulated as follows.

- (A?) could Kiwi be bigger than it is?  
 (B!) of course not, Kiwi must be exactly as big as it is!

The *contents* of (A?) and (B!) can be phrased as follows.

- (a) possibly, Kiwi is bigger than it is.  
 (b) necessarily, Kiwi is exactly as big as it is.

Recall also that we analyze

$a$  is bigger than  $b$

to be equivalent to:

$a$ 's size is greater than  $b$ 's size       $[s(a) > s(b)]$

and we analyze

$a$  is exactly as big as  $b$

to be equivalent to:

$a$ 's size is identical to  $b$ 's size       $[s(a) = s(b)]$

Thus, symbolizing 'Kiwi' by ' $\kappa$ ', we *might* rephrase (a) and (b) as follows.

- (1) possibly,  $s(\kappa) > s(\kappa)$   
 i.e., there is an accessible world in which  $\kappa$ 's size is greater than  $\kappa$ 's size.  
 (2) necessarily,  $s(\kappa) = s(\kappa)$   
 i.e., in every accessible world,  $\kappa$ 's size is identical to  $\kappa$ 's size.

Clearly, (1) is false, and (2) is true [at least, supposing  $s(\kappa)$  exists at every accessible world.] But presumably the questioner does not intend the question that way. So, how do we symbolize what the questioner really means? Probably the best way to analyze (A?) is to restate it using the modifier 'actually' as follows.

- (a2) possibly, Kiwi is bigger than it *actually* is.

Or, using the intermediate notion of size to write this, we have:

- (a3) possibly, Kiwi's size is greater than Kiwi's *actual* size.

Using the symbol ‘ $\bigcirc$ ’ to abbreviate the word ‘actual’, we can symbolize (a3) as follows.

- (a) possibly,  $\kappa$ 's size is greater than  $\kappa$ 's actual size  
 possibly,  $s(\kappa) > \bigcirc s(\kappa)$   
 $\diamond[s(\kappa) \bigcirc s(\kappa)]$

If we state this in unfettered world theory, it goes as follows.

there is an accessible world  $w$  such that  $\kappa$ 's size at  $w$  is greater than  $\kappa$ 's size at the actual world.

### 3. Syntactic Analysis of ‘Actual’

Unfortunately, the above symbolization is little more than stenography unless we provide a precise syntactic account of ‘ $\bigcirc$ ’. In this regard there are two plausible analyses.

According to one analysis, ‘actual’ is categorially complex, being a special kind of adverb; in particular, according to this analysis, ‘actual’ a one-place functor that takes a function sign (of degree  $k$ ) as input and delivers a function sign (of degree  $k$ ) as output. In that case, its official category is given as follows. [See chapter on Categorical Grammar.]

$$\text{cat}(\bigcirc) = (\mathbb{N}^k \rightarrow \mathbb{N}) \rightarrow (\mathbb{N}^k \rightarrow \mathbb{N})$$

For example, the functor ‘actual’ takes ‘length’ – a one-place function sign – and delivers ‘actual length’ – also a one-place function sign.

According to the second analysis, ‘actual’ is a special one-place function sign, which means its category is given as follows.

$$\text{cat}(\bigcirc) = \mathbb{N} \rightarrow \mathbb{N}$$

Although both analyses produce the same surface structures, they differ concerning phrase structures, as illustrated in the following two analyses.

the actual size of Kiwi	surface expression
[actual(size)](Kiwi)	function sign adverb
actual(size(Kiwi))	function sign

Here, the phrase structures are represented algebraically; in particular, the parentheses mark the order of functor application, and hence the tree structure. In the first analysis, one first applies the expression ‘actual’ to ‘size’, which results in ‘actual size’, which in turn is applied to ‘Kiwi’. In the second analysis, one first applies ‘size’ to ‘Kiwi’, which results in ‘size of Kiwi’, to which one next applies ‘actual’.

Theoretically speaking, the first analysis hypothesizes an intermediate grammatical expression – ‘actual size’ – which is not hypothesized in the second analysis.

Each analysis has certain theoretical advantages. The first analysis seems *grammatically* more plausible because it treats ‘actual size’ as grammatically autonomous. The second analysis has logical advantages, because it maintains categorial simplicity. Logicians, on the whole, prefer not to jump to higher-order functors unless there is no other choice.

Preferring categorial simplicity, and accepting the grammatical trade-off, we choose to follow the second analysis in what follows. In particular, we propose the following first-order syntactic analysis of ‘actual’.

- (d1)  $\bigcirc$  is a special one-place function-sign;  
 In other words, we have the following additional term-forming rule.  
 if  $\tau$  is a singular-term, then so is:  $\bigcirc\tau$

#### 4. Direct Comparison versus Mediated Comparison

In the previous sections we have analyzed the Kiwi problem using the notion of size, which introduces an intermediate class of abstracts objects – sizes – which are structurally modelled by the positive real numbers.

We don't have to rely on this intermediate class of objects. Since, the ‘actual’ modifier is a general singular-term modifier, we can apply it to all singular-terms, including proper nouns, such as ‘Kiwi’. This allows us to go back and reformulate our original sentence.

Kiwi might be bigger than it actually is.

In particular, we can simply use the two-place predicate ‘\_\_ is bigger than \_\_’, and symbolize it as follows.<sup>1</sup>

$\diamond B[K, \bigcirc K]$

possibly, Kiwi is bigger than the actual Kiwi

or

possibly, Kiwi is bigger than Kiwi as it actually is

#### 5. Semantic Analysis of ‘Actual’ – First Attempt

In the present section, we propose semantic/derivation rules for the  $\bigcirc$ -operator. The following constitute a fairly natural pair of rules.

$\bigcirc\bigcirc$	$\bigcirc I$
$\frac{c = \bigcirc\tau}{c = \tau} \quad /i$	$\frac{c = \tau}{c = \bigcirc\tau} \quad /0$
$c$ is any constant, and $\tau$ is any closed singular-term.	

Combining these two rules, we have the following semantic constraint:

$c$  is *the actual*  $\tau$  at world  $i$   
 iff  
 $c$  is the  $\tau$  at *the actual world* 0.

<sup>1</sup> Another use of the ‘actual’ modifier involves comparing an actual person with the corresponding character in a story “about” that actual person. Pieces of historical fiction are presumably “about” real people. For example, the *actual Socrates* (supposing there was such a person) might be compared with the *Platonic Socrates* (by which I mean the character in the Platonic dialogues).

Or, if we wish to state matters in more fancy semantic notation, we can write the following.

$$v(\bigcirc\tau / i) = v(\tau / 0)$$

Here,  $v(\varepsilon/i)$  is the *semantic value* (interpretation) of expression  $\varepsilon$  at world  $i$ .

In writing down the rules above, we presume that the sequence  $i$  originates from 0. This is a safe presumption for all our derivations, since every index has 0 as its origin. However, for purposes of semantic generality, we strengthen the proposed rules as follows.

$\bigcirc\bigcirc$	$\bigcirc\text{I}$
$\frac{c = \bigcirc\tau}{c = \tau} \quad /i$	$\frac{c = \tau}{c = \bigcirc\tau} \quad /o(i)$
$c$ is any constant, and $\tau$ is any closed singular-term.	

Here  $o(i)$  is the origin, or initial element, of the sequence  $i$ , which is permitted to be any possible world. In other words, for semantic purposes, any possible world may be treated as the "actual" world; there is nothing *semantically* special about the actual world.

## 6. Flexible Actuality

Let us return to Russell's example. In answering the question whether Kiwi might be bigger than it is, the yacht owner (presumably) means the following.

- (b) necessarily,  $s(\kappa) = s(\kappa)$   
 $\square[s(\kappa) = s(\kappa)]$

This is a logical truth, and not very interesting. However, the yacht owner might mean something considerably stronger, and hence more interesting, namely,

- (b+) necessarily,  $s(\kappa) = \bigcirc s(\kappa)$ ,  
 $\square[s(\kappa) = \bigcirc s(\kappa)]$ ,

which translates into world theory as:

in every accessible world,  $\kappa$ 's size is equal to  $\kappa$ 's actual size;  
 i.e.,  
 for every accessible world  $\omega$ ,  $\kappa$ 's size at  $\omega$  is equal to  $\kappa$ 's size at the actual world.

In other words,  $\kappa$  must have the size it, *as a matter of fact*, has; it can/could have no other size. Let us describe this by saying that  $\kappa$ 's size is *rigid*. The following is the proposed schematic definition of rigidity.

- (d2)  $\text{rigid}[\tau]_{=af} \quad \square[\tau = \bigcirc\tau]$

Here,  $\tau$  is any closed singular-term.

The following is our principal application so far.

Kiwi's size is rigid  
 $\text{rigid}[s(\kappa)]$   
 $\Box[s(\kappa) = \bigcirc s(\kappa)]$

Now, the problem is that we can modally qualify the latter sentences; for example, we might say:

[Although Kiwi's size is not rigid, ]

(m) Kiwi's size *might* be rigid

Important: the remoteness of the envisaged circumstance should not distract us when we are evaluating its content. The natural symbolization of (3) construes it as a straightforward adverbial modification of (2), obtained by prefixing a ‘ $\Diamond$ ’.

(s)  $\Diamond\Box[s(\kappa) = \bigcirc s(\kappa)]$

If we translate (s) into world theory talk, we obtain the following.

there is an accessible world, call it 01, such that  $\kappa$ 's size at every world accessible to 01 is the same as  $\kappa$ 's size at the original world 0.

But in saying that Kiwi's size *might* be rigid, we mean to say:

there is an accessible world, call it 01, such that  $\kappa$ 's size at every world accessible to 01 is the same as  $\kappa$ 's size at the intermediate world 01.

The problem is that the  $\bigcirc$ -operator shifts the evaluation all the way back to the "actual" world, which is *too far!* In describing rigidity, we want ‘actual’ to shift the evaluation back to the intermediate world.

For this reason, we discard our earlier semantic analysis of ‘ $\bigcirc$ ’ in favor of the following, revised, analysis.

$\bigcirc\bigcirc$	$\bigcirc\text{I}$
$\frac{c = \bigcirc\tau \quad /i}{c = \tau \quad /i-1}$	$\frac{c = \tau \quad /i-1}{c = \bigcirc\tau \quad /i}$
<p><math>c</math> is any constant, and <math>\tau</math> is any closed singular-term;  <math>i-1</math> is the immediate predecessor of <math>i</math></p>	
<p><math>i-1 =</math> the immediate <i>predecessor</i> of index <math>i</math>, which is the sequence obtained from <math>i</math> by removing its last element, <u>provided</u> <math>i</math> has at least two elements;</p> <p><math>i-1</math> is undefined if <math>i</math> has only one element.</p>	

In other words,

$c$  is the actual  $\tau$  at world  $i$   
 iff  
 $c$  is the  $\tau$  at the world prior to  $i$ .

Since ‘actual’ is a one-place function sign, we can repeatedly apply it, to obtain expressions like the following.

$\bigcirc s(\kappa)$   
 $\bigcirc\bigcirc s(\kappa)$   
 $\bigcirc\bigcirc\bigcirc s(\kappa)$   
 etc.

It seems natural to employ algebraic notation to reduce the symbolic clutter, which is summarized in the following inductive definition.

Df	$\bigcirc^1\tau = \bigcirc\tau;$
	$\bigcirc^{m+1}\tau = \bigcirc\bigcirc^m\tau$

Similarly, in the semantics, we can define a corresponding sequence of priority functions, as follows.

Df	$i-2 = (i-1)-1$
	$i-(m+1) = (i-m)-1$

We can then provide a general semantic definition as follows.

$\bigcirc^m\mathbf{O}$	$\bigcirc^m\mathbf{I}$
$\frac{c = \bigcirc^m\tau \quad /i}{c = \tau \quad /i-m}$	$\frac{c = \tau \quad /i-m}{c = \bigcirc^m\tau \quad /i}$
$c$ is any constant, and $\tau$ is any closed singular-term; $i-m$ is the $m$ -th predecessor of $i$	

## 7. Rigid Actuality Revisited

We started with one account of actuality, rigid actuality, and discarded it in favor of an account in terms of flexible actuality. Even having seen how rigid actuality can get us into trouble, we might still wish to retain the expressive resources of rigid actuality in addition to the expressive resources of flexible actuality.

For that reason, we re-introduce rigid actuality, but with a new symbol – ‘ $\odot$ ’ – which is semantically characterized by the following rules.

$\odot\mathbf{O}$	$\odot\mathbf{I}$
$\frac{c = \odot\tau \quad /i}{c = \tau \quad /o(i)}$	$\frac{c = \tau \quad /o(i)}{c = \odot\tau \quad /i}$
$c$ is any constant, and $\tau$ is any closed singular-term; $o(i)$ is the origin of $i$	

## 8. The Equivalence of $\bigcirc$ -Modified Terms and Scoped-Terms

We have now seen two solutions to the Kiwi problem – one involving scoped singular-terms, the other involving the modifier ‘actual’. In this section, we prove that  $\bigcirc$ -modified terms and scoped-terms are interchangeable, under prescribed circumstances.

Toward that end, we provide four *schematic* derivations. In particular, in the following,  $\tau$  is any closed singular-term,  $v$  is any variable, and  $\Phi$  is any formula in which  $v$  does not occur within the scope of any modal operator [the latter restriction is critical for applying LL]. Also,  $i$  is any index, and  $m$  is any numeral.

(1)	$\exists x[x = \tau] \ \& \ \diamond\Phi[\bigcirc\tau / v]$	$/i$	Pr
(2)	SHOW: $(\tau/v)\diamond\Phi$	$/i$	Df $(\tau/v)$
(3)	SHOW: $\exists v\{v=\tau \ \& \ \diamond\Phi\}$	$/i$	DD
(4)	$a=\tau$	$/i$	1a, $\exists O$
(5)	SHOW: $\diamond\Phi[a/v]$	$/i$	DD
(6)	$\Phi[\bigcirc\tau / v]$	$/i+m$	1b, $\diamond O$
(7)	$a = \bigcirc\tau$	$/i+m$	4, $\bigcirc I$
(8)	$\Phi[a/v]$	$/i+m$	6,7,LL
(9)	$\diamond\Phi[a/v]$	$/i$	8, $\diamond I$
(10)	$\exists v\{v=\tau \ \& \ \diamond\Phi\}$	$/i$	4,5,QL
(1)	$(\tau/v)\diamond\Phi$	$/i$	Pr
(2)	SHOW: $\exists x[x = \tau] \ \& \ \diamond\Phi[\bigcirc\tau / v]$	$/i$	8,9,SL
(3)	$\exists v\{v=\tau \ \& \ \diamond\Phi\}$	$/i$	1,Df $(\tau/v)$
(4)	$a=\tau \ \& \ \diamond\Phi[a/v]$	$/i$	3, $\exists O$
(5)	$\Phi[a/v]$	$/i+m$	4b, $\diamond O$
(6)	$a = \bigcirc\tau$	$/i+m$	4a, $\bigcirc I$
(7)	$\Phi[\bigcirc\tau / v]$	$/i+m$	5,6,LL
(8)	$\exists x[x = \tau]$	$/i$	4a,QL
(9)	$\diamond\Phi[\bigcirc\tau / v]$	$/i$	7, $\diamond I$
(1)	$\exists x[x = \tau] \ \& \ \square\Phi[\bigcirc\tau / v]$	$/0$	Pr
(2)	SHOW: $(\tau/v)\square\Phi$	$/0$	Df $(\tau/v)$
(3)	SHOW: $\exists v\{v=\tau \ \& \ \square\Phi\}$	$/0$	DD
(4)	$a=\tau$	$/0$	1a, $\exists O$
(5)	SHOW: $\square\Phi[a/v]$	$/0$	ND
(6)	SHOW: $\Phi[a/v]$	$/01$	DD
(7)	$\Phi[\bigcirc\tau / v]$	$/01$	1b, $\square O$
(8)	$a = \bigcirc\tau$	$/01$	4, $\bigcirc I$
(9)	$\Phi[a/v]$	$/01$	7,8,LL
(10)	$\exists v\{v=\tau \ \& \ \square\Phi\}$	$/0$	4,5,QL
(1)	$(\tau/v)\square\Phi$	$/0$	Pr
(2)	SHOW: $\exists x[x = \tau] \ \& \ \square\Phi[\bigcirc\tau / v]$	$/0$	4a,5,QL
(3)	$\exists v\{v=\tau \ \& \ \square\Phi\}$	$/0$	1,Df $(\tau/v)$
(4)	$a=\tau \ \& \ \square\Phi[a/v]$	$/0$	3, $\exists O$
(5)	SHOW: $\square\Phi[\bigcirc\tau / v]$	$/0$	ND
(6)	SHOW: $\Phi[\bigcirc\tau / v]$	$/01$	DD
(7)	$a = \bigcirc\tau$	$/01$	4a, $\bigcirc I$
(8)	$\Phi[a/v]$	$/01$	4b, $\square O$
(9)	$\Phi[\bigcirc\tau / v]$	$/01$	7,8,LL

## 9. ‘Actually’ – Actuality As Applied to Predicates and Sentences

So far, ‘ $\circ$ ’ is used as a function-sign-modifying adverb. It serves equally well as a predicate-modifying adverb. For example, we can adverbially modify the one-place predicate ‘...is large’ to obtain the one-place predicate ‘...is actually large’. According to this syntactic analysis, the category of actually (symbolized again by ‘ $\circ$ ’) is given as follows.

$$\text{cat}(\circ) = (\mathbf{N}^k \rightarrow \mathbf{S}) \rightarrow (\mathbf{N}^k \rightarrow \mathbf{S})$$

In other words, ‘ $\circ$ ’ takes a  $k$ -place predicate and yields a  $k$ -place predicate.

Symbolizing goes as follows.

$$\begin{array}{ll} \alpha \text{ is large:} & \mathbf{L}[\alpha] \\ \alpha \text{ is actually large:} & [\circ(\mathbf{L})][\alpha] \end{array}$$

Notice, however, that the sentence ‘Kiwi is actually large’ can also be parsed in a way that construes ‘actually’ as a *sentential adverb*, on a par with the modal operator ‘necessarily’. In this case, its category is given as follows,

$$\text{cat}(\circ) = \mathbf{S} \rightarrow \mathbf{S}$$

and the syntactic analysis is given as follows.

$$\begin{array}{ll} \mathbf{K} \text{ is actually large} & \\ \text{actually, } \mathbf{K} \text{ is large:} & \circ\mathbf{L}[\mathbf{K}] \end{array}$$

As with the modifier ‘actual’, in choosing a category for ‘actually’, we opt for categorial simplicity. When combined with our earlier syntactic rule for ‘ $\circ$ ’, we have the following rules of formation.

$$\begin{array}{l} \text{if } \tau \text{ is a singular-term, then so is: } \circ(\tau) \\ \text{if } \Phi \text{ is a formula, then so is: } \circ\Phi \end{array}$$

This of course means that the symbol ‘ $\circ$ ’ does not have a unique category – it is categorially ambiguous. We accept this ambiguity, since we can always tell the category from the context.

## 10. Examples of ‘Actually’ from Ordinary Language

We have already seen how the word ‘actually’ can be used to clarify the Russell example.

I expected Kiwi to be bigger than it *actually* is.

As we analyzed this sentence earlier, we employed the adverb ‘the actual’ which yielded the following.

I expected the size of Kiwi to be bigger than its actual size.

We now consider other applications from ordinary language in which ‘actually’ is understood as a sentential adverb. We will present an example that is similar to the Russell example.

- (q?) could a Republican be a Democrat?
- (a!) of course not, necessarily no Republican is a Democrat!

First, let us use the adjectives ‘Republican’ and ‘Democrat’ to be mutually exclusive, so that we have the following meaning postulate (a special kind of axiom).

$$(m) \quad \Box \forall x(Rx \rightarrow \sim Dx)$$

Here, ‘ $\Box$ ’ is the absolute necessity operator, which is governed by System L.

Now, the issue above concerns the content of the following sentence.

$$(s) \quad \text{some Republican might be a Democrat}$$

There are two natural readings of this sentence, according to whether ‘might’ or ‘some’ has wide scope.

$$\begin{array}{ll} \exists x(Rx \ \& \ \Diamond Dx) & \text{‘some R’ is wide} \\ \Diamond \exists x(Rx \ \& \ Dx) & \text{‘might’ is wide} \end{array}$$

Using spatial terminology to talk about possible worlds, the first one says:

there is someone who is a Republican here but is a Democrat elsewhere.

The second one says:

somewhere there is a person who is both a Republican and a Democrat.

Whereas the first one is consistent with our earlier exclusion principle (that nothing is both R and D), the latter one is not.

So far, we do not require an actuality adverb to clarify matters. This changes, however, as soon as we change the quantifier to ‘every’.

$$(e?) \quad \text{could every Republican be a Democrat?}$$

In analyzing the content of this sentence, we have the usual ambiguity of scope between ‘every R’ and ‘might’.

$$\begin{array}{ll} (1) \quad \forall x(Rx \rightarrow \Diamond Dx) & \text{‘every R’ is wide} \\ (2) \quad \Diamond \forall x(Rx \rightarrow Dx) & \text{‘might’ is wide} \end{array}$$

If we analyze these world-theoretically, we have the following translations.

$$(1) \quad \text{for anyone } x: \\ \quad \text{if } x \text{ is a Republican at } i, \\ \quad \text{then there is an } i\text{-accessible world } j \text{ such that} \\ \quad \quad x \text{ is a Democrat at } j.$$

or: every Republican is a Democrat somewhere or other

$$(2) \quad \text{there is an } i\text{-accessible world } j \text{ such that} \\ \quad \text{for anyone } x, \\ \quad \text{if } x \text{ is Republican at } j, \\ \quad \quad \text{then } x \text{ is a Democrat at } j.$$

or: somewhere, every Republican is a Democrat.

If we combine the second reading with our meaning postulates, we do not obtain a contradiction, but only the following.

$$\diamond \sim \exists x R x$$

which says that there is a possible world with no Republicans.

## 11. De Re Plurals and Actuality

There are other readings of ‘every Republican might be a Democrat’. One of them employs *de re* plurals in a manner similar to Bricker.<sup>2</sup>

the Republicans are such that, it might be that everyone of them is a Democrat

As with any irreducible plural, this sentence exceeds the expressive bounds of first-order logic. So we postpone a careful analysis until a later chapter on higher-order modal logic.

However, we can express something equivalent to the *de re* plural reading by using the modifier ‘actual’.<sup>3</sup>

it might be that every actual Republican is a Democrat

it might be that everyone who is actually Republican is a Democrat

$$\diamond \forall x (\bigcirc R x \rightarrow B x)$$

Here, ‘ $\bigcirc$ ’ is a one-place sentential adverb. If we express this in world-theory, then we have the following.

there is an *i*-accessible world *j* such that  
for anyone *x*,  
if *x* is a Republican at *i*,  
then *x* is Democrat at *j*.

$$\exists j \{ i \prec j \ \& \ \forall x ( [R x / i] \rightarrow [D x / j] ) \}$$

Once we see that ‘*Rx*’ can be adverbially modified, we also see that ‘*Dx*’ can be so modified, which yields the following formula.<sup>4</sup>

$$\diamond \forall x (R x \rightarrow \bigcirc D x)$$

there is an *i*-accessible world *j* such that  
for anyone *x*,  
if *x* is a Republican at *j*,  
then *x* is a Democrat at *i*.

$$\exists j \{ i \prec j \ \& \ \forall x ( [R x / j] \rightarrow [D x / i] ) \}$$

<sup>2</sup> Bricker considers *de re* plural quantification. "Quantified Modal Logic and the Plural De Re," *Contemporary Perspectives in the Philosophy of Language*, II, *Midwest Studies in Philosophy* 14, French, Uehling, and Wettstein (eds.), University of Notre Dame Press (1989), 372-394.

<sup>3</sup> Bricker considers an analysis of *de re* plurals using ‘actual’, which uses the concept of rigid actuality; he rejects this analysis for precisely the same reasons we reject rigid actuality in general. However, an analysis of *de re* plurals using flexible actuality is not subject to Bricker's objection.

<sup>4</sup> Similarly, if we opt for the second-order treatment, we can treat ‘Democrats’ as a *de re* plural.

This is somewhat difficult to state in colloquial English. The "straight" reading,

every Republican might actually be a Democrat,

means something entirely different. The following sentence comes closer.

it might be that: every Republican is some actual Democrat

World-theoretically, this means:

there is an accessible world where:

every one who is a Republican there

is (=) someone who is a Democrat here.

Basically, we have two plurals [sets, if you like];

$R_j$  : those who are R at  $j$   
 $D_i$  : those who are D at  $i$

We are saying every member of  $R_j$  is a member of  $D_i$  [ $R_j \subseteq D_i$ ].

## 12. Another Example of ‘Actually’

Davies and Humberstone<sup>5</sup> consider another ordinary language example that can be treated using the actuality adverb. Consider the following example.

(0) Jay wants to kill a fly.

There are a number of readings of this sentence.

### Reading #1:

(1) Jay, an inventor, is working on a project, but this activity has attracted a fly, a particularly pesky fly, which Jay wants to kill.

In this case, we read ‘a fly’ as a *de re singular*, and symbolize it as follows, where ‘ $\llbracket JW \rrbracket$ ’ is read ‘Jay wants it to be the case that \_\_\_’, which is a box-operator

$\exists x \{ Fx \ \& \ \llbracket JW \rrbracket KJx \}$                       [‘some F’ has wide scope, and is singular]

### Reading #2:

(2) Jay, our inventor, has just designed a new-fangled fly-swatter, which he wants to test. In order to prove his product, he needs to kill a fly. So he wants to kill a fly – any fly will do, so long as it is a confirmed kill.

In this case, we read ‘a fly’ as a *de dicto singular*, and symbolize it as follows.

$\llbracket JW \rrbracket \exists x \{ Fx \ \& \ KJx \}$                       [‘Jay wants ...’ has wide scope]

Notice the following logical fact: just using the deductive resources of MPL(k), we can deduce the following.

<sup>5</sup> +++reference+++

$$[[JW]] \exists x Fx$$

This says:

Jay wants there to be at least one fly.

The reason is simple: if there are no flies, there are no flies to kill, but Jay needs to kill a fly in order to prove his invention.

### Reading #3:

- (3) Jay, our famous fly-swatter inventor, has developed a proving facility, in which he tests fly-swatters. Let us suppose that one day Jay decides that there are one too many flies in his proving facility – so he wants to kill one of them. So he wants to kill a fly. On the other hand, there is no particular fly he wants to kill; concerning this matter he is completely indifferent. On still another (!) hand, the fly he kills must be one of the flies in the proving facility, or the carnage will not result in the desired reduction in his fly population.

This example seems, offhand, not radically different from Example #2. We simply need to add reference to the proving facility, thus:

$$(r) \quad [[JW]] \exists x \{ Fx \ \& \ Px \ \& \ KJx \}$$

This is not quite right however. Jay wants to kill a fly in order to reduce the fly population. His wishes are considerably more specific than conveyed by (r). Reading (r) allows situations in which a "wild" fly infiltrates the experimental population, and is killed. This is not enough; Jay wants to kill one of his flies.

In order to accomplish this reading, we can do one of two things.

First, we can adopt a *de re* plural approach, in which case we translate the sentence as follows.

- (1) the flies in Jay's proving facility are such that, Jay wants to kill one of them

Again, this requires the resources of second order logic.

Second, we can use the actuality adverb, in which case we translate the sentence as follows.

- (2)  $[[JW]] \exists x \{ \bigcirc(Fx \ \& \ Px) \ \& \ KJx \}$

### Reading #4:

- (4) The reason Jay is designing fly-swatters in the first place is that he hates flies. He wants to kill some flies, because he wants there to be no flies. On the other hand, he realizes that there are billions of flies, so personally killing them all is impossible, so he does not want to kill all flies, but he does want to do his share.

If we attempt to write this as follows

$$[[JW]] \exists x \{ Fx \ \& \ KJx \}$$

we are forced to the conclusion,

$$\llbracket JW \rrbracket \exists x Fx,$$

as before. But, our additional premise states that Jay wants there to be no flies, which is symbolized as follows.

$$\llbracket JW \rrbracket \sim \exists x Fx$$

But, supposing Jay's wishes are consistent, which among other things entails the following

$$\llbracket JW \rrbracket \sim \exists x Fx \rightarrow \sim \llbracket JW \rrbracket \exists x Fx,$$

we have a contradiction.

If we add the resources of actuality to our analysis, then we can symbolize reading #4 as follows.

$$\llbracket JW \rrbracket \exists x \{ \bigcirc Fx \ \& \ KJx \}$$

This has the following consequence.

$$\llbracket JW \rrbracket \exists x \bigcirc Fx$$

A "straight", but incorrect, reading of this goes thus.

Jay wants there to be an actual fly.

This reading is misleading, because the original formula is merely a logical consequence of:

$$\exists x Fx$$

In other words,

if there is a fly at the actual world,  
then  
at every (Jay-wish-) accessible world, there is a fly at the actual world.

This is analogous to saying the following very odd, but nevertheless true, statement.

if there is a tower in London,  
then  
wherever you go, there is a tower in London.

### 13. Official Rules of Derivation for the Actuality Adverbs

Now, let us look at the derivation rules, and hence the semantics, for the actuality operators in their sentential adverb guise. Note carefully that we are using each actuality operator as two different grammatical categories. First, as earlier, we use ‘ $\circ$ ’ and ‘ $\odot$ ’ as singular-term “adverbs”. Now, we also use them as sentential adverbs. [Note, in each of the following, it is understood that all indices are old.]

$\circ\circ/\circ I$			
$c = \circ\tau$	$/i$	$\circ\Phi$	$/i$
$\underline{\underline{c = \tau}}$		$\underline{\underline{\Phi}}$	
$c = \tau$	$/i-1$	$\Phi$	$/i-1$
<p><math>c</math> is any constant, <math>\tau</math> is any closed singular-term, <math>\Phi</math> is any formula, <math>i</math> is any old index, and <math>i-1</math> is the <i>immediate predecessor</i> of <math>i</math>, which is the sequence obtained from <math>i</math> by removing its last element, <i>provided</i> <math>i</math> has at least two elements, and which is undefined otherwise.</p>			

$\odot\odot/\odot I$			
$c = \odot\tau$	$/i$	$\odot\Phi$	$/i$
$\underline{\underline{c = \tau}}$		$\underline{\underline{\Phi}}$	
$c = \tau$	$/o(i)$	$\Phi$	$/o(i)$
<p><math>c</math> is any constant, <math>\tau</math> is any closed singular-term, <math>\Phi</math> is any formula, <math>i</math> is any old index, and <math>o(i)</math> is the origin of <math>i</math>.</p>			

## 14. Appendix – Pitfalls in the Logic of Actuality

### 1. Introduction

In this appendix, we discuss serious problems with the logic of actuality so far proposed. Since the problems do not particularly concern predicate logic (or higher), for the sake of simplicity we concentrate on modal sentential logic, augmented by the actuality operator. The logical systems considered all have the nominal form  $\Sigma+\circ$ , where  $\Sigma$  is one of our sentential modal systems (KD, K4, etc.), and ‘ $\circ$ ’ is the flexible actuality operator.

### 2. A Problem with Adding ‘ $\circ$ ’ to Modal Systems that Extend K

Recall that one can strengthen System K by adding further sentential rules, which include the characteristic rules for T, S4, S5, etc. We can take any of these systems and add the flexible actuality operator  $\circ$ . Alternatively we can take  $K+\circ$  and add various characteristic rules.

For starters, let us consider adding the t-rules, which would be required in any modal logic that concerns a *factive* operator – including most importantly *necessity*, but also *regret* and *knowledge*.

$\frac{\Box O(t)}{\Box \Phi} \quad /i$ $\frac{\Box \Phi}{\Phi} \quad /i$	$\frac{\Diamond I(t)}{\Phi} \quad /i$ $\frac{\Phi}{\Diamond \Phi} \quad /i$
--	---

Now, if we simply add these rules to  $K+\Box$ , we are in for a *big* surprise. For now we can construct the following derivation.

**Ouch!!!**

(1)	SHOW: $P \rightarrow \Box P$	/0	CD
(2)	P	/0	As
(3)	SHOW: $\Box P$	/0	ND
(4)	SHOW: P	/01	DD
(5)	$\Box P$	/01	2, $\Box I$
(6)	$\Diamond \Box P$	/01	5, $\Diamond I(t)$
(7)	$\Box P$	/012	6, $\Diamond O$
(8)	P	/01	7, $\Box O$

By duality, we can similarly prove  $\lceil \Diamond P \rightarrow P \rceil$ .

This is, to put it succinctly, a disaster! At the very least, we have formulas, which are not theorems of System T without the  $\Box$ -operator, but which are theorems of the augmented system. This is bad enough! At the worst, we have a completely trivial logical system. Recall that the following are already theses of System T [KT].

$$\Box P \rightarrow P$$

$$P \rightarrow \Diamond P$$

Putting them together with the earlier ones, we have the following as theses.

(trv)  $P \leftrightarrow \Box P$   
 $P \leftrightarrow \Diamond P$

Any modal system that has (trv) as theses is said to be *trivial*, since such a system makes no modal distinctions – for example, truth is the same as necessity is the same as possibility. Thus, we have shown that simply adding the t-rules to  $K+\Box$  results in a trivial system. This, of course, is unacceptable.

The trivialization of modal logic is a particularly disastrous example of a general issue arising in formal systems. Logic is built, heuristically at least, from the ground up. We start with the SL connectives, then we add modal operators, then we add actuality, and so forth. Or, perhaps, we start with the SL connectives, then we add quantifiers, then we add identity, then we add descriptions, and so forth. The critical requirement (*desideratum*) in this construction process is described as follows.

(des) Whenever we introduce a "new" logical symbol, and correspondingly new logical rules, we **must not** produce undesirable results in regard to the "old" logical symbols.

Another way to describe this requirement, which is called the Requirement of Non-Creativity [in relation to logical systems], is described as follows.

**Requirement of Non-Creativity**

New rules **must not** generate any new theorems that *do not involve* the new symbols.

For example, introducing the  $\Box$ -rules and  $\Diamond$ -rules should not generate any new theorems that do not pertain to ' $\Box$ ' or ' $\Diamond$ '. Similarly, introducing the  $\forall$ -rules and the  $\exists$ -rules should not generate any new theorems that do not pertain to ' $\forall$ ' or ' $\exists$ '.

**3. Modal Depth**

By way of avoiding disaster, we must somehow go back and rewrite our rules so as to prevent derivations such as the above. Towards this end we offer the following inductive definition of *modal depth*, herein abbreviated by ' $\partial$ '.

**Base Case:**

$$\partial(\mathbb{A}) = 0, \text{ for any atomic formula } \mathbb{A};$$

**Inductive Cases:**

$$\begin{aligned} \partial(\sim\Phi) &= \partial(\Phi) \\ \partial(\Box\Phi) &= \partial(\Phi)+1 \\ \partial(\Diamond\Phi) &= \partial(\Phi)+1 \\ \partial(\Phi\odot\Psi) &= \min\{\partial(\Phi),\partial(\Psi)\} \text{ where } \odot \text{ is any two-place SL-connective} \\ \partial(\bigcirc\Phi) &= \min\{-1, \partial(\Phi)-1\} \end{aligned}$$

**4. The New T-Rules**

With the notion of modal depth in hand, we propose the following rewrite of the t-rules.

$\frac{\Box\Phi \quad /i}{\Phi} \quad /i$	$\frac{\Phi \quad /i}{\Diamond\Phi} \quad /i$
$\Phi \text{ must have modal depth } \geq 0$	

When we examine our earlier problematic derivation, we notice that line (6) is no longer admissible.

(1)	SHOW: $P \rightarrow \Box P$	/0	CD	
(2)	P	/0	As	
(3)	SHOW: $\Box P$	/0	ND	
(4)	SHOW: P	/01	DD	
(5)	$\bigcirc P$	/01	2, $\bigcirc I$	$\partial(\bigcirc P) = -1$
(6)	$\Diamond \bigcirc P$	/01	5, $\Diamond I(t)$	****

## 5. What About Other Rules?

We have seen how the interaction of  $\Box$  and  $\Diamond$  leads to difficulties when we attempt to add the  $t$ -rules to our system. What about other rules? We consider a few of them.

### 1. B-Rules

$\frac{\Box O(b) \quad \Box \Phi \quad /i+m \text{ (old)}}{\Phi \quad /i}$	$\frac{\Diamond I(b) \quad \Phi \quad /i}{\Diamond \Phi \quad /i+m \text{ (old)}}$
--	--

These rules are problematic, as evidenced by the following derivation.

(1)	SHOW: $\Diamond \Diamond P \rightarrow P$	/0	CD	
(2)	$\Diamond \Diamond P$	/0	As	
(3)	SHOW: $P$	/0	ND	
(4)	$P$	/012	$2, \Diamond O2$	
(5)	SHOW: $\Box \Diamond P$	/012	ND	
(6)	SHOW: $\Box P$	/0123	$4, \Box I$	
(7)	$\Box P$	/01	$5, \Box O(b)$	***
(8)	$P$	/0	$7, \Box O$	

The formula  $\Diamond \Diamond P \rightarrow P$  is not provable in System KB, so introducing  $\Box$  is problematic, unless we restrict the  $b$ -rules. The offending line is (7), which is the only line that utilizes  $\Box O(b)$ . Notice, however, that the inference in question involves the formula,  $\Box P$ , which has depth  $-1$ , and accordingly can be disallowed by adding the depth-restriction to the  $b$ -rules.

### 2. 4-Rules

$\frac{\Box O(4) \quad \Box \Phi \quad /i}{\Phi \quad /i+m+n \text{ (old)}}$	$\frac{\Diamond I(4) \quad \Phi \quad /i+m+n \text{ (old)}}{\Diamond \Phi \quad /i}$
--	--

This rule is problematic, as evidenced by the following derivation.

(1)	SHOW: $\Diamond(P \ \& \ \Diamond Q) \rightarrow P$	/0	CD	
(2)	$\Diamond(P \ \& \ \Diamond Q)$	/0	As	
(3)	SHOW: $P$	/0	DD	
(4)	$P \ \& \ \Diamond Q$	/01	$2, \Diamond O$	
(5)	$Q$	/012	$4b, \Diamond O$	
(6)	$\Box P$	/012	$4a, \Box I$	
(7)	$\Diamond \Box P$	/0	$6, \Diamond I(4)$	***
(8)	$\Box P$	/03	$7, \Diamond O$	
(9)	$P$	/0	$8, \Box O$	

The formula in question is not valid in K4, so we have a problem. On the other hand, line (7) involves the formula  $\Box P$ , which has depth  $-1$ , and accordingly can be disallowed by adding the depth-restriction to the  $4$ -rules.

**3. 5-Rules**

$\begin{array}{l} \Box O(5) \\ \quad \Box \Phi \qquad /i+m \text{ (old)} \\ \quad \hline \quad \Phi \qquad /i+n \text{ (old)} \end{array}$	$\begin{array}{l} \Diamond I(5) \\ \quad \Phi \qquad /i+m \text{ (old)} \\ \quad \hline \quad \Diamond \Phi \qquad /i+n \text{ (old)} \end{array}$
--	--

The 5-rules are problematic. Consider the following derivation.

(1)	SHOW: $\Diamond P \rightarrow P$	/0	CD	
(2)	$\Diamond P$	/0	As	
(3)	SHOW: P	/0	DD	
(4)	P	/01	2, $\Diamond O$	
(5)	SHOW: $\Box \Diamond P$	/01	ND	
(6)	SHOW: $\Diamond \Box P$	/012	DD	
(7)	$\Diamond \Box P$	/012	4, $\Diamond I$	
(8)	$\Box P$	/01	5, $\Box O(5)$	***
(9)	P	/0	8, $\Box O$	

As before, this is a highly undesirable result, so we need to block the inference. On the other hand, line (8) involves the formula  $\Box P$ , which has depth  $-1$ , and accordingly can be disallowed by adding the depth-restriction to the 5-rules.

## 6. Summary of Adjusted Rules for K+ Systems with Actuality

With the above results in mind, we propose the following adjustments for the modal rules for K+ systems with actuality.

$\frac{\Box O(k) \quad \Box \Phi \quad /i}{\Phi \quad /i+m \text{ (old)}}$	$\frac{\Diamond I(k) \quad \Phi \quad /i+m \text{ (old)}}{\Diamond \Phi \quad /i}$
$\frac{\Box O(d) \quad \Box \Phi \quad /i}{\Phi \quad /i+n \text{ (new)}}$	$\frac{\Diamond I(d) \quad \Phi \quad /i+n \text{ (new)}}{\Diamond \Phi \quad /i}$
no further restriction	
$\frac{\Box O(t) \quad \Box \Phi \quad /i}{\Phi \quad /i}$	$\frac{\Diamond I(t) \quad \Phi \quad /i}{\Diamond \Phi \quad /i}$
$\frac{\Box O(b) \quad \Box \Phi \quad /i+m \text{ (old)}}{\Phi \quad /i}$	$\frac{\Diamond I(b) \quad \Phi \quad /i}{\Diamond \Phi \quad /i+m \text{ (old)}}$
$\frac{\Box O(4) \quad \Box \Phi \quad /i}{\Phi \quad /i+m+n \text{ (old)}}$	$\frac{\Diamond I(4) \quad \Phi \quad /i+m+n \text{ (old)}}{\Diamond \Phi \quad /i}$
$\frac{\Box O(5) \quad \Box \Phi \quad /i+m \text{ (old)}}{\Phi \quad /i+n \text{ (old)}}$	$\frac{\Diamond I(5) \quad \Phi \quad /i+m \text{ (old)}}{\Diamond \Phi \quad /i+n \text{ (old)}}$
$\Phi$ must have modal depth $\geq 0$	

## 15. Derivation Exercises

**Directions:** for each of the following argument forms, construct a formal derivation of the conclusion (marked by ‘/’) from the premises (if any) in QML(*k*)+ $\bigcirc$ . In cases in which two formulas are separated by ‘//’, construct a derivation of each formula from the other.

1. /  $\forall x\{\diamond[x = \imath y\bigcirc Fy] \rightarrow x = \imath yFy\}$
2.  $\diamond[k \neq \bigcirc k]$  ;  $\exists x[x=k] \ \& \ \square\exists x[x=k]$  /  $(k/x)\diamond[k \neq x]$
3.  $(k/x)\diamond[k \neq x]$  /  $\diamond[k \neq \bigcirc k]$
4.  $\square\exists x[x = k]$  ;  $\diamond\square[k = \bigcirc k]$  /  $\diamond(k/x)\square[k = x]$
5.  $\diamond(k/x)\square[k = x]$  /  $\diamond\square[k = \bigcirc k]$
6.  $\exists x[x = k]$  ;  $\diamond[k \neq \bigcirc k]$  /  $\diamond[k \neq \bigcirc k]$
7.  $\exists x[x = k]$  ;  $\diamond[k \neq \bigcirc k]$  /  $\diamond[k \neq \bigcirc k]$
8. /  $\square\forall x\{x=\bigcirc k \leftrightarrow x=\bigcirc k\}$
9.  $\square\exists x[x = k]$  ;  $\diamond\square[k = \bigcirc k]$  /  $\exists x\diamond\{k=x \ \& \ \square[k=x]\}$
10.  $\square\exists y[y = \imath xFx]$  ;  $\diamond\square[\imath x\bigcirc Fx = \imath xFx]$  /  $\diamond\exists y\square[y = \imath xFx]$
11.  $\diamond\exists x(Fx \ \& \ \square Gx)$  /  $\diamond\square\exists x(\bigcirc Fx \ \& \ Gx)$
12.  $\square\forall x(Fx \rightarrow \square Fx)$  ;  $\diamond\square\exists x(\bigcirc Fx \ \& \ Gx)$  /  $\diamond\square\exists x\{Fx \ \& \ Gx\}$
13.  $\square\forall x(Fx \rightarrow \square Fx)$  ;  $\square\square\exists x(\bigcirc Fx \ \& \ Gx)$  /  $\square\square\exists x\{Fx \ \& \ Gx\}$
14.  $\square\exists x[x = \imath xFx]$  /  $\square\square\exists x\{\imath x\bigcirc Fx = \imath xFx \rightarrow x = \imath xFx\}$
15.  $\exists x[x = s(k)]$  ;  $\square[s(k) = \bigcirc s(k)]$  /  $(s(k)/x)\square[s(k) = x]$
16.  $(s(k)/x)\square[s(k) = x]$  /  $\square[s(k) = \bigcirc s(k)]$
17.  $\exists x[x = s(k)]$  ;  $\square[s(k) = \bigcirc s(k)]$  /  $(s(k)/x)\square[s(k) = x]$
18.  $(s(k)/x)\diamond[s(k) \neq x]$  /  $\diamond[s(k) \neq \bigcirc s(k)]$
19.  $\exists x[x = s(k)]$  ;  $\diamond[s(k) \neq \bigcirc s(k)]$  /  $(s(k)/x)\diamond[s(k) \neq x]$
20.  $\diamond(j/x)\square(x=j \ \& \ Fx)$  /  $\diamond\square\{F[\bigcirc j] \ \& \ Fj\}$
21.  $\square\diamond\square R[\bigcirc\bigcirc k, \bigcirc k]$  /  $\square\forall x\{x=k \rightarrow \diamond\forall y[y=k \rightarrow \square Rxy]\}$
22.  $\square\exists x[x = \imath x\bigcirc Fx]$  ;  $\square\exists x[x = \bigcirc\imath xFx]$  /  $\square[\imath x\bigcirc Fx = \bigcirc\imath xFx]$
23.  $\exists x[x = k]$  /  $(k/x)\diamond B[k, x] \leftrightarrow \diamond B[k, \bigcirc k]$
24. /  $\forall x\square\{\bigcirc[x = \imath xFx] \rightarrow x = \imath x\bigcirc Fx\}$
25. /  $\forall x\square\{x = \bigcirc\imath xFx \rightarrow x = \imath x\bigcirc Fx\}$
26. /  $\square\forall x\{\bigcirc[x = \tau] \leftrightarrow x = \bigcirc\tau\}$
27.  $\square\exists x[x = \imath xFx]$  /  $\square\square\exists x\{\imath x\bigcirc Fx = \imath xFx \rightarrow x = \imath xFx\}$
28. /  $\forall x\{\diamond[x = \imath x\bigcirc Fx] \rightarrow x = \imath xFx\}$
29.  $\square\square\exists x(\bigcirc\bigcirc Fx \ \& \ \sim\bigcirc Fx)$  ;  $\diamond P$  /  $\square\exists x(\bigcirc Fx \ \& \ \sim Fx)$
30.  $\diamond F[\bigcirc k]$  ;  $\exists x[x = k]$  /  $(k/x)\diamond Fx$
31.  $\square\diamond\square R[\bigcirc\bigcirc k, \bigcirc k]$  ;  $\square\exists x[x = k]$  ;  $\square\square\exists x[x = k]$  /  $\square(k/x)\diamond(k/y)\square Rxy$
32.  $\square\exists x[x = k]$  ;  $\square\diamond[k = \bigcirc k]$  /  $\square(k/x)\diamond[k = x]$
33. /  $\forall x\{\square\diamond[x = \imath x\bigcirc Fx] \rightarrow \square[x = \imath xFx]\}$
34. /  $\forall x\{\diamond\diamond[x = \imath x\bigcirc Fx] \rightarrow \diamond[x = \imath xFx]\}$
35.  $\square\exists x[x = \imath xFx]$  ;  $\square\square[\imath x\bigcirc Fx = \imath xFx]$  /  $\square\exists x\square[x = \imath xFx]$
36.  $\square\forall x(Fx \rightarrow \square Fx)$  ;  $\square\square\exists x(\bigcirc Fx \ \& \ Gx)$  /  $\square\square\exists x\{Fx \ \& \ Gx\}$
37.  $\square F[\bigcirc k]$  ;  $\exists x[x = k]$  /  $(k/x)\square Fx$
38.  $\exists x\diamond Fx$  ;  $\square\exists x\forall y(Fy \rightarrow y=x)$  /  $\diamond\exists x\square[x = \bigcirc\imath xFx]$
39.  $\square\exists x(\bigcirc Fx \ \& \ Gx)$  ;  $\exists x[x = \imath xFx]$  /  $\exists x\square Gx$
40.  $\square\exists y[y = \imath xFx]$  ;  $\square\diamond[\imath x\bigcirc Fx = \imath xFx]$  /  $\square\exists y\diamond[y = \imath xFx]$

## 16. Answers to Selected Exercises

## #1

(1)	SHOW: $\forall x\{\Diamond[x = \imath y \circ Fy] \rightarrow x = \imath y Fy\}$ /0UCD	
(2)	$\Diamond[a = \imath y \circ Fy]$	/0 As
(3)	SHOW: $a = \imath y Fy$	/0 4,1I
(4)	SHOW: $\forall y\{Fy \leftrightarrow y=a\}$	/0 UD
(5)	SHOW: $Fb \leftrightarrow b=a$	/0 6,15,SL
(6)	SHOW: $Fb \rightarrow b=a$	/0 CD
(7)	$Fb$	/0 As
(8)	SHOW: $b=a$	/0 DD
(9)	$a = \imath y \circ Fy$	/01 2, $\Diamond O$
(10)	$\forall y\{\circ Fy \leftrightarrow y=a\}$	/01 9,1O
(11)	$\circ Fb$	/01 7, $\circ I$
(12)	$\circ Fb \leftrightarrow b=a$	/01 10, $\forall O$
(13)	$b=a$	/01 11,12,SL
(14)	$b=a$	/0 13, =Rep
(15)	SHOW: $b=a \rightarrow Fb$	/0 CD
(16)	$b=a$	/0 As
(17)	SHOW: $Fb$	/0 DD
(18)	$a = \imath y \circ Fy$	/01 2, $\Diamond O$
(19)	$\forall y\{\circ Fy \leftrightarrow y=a\}$	/01 18,1O
(20)	$\circ Fa$	/01 19, $\circ O$
(21)	$Fa$	/0 20, $\circ O$
(22)	$Fb$	/0 16,21,LL

## #2

(1)	$\Diamond[k \neq \circ k]$	/0 Pr
(2)	$\exists x[x=k] \ \& \ \Diamond \exists x[x=k]$	/0 Pr
(3)	SHOW: $(k/x)\Diamond[k \neq x]$	/0 Df ( $\tau/v$ )
(4)	SHOW: $\exists x\{x=k \ \& \ \Diamond[k \neq x]\}$	/0 13, $\exists I$
(5)	$k \neq \circ k$	/01 1, $\Diamond O$
(6)	$\exists x[x=k]$	/01 2b, $\square O$
(7)	$a=k$	/01 6, $\exists O$
(8)	$a \neq \circ k$	/01 5,7,LL
(9)	$b=k$	/0 2a, $\exists O$
(10)	$b = \circ k$	/01 9, $\circ I$
(11)	$k \neq b$	/01 5,10,LL
(12)	$\Diamond[k \neq b]$	/0 11, $\Diamond I$
(13)	$b = k \ \& \ \Diamond[k \neq b]$	/0 10,12,SL

## #3

(1)	$(k/x)\Diamond[k \neq x]$	/0 Pr
(2)	SHOW: $\Diamond[k \neq \circ k]$	/0 DD
(3)	$\exists x\{x=k \ \& \ \Diamond[k \neq x]\}$	/0 1, Df ( $\tau/v$ )
(4)	$a=k \ \& \ \Diamond[k \neq a]$	/0 3, $\exists O$
(5)	$k \neq a$	/01 4b, $\Diamond O$
(6)	$a = \circ k$	/01 4a, $\circ I$
(7)	$k \neq \circ k$	/01 5,6,LL
(8)	$\Diamond[k \neq \circ k]$	/0 7, $\Diamond I$

## #4

(1)	$\square \exists x[x = k]$	/0 Pr
(2)	$\Diamond \square[k = \circ k]$	/0 Pr
(3)	SHOW: $\Diamond(k/x)\square[k = x]$	/0 15, $\Diamond I$
(4)	$\square[k = \circ k]$	/01 2, $\Diamond O$
(5)	$\exists x[x = k]$	/01 1, $\square O$
(6)	$a = k$	/01 5, $\exists O$
(7)	$k = a$	/01 6, Sym=
(8)	SHOW: $\square[k = a]$	/01 ND
(9)	SHOW: $k = a$	/012 DD
(10)	$a = \circ k$	/012 6, $\circ I$
(11)	$k = \circ k$	/012 4, $\square O$
(12)	$k = a$	/012 10,11,LL
(13)	$k = a \ \& \ \square[k = a]$	/01 7,8,SL
(14)	$\exists x\{k=x \ \& \ \square[k=x]\}$	/01 13, $\exists I$
(15)	$(k/x)\square[k=x]$	/01 14, Df ( $\tau/v$ )

## #5

(1)	$\Diamond(k/x)\square[k = x]$	/0 Pr
(2)	SHOW: $\Diamond \square[k = \circ k]$	/0 6, $\Diamond I$
(3)	$(k/x)\square[k = x]$	/01 1, $\Diamond O$
(4)	$\exists x\{x=k \ \& \ \square[k = x]\}$	/01 3, Df ( $\tau/v$ )
(5)	$a=k \ \& \ \square[k = a]$	/01 4, $\exists O$
(6)	SHOW: $\square[k = \circ k]$	/01 ND
(7)	SHOW: $k = \circ k$	/012 DD
(8)	$k = a$	/012 5b, $\square O$
(9)	$a = \circ k$	/012 5a, $\circ I$
(10)	$k = \circ k$	/012 8,9,LL

## #6

(1)	$\exists x[x = k]$	/0 Pr
(2)	$\Diamond[k \neq \circ k]$	/0 Pr
(3)	SHOW: $\Diamond[k \neq \circ k]$	/0 5, $\Diamond I$
(4)	$k \neq \circ k$	/01 2, $\Diamond O$
(5)	SHOW: $k \neq \circ k$	/01 ID
(6)	$k = \circ k$	/01 As
(7)	SHOW: $\ast$	/* 8,12,SL
(8)	$\circ k \neq \circ k$	/01 4,6,LL
(9)	$a = k$	/0 1, $\exists O$
(10)	$a = \circ k$	/01 9, $\circ I$
(11)	$a = \circ k$	/01 9, $\circ I$
(12)	$\circ k = \circ k$	/01 10,11,LL

## #7

(1)	$\exists x[x = k]$	/0 Pr
(2)	$\Diamond[k \neq \circ k]$	/0 Pr
(3)	SHOW: $\Diamond[k \neq \circ k]$	/0 9, $\Diamond I$
(4)	$k \neq \circ k$	/01 2, $\Diamond O$
(5)	$a = k$	/0 1, $\exists O$
(6)	$a = \circ k$	/01 5, $\circ I$
(7)	$a = \circ k$	/01 5, $\circ I$
(8)	$\circ k = \circ k$	/01 6,7,LL
(9)	$k \neq \circ k$	/01 4,8,LL

## #8

(1)	SHOW: $\Box \forall x \{x = \bigcirc K \leftrightarrow x = \bigcirc K\}$	/0	ND
(2)	SHOW: $\forall x \{x = \bigcirc K \leftrightarrow x = \bigcirc K\}$	/01	UD
(3)	SHOW: $a = \bigcirc K \leftrightarrow a = \bigcirc K$	/01	SL
(4)	SHOW: $\rightarrow$	/01	CD
(5)	$a = \bigcirc K$	/01	As
(6)	SHOW: $a = \bigcirc K$	/01	7, $\bigcirc$ I
(7)	$a = K$	/0	5, $\bigcirc$ O
(8)	SHOW: $\leftarrow$	/01	CD
(9)	$a = \bigcirc K$	/01	As
(10)	SHOW: $a = \bigcirc K$	/01	11, $\bigcirc$ I
(11)	$a = K$	/0	9, $\bigcirc$ O

## #9

(1)	$\Box \exists x \{x = K\}$	/0	Pr
(2)	$\Diamond \Box [K = \bigcirc K]$	/0	Pr
(3)	SHOW: $\exists x \Diamond \{K = x \ \& \ \Box [K = x]\}$	/0	14, $\exists$ I
(4)	$\Box [K = \bigcirc K]$	/01	2, $\Diamond$ O
(5)	$\exists x \{x = K\}$	/01	1, $\Box$ O
(6)	$a = K$	/01	5, $\exists$ O
(7)	$K = a$	/01	6, Sym=
(8)	SHOW: $\Box [K = a]$	/01	ND
(9)	SHOW: $K = a$	/012	DD
(10)	$K = \bigcirc K$	/012	4, $\Box$ O
(11)	$a = \bigcirc K$	/012	6, $\bigcirc$ I
(12)	$K = a$	/012	10, 11, LL
(13)	$K = a \ \& \ \Box [K = a]$	/01	7, 8, SL
(14)	$\Diamond \{K = a \ \& \ \Box [K = a]\}$	/0	13, $\Diamond$ I

## #10

(1)	$\Box \exists y \{y = \neg x Fx\}$	/0	Pr
(2)	$\Diamond \Box [\neg x \bigcirc Fx = \neg x Fx]$	/0	Pr
(3)	SHOW: $\Diamond \exists y \Box [y = \neg x Fx]$	/0	7, $\Diamond$ I
(4)	$\Box [\neg x \bigcirc Fx = \neg x Fx]$	/01	2, $\Diamond$ O
(5)	$\exists y \{y = \neg x Fx\}$	/01	1, $\Box$ O
(6)	$a = \neg x Fx$	/01	5, $\exists$ O
(7)	SHOW: $\exists y \Box [y = \neg x Fx]$	/01	7, $\exists$ I
(8)	SHOW: $\Box [a = \neg x Fx]$	/01	ND
(9)	SHOW: $a = \neg x Fx$	/012	DD
(10)	$\neg x \bigcirc Fx = \neg x Fx$	/012	4, $\Box$ O
(11)	SHOW: $a = \neg x \bigcirc Fx$	/012	12, $\neg$ I
(12)	SHOW: $\forall x \{\bigcirc Fx \leftrightarrow x = a\}$	/012	UD
(13)	SHOW: $\bigcirc Fb \leftrightarrow b = a$	/012	14, 21, SL
(14)	SHOW: $\bigcirc Fb \rightarrow b = a$	/012	CD
(15)	$\bigcirc Fb$	/012	As
(16)	SHOW: $b = a$	/012	DD
(17)	$Fb$	/01	15, $\bigcirc$ O
(18)	$\forall x \{Fx \leftrightarrow x = a\}$	/01	6, $\neg$ O
(19)	$Fb \leftrightarrow b = a$	/01	18, $\forall$ O
(20)	$b = a$	/01	17, 19, SL
(21)	SHOW: $b = a \rightarrow \bigcirc Fb$	/012	CD
(22)	$b = a$	/012	As
(23)	SHOW: $\bigcirc Fb$	/012	24, $\bigcirc$ I
(24)	SHOW: $Fb$	/01	DD
(25)	$\forall x \{Fx \leftrightarrow x = a\}$	/01	6, $\neg$ O
(26)	$Fa$	/01	OO
(27)	$Fb$	/01	22, 26, LL
(28)	$a = \neg x Fx$	/012	10, 11, LL

## #11

(1)	$\Diamond \exists x \{Fx \ \& \ \Box Gx\}$	/0	Pr
(2)	SHOW: $\Diamond \Box \exists x \{\bigcirc Fx \ \& \ Gx\}$	/0	5, $\Diamond$ I
(3)	$\exists x \{Fx \ \& \ \Box Gx\}$	/01	1, $\Diamond$ O
(4)	$Fa \ \& \ \Box Ga$	/01	3, $\exists$ O
(5)	SHOW: $\Box \exists x \{\bigcirc Fx \ \& \ Gx\}$	/01	ND
(6)	SHOW: $\exists x \{\bigcirc Fx \ \& \ Gx\}$	/012	9, $\exists$ I
(7)	$\bigcirc Fa$	/012	4a, $\bigcirc$ I
(8)	$Ga$	/012	4b, $\Box$ O
(9)	$\bigcirc Fa \ \& \ Ga$	/012	7, 8, SL

## #12

(1)	$\Box \forall x \{Fx \rightarrow \Box Fx\}$	/0	Pr
(2)	$\Diamond \Box \exists x \{\bigcirc Fx \ \& \ Gx\}$	/0	Pr
(3)	SHOW: $\Diamond \Box \exists x \{Fx \ \& \ Gx\}$	/0	4, $\Diamond$ I
(4)	$\Box \exists x \{\bigcirc Fx \ \& \ Gx\}$	/01	2, $\Diamond$ O
(5)	SHOW: $\Box \exists x \{Fx \ \& \ Gx\}$	/01	6, $\Diamond$ I
(6)	SHOW: $\exists x \{Fx \ \& \ Gx\}$	/012	13, $\exists$ I
(7)	$\exists x \{\bigcirc Fx \ \& \ Gx\}$	/012	4, $\Box$ O
(8)	$\bigcirc Fa \ \& \ Ga$	/012	7, $\exists$ O
(9)	$Fa$	/01	8a, $\bigcirc$ O
(10)	$\forall x \{Fx \rightarrow \Box Fx\}$	/01	1, $\Box$ O
(11)	$\Box Fa$	/01	9, 10, QL
(12)	$Fa$	/012	11, $\Box$ O
(13)	$Fa \ \& \ Ga$	/012	8b, 12, SL

## #13

(1)	$\Box \forall x \{Fx \rightarrow \Box Fx\}$	/0	Pr
(2)	$\Box \Box \exists x \{\bigcirc Fx \ \& \ Gx\}$	/0	Pr
(3)	SHOW: $\Box \Box \exists x \{Fx \ \& \ Gx\}$	/0	ND2
(4)	SHOW: $\exists x \{Fx \ \& \ Gx\}$	/012	6b, 10, QL
(5)	$\exists x \{\bigcirc Fx \ \& \ Gx\}$	/012	2, $\Box$ O2
(6)	$\bigcirc Fa \ \& \ Ga$	/012	5, $\exists$ O
(7)	$Fa$	/01	6a, $\bigcirc$ O
(8)	$\forall x \{Fx \rightarrow \Box Fx\}$	/01	1, $\Box$ O
(9)	$\Box Fa$	/01	9, 10, QL
(10)	$Fa$	/012	11, $\Box$ O

## #14

(1)	$\Box \exists x \{x = \neg x Fx\}$	/0	Pr
(2)	SHOW: $\Box \Box \exists x \{\neg x \bigcirc Fx = \neg x Fx \rightarrow x = \neg x Fx\}$	/0	ND2
(3)	SHOW: $\exists x \{\neg x \bigcirc Fx = \neg x Fx \rightarrow x = \neg x Fx\}$	/012	ID
(4)	$\sim \exists x \{\neg x \bigcirc Fx = \neg x Fx \rightarrow x = \neg x Fx\}$	/012	As
(5)	SHOW: $\times$	/*	12, 26, SL
(6)	$\exists x \{x = \neg x Fx\}$	/01	1, $\Box$ O
(7)	$a = \neg x Fx$	/01	6, $\exists$ O
(8)	$\forall x \{Fx \leftrightarrow x = a\}$	/01	7, $\neg$ O
(9)	$a = \bigcirc \neg x Fx$	/012	5, $\bigcirc$ I
(10)	$\sim \{\neg x \bigcirc Fx = \neg x Fx \rightarrow a = \neg x Fx\}$	/012	4, QL
(11)	$\neg x \bigcirc Fx = \neg x Fx$	/012	10, SL
(12)	$a \neq \neg x Fx$	/012	10, SL
(13)	SHOW: $a = \neg x \bigcirc Fx$	/012	14, $\neg$ I
(14)	SHOW: $\forall x \{\bigcirc Fx \leftrightarrow x = a\}$	/012	UD
(15)	SHOW: $\bigcirc Fb \leftrightarrow b = a$	/012	16, 21, SL
(16)	SHOW: $\bigcirc Fb \rightarrow b = a$	/012	CD
(17)	$\bigcirc Fb$	/012	As
(18)	SHOW: $b = a$	/012	20, =Rep
(19)	$Fb$	/01	17, $\bigcirc$ O
(20)	$b = a$	/01	8, 19, QL
(21)	SHOW: $b = a \rightarrow \bigcirc Fb$	/012	CD
(22)	$b = a$	/012	As
(23)	SHOW: $\bigcirc Fb$	/012	25, $\bigcirc$ I
(24)	$b = a$	/01	22, =Rep
(25)	$Fb$	/01	8, QL
(26)	$a = \neg x Fx$	/012	11, 13, LL

## #15

(1)	$\exists x[x=s(K)]$	/0	Pr
(2)	$\Box[s(K)=\bigcirc s(K)]$	/0	Pr
(3)	SHOW: $(s(K)/x)\Box[s(K)=x]$	/0	Df ( $\tau/v$ )
(4)	SHOW: $\exists x\{x=s(K) \ \& \ \Box[s(K)=x]\}$	/0	5,6,QL
(5)	$a = s(K)$	/0	1, $\exists$ O
(6)	SHOW: $\Box[s(K)=a]$	/0	ND
(7)	SHOW: $s(K)=a$	/01	8,9,LL
(8)	$s(K) = \bigcirc s(K)$	/01	2, $\Box$ O
(9)	$a = \bigcirc s(K)$	/01	5, $\bigcirc$ I

## #16

(1)	$(s(K)/x)\Box[s(K)=x]$	/0	Pr
(2)	SHOW: $\Box[s(K)=\bigcirc s(K)]$	/0	ND
(3)	SHOW: $s(K) = \bigcirc s(K)$	/01	6,7,LL
(4)	$\exists x\{x=s(K) \ \& \ \Box[s(K)=x]\}$	/0	1,Df ( $\tau/v$ )
(5)	$a=s(K) \ \& \ \Box[s(K)=a]$	/0	4, $\exists$ O
(6)	$a = \bigcirc s(K)$	/01	5a, $\bigcirc$ I
(7)	$s(K) = a$	/01	5b, $\Box$ O

## #17

(1)	$\exists x[x=s(K)]$	/0	Pr
(2)	$\Box[s(K)=\bigcirc s(K)]$	/0	Pr
(3)	SHOW: $(s(K)/x)\Box[s(K)=x]$	/0	Df ( $\tau/v$ )
(4)	SHOW: $\exists x\{x=s(K) \ \& \ \Box[s(K)=x]\}$	/0	5,6,QL
(5)	$a = s(K)$	/0	1, $\exists$ O
(6)	SHOW: $\Box[s(K)=a]$	/0	ND
(7)	SHOW: $s(K) = a$	/01	DD
(8)	$s(K) = \bigcirc s(K)$	/01	2, $\Box$ O
(9)	$a = \bigcirc s(K)$	/01	5, $\bigcirc$ I
(10)	$s(K) = a$	/01	8,9,LL

## #18

(1)	$(s(K)/x)\Diamond[s(K)\neq x]$	/0	Pr
(2)	SHOW: $\Diamond[s(K)\neq\bigcirc s(K)]$	/0	7, $\Diamond$ I
(3)	$\exists x\{x=s(K) \ \& \ \Diamond[s(K)\neq x]\}$	/0	1,Df ( $\tau/v$ )
(4)	$a=s(K) \ \& \ \Diamond[s(K)\neq a]$	/0	3, $\exists$ O
(5)	$s(K)\neq a$	/01	4b, $\Diamond$ O
(6)	$a = \bigcirc s(K)$	/01	4a, $\bigcirc$ I
(7)	$s(K)\neq\bigcirc s(K)$	/01	5,6,LL

## #19

(1)	$\exists x[x=s(K)]$	/0	Pr
(2)	$\Diamond[s(K)\neq\bigcirc s(K)]$	/0	Pr
(3)	SHOW: $(s(K)/x)\Diamond[s(K)\neq x]$	/0	Df ( $\tau/v$ )
(4)	SHOW: $\exists x\{x=s(K) \ \& \ \Diamond[s(K)\neq x]\}$	/0	5,9,QL
(5)	$a = s(K)$	/0	1, $\exists$ O
(6)	$s(K) \neq \bigcirc s(K)$	/01	2, $\Diamond$ O
(7)	$a = \bigcirc s(K)$	/01	5, $\bigcirc$ I
(8)	$s(K) \neq a$	/01	6,7,LL
(9)	$\Diamond[s(K) \neq a]$	/0	8, $\Diamond$ I

## #20

(1)	$\Diamond(J/x)\Box(x=J \ \& \ Fx)$	/0	Pr
(2)	SHOW: $\Diamond\Box\{F[\bigcirc J] \ \& \ FJ\}$	/0	ND
(3)	$(J/x)\Box(x=J \ \& \ Fx)$	/01	1, $\Diamond$ O
(4)	SHOW: $\Box\{F[\bigcirc J] \ \& \ FJ\}$	/01	ND
(5)	SHOW: $F[\bigcirc J] \ \& \ FJ$	/012	10,11,SL
(6)	$\exists x\{x=J \ \& \ \Box(x=J \ \& \ Fx)\}$	/01	3,Df ( $\tau/v$ )
(7)	$a=J \ \& \ \Box(a=J \ \& \ Fa)$	/01	6, $\exists$ O
(8)	$a=J \ \& \ Fa$	/012	7b, $\Box$ O
(9)	$FJ$	/012	8a,8b,LL
(10)	$a = \bigcirc J$	/012	7a, $\bigcirc$ I
(11)	$F[\bigcirc J]$	/012	8b,10,LL

## #21

(1)	$\Box\Diamond\Box R[\bigcirc\bigcirc K, \bigcirc K]$	/0	Pr
(2)	SHOW: $\Box\forall x\{x=K \rightarrow \Diamond\forall y[y=K \rightarrow \Box Rxy]\}$	/0	ND
(3)	SHOW: $\forall x\{x=K \rightarrow \Diamond\forall y[y=K \rightarrow \Box Rxy]\}$	/01	UCD
(4)	$a=K$	/01	As
(5)	SHOW: $\Diamond\forall y[y=K \rightarrow \Box Ray]$	/01	8, $\Diamond$ I
(6)	$\Diamond\Box R[\bigcirc\bigcirc K, \bigcirc K]$	/01	1, $\Box$ O
(7)	$\Box R[\bigcirc\bigcirc K, \bigcirc K]$	/012	6, $\Diamond$ O
(8)	SHOW: $\forall y[y=K \rightarrow \Box Ray]$	/012	UCD
(9)	$b=K$	/012	As
(10)	SHOW: $\Box Rab$	/012	ND
(11)	SHOW: $Rab$	/0123	12-14,LL2
(12)	$R[\bigcirc\bigcirc K, \bigcirc K]$	/0123	7, $\Box$ O
(13)	$a=\bigcirc\bigcirc K$	/0123	4, $\bigcirc$ I2
(14)	$b=\bigcirc K$	/0123	9, $\bigcirc$ I

## #22

(1)	$\Box\exists x[x=1x\bigcirc Fx]$	/0	Pr
(2)	$\Box\exists x[x=01x\bigcirc Fx]$	/0	Pr
(3)	SHOW: $\Box[1x\bigcirc Fx = 01x\bigcirc Fx]$	/0	ND
(4)	SHOW: $1x\bigcirc Fx = 01x\bigcirc Fx$	/01	7,8,14,LL2
(5)	$\exists x[x=1x\bigcirc Fx]$	/01	1, $\Box$ O
(6)	$\exists x[x=01x\bigcirc Fx]$	/01	2, $\Box$ O
(7)	$a = 1x\bigcirc Fx$	/01	5, $\exists$ O
(8)	$b = 01x\bigcirc Fx$	/01	6, $\exists$ O
(9)	$b = 1x\bigcirc Fx$	/0	8, $\bigcirc$ O
(10)	$\forall x(Fx \leftrightarrow x=b)$	/0	9, $\bigcirc$ I
(11)	$Fb$	/0	10, $\bigcirc$ O
(12)	$\bigcirc Fb$	/01	11, $\bigcirc$ I
(13)	$\forall x\{\bigcirc Fx \leftrightarrow x=a\}$	/01	7, $\bigcirc$ O
(14)	$b=a$	/01	12,13,QL

## #23

(1)	$\exists x[x=K]$	/0	Pr
(2)	SHOW: $(K/x)\Diamond B[K,x] \leftrightarrow \Diamond B[K,\bigcirc K]$	/0	SL
(3)	SHOW: $\rightarrow$	/0	CD
(4)	$(K/x)\Diamond B[K,x]$	/0	As
(5)	SHOW: $\Diamond B[K,\bigcirc K]$	/0	9, $\Diamond$ I
(6)	$\exists x\{x=K \ \& \ \Diamond B[K,x]\}$	/0	3,Df ( $\tau/v$ )
(7)	$a=K \ \& \ \Diamond B[K,a]$	/0	5, $\exists$ O
(8)	$B[K,a]$	/01	7b, $\Diamond$ O
(9)	$a=\bigcirc K$	/01	7a, $\bigcirc$ I
(10)	$B[K,\bigcirc K]$	/01	8,9,LL
(11)	SHOW: $\leftarrow$	/0	CD
(12)	$\Diamond B[K,\bigcirc K]$	/0	As
(13)	SHOW: $(K/x)\Diamond B[K,x]$	/0	Df ( $\tau/v$ )
(14)	SHOW: $\exists x\{x=K \ \& \ \Diamond B[K,x]\}$	/0	16,19,QL
(15)	$B[K,\bigcirc K]$	/01	11, $\Diamond$ O
(16)	$a=K$	/0	1, $\exists$ O
(17)	$a=\bigcirc K$	/01	16, $\bigcirc$ I
(18)	$B[K,a]$	/01	15,17,LL
(19)	$\Diamond B[K,a]$	/0	18, $\Diamond$ I

## #24

(1)	SHOW: $\forall x \Box \{ \Box [x = 1xFx] \rightarrow x = 1x \Box OFx \}$	/0	ND
(2)	SHOW: $\Box \{ \Box [a = 1xFx] \rightarrow a = 1x \Box OFx \}$	/0	ND
(3)	SHOW: $\Box [a = 1xFx] \rightarrow a = 1x \Box OFx$	/01	CD
(4)	$\Box [a = 1xFx]$	/01	As
(5)	SHOW: $a = 1x \Box OFx$	/01	6,1I
(6)	SHOW: $\forall x \{ \Box OFx \leftrightarrow x = a \}$	/01	UD
(7)	SHOW: $\Box Fb \leftrightarrow b = a$	/01	10,15,SL
(8)	$a = 1xFx$	/0	4,OO
(9)	$\forall x \{ Fx \leftrightarrow x = a \}$	/0	8,1O
(10)	SHOW: $\rightarrow$	/01	CD
(11)	$\Box Fb$	/01	As
(12)	SHOW: $b = a$	/01	14,=Rep
(13)	$Fb$	/0	11,OO
(14)	$b = a$	/0	9,13,QL
(15)	SHOW: $\leftarrow$	/01	CD
(16)	$b = a$	/01	As
(17)	SHOW: $\Box Fb$	/01	19,OI
(18)	$b = a$	/0	As
(19)	$Fb$	/0	9,18,QL

## #25

(1)	SHOW: $\forall x \Box \{ x = \Box 1xFx \rightarrow x = 1x \Box OFx \}$	/0	UD
(2)	SHOW: $\Box \{ a = \Box 1xFx \rightarrow a = 1x \Box OFx \}$	/0	ND
(3)	SHOW: $a = \Box 1xFx \rightarrow a = 1x \Box OFx$	/01	CD
(4)	$a = \Box 1xFx$	/01	As
(5)	SHOW: $a = 1x \Box OFx$	/01	6,1I
(6)	SHOW: $\forall x \{ \Box OFx \leftrightarrow x = a \}$	/01	UD
(7)	SHOW: $\Box Fb \leftrightarrow b = a$	/01	10,15,SL
(8)	$a = 1xFx$	/0	4,DfO
(9)	$\forall x \{ Fx \leftrightarrow x = a \}$	/0	8,1O
(10)	SHOW: $\rightarrow$	/01	CD
(11)	$\Box Fb$	/01	As
(12)	SHOW: $b = a$	/01	14,=Rep
(13)	$Fb$	/0	11,DfO
(14)	$b = a$	/0	9,13,QL
(15)	SHOW: $\leftarrow$	/01	CD
(16)	$b = a$	/01	As
(17)	SHOW: $\Box Fb$	/01	19,DfO
(18)	$b = a$	/0	As
(19)	$Fb$	/0	9,18,QL

## #26

(1)	SHOW: $\Box \forall x \{ \Box [x = T] \leftrightarrow x = \Box T \}$	/0	ND
(2)	SHOW: $\forall x \{ \Box [x = T] \leftrightarrow x = \Box T \}$	/01	UD
(3)	SHOW: $\Box [a = T] \leftrightarrow a = \Box T$	/01	SL
(4)	SHOW: $\rightarrow$	/01	CD
(5)	$\Box [a = T]$	/01	As
(6)	SHOW: $a = \Box T$	/01	7,OI
(7)	$a = T$	/0	5,OO
(8)	SHOW: $\leftarrow$	/01	CD
(9)	$a = \Box T$	/01	As
(10)	SHOW: $\Box [a = T]$	/01	11,OI
(11)	$a = T$	/0	9,OO

## #27

(1)	$\Box \exists x [x = 1xFx]$	/0	Pr
(2)	SHOW: $\Box \Box \exists x \{ 1x \Box OFx = 1xFx \rightarrow x = 1xFx \}$	/0	ND2
(3)	SHOW: $\exists x \{ 1x \Box OFx = 1xFx \rightarrow x = 1xFx \}$	/012	ID
(4)	$\sim \exists x \{ 1x \Box OFx = 1xFx \rightarrow x = 1xFx \}$	/012	As
(5)	SHOW: $\times$	/*	12,26,SL
(6)	$\exists x [x = 1xFx]$	/01	1,OO
(7)	$a = 1xFx$	/01	6,EO
(8)	$\forall x \{ Fx \leftrightarrow x = a \}$	/01	7,1O
(9)	$a = \Box 1xFx$	/012	5,OI
(10)	$\sim \{ 1x \Box OFx = 1xFx \rightarrow a = 1xFx \}$	/012	4,QL
(11)	$1x \Box OFx = 1xFx$	/012	10,SL
(12)	$a \neq 1xFx$	/012	10,SL
(13)	SHOW: $a = 1x \Box OFx$	/012	14,1I
(14)	SHOW: $\forall x \{ \Box Fx \leftrightarrow x = a \}$	/012	UD
(15)	SHOW: $\Box Fb \leftrightarrow b = a$	/012	16,21,SL
(16)	SHOW: $\Box Fb \rightarrow b = a$	/012	CD
(17)	$\Box Fb$	/012	As
(18)	SHOW: $b = a$	/012	20,=Rep
(19)	$Fb$	/01	17,OO
(20)	$b = a$	/01	8,19,QL
(21)	SHOW: $b = a \rightarrow \Box Fb$	/012	CD
(22)	$b = a$	/012	As
(23)	SHOW: $\Box Fb$	/012	25,OI
(24)	$b = a$	/01	22,=Rep
(25)	$Fb$	/01	8,QL
(26)	$a = 1xFx$	/012	11,13,LL

## #28

(1)	SHOW: $\forall x \{ \Diamond [x = 1x \Box OFx] \rightarrow x = 1xFx \}$	/0	UCD
(2)	$\Diamond [a = 1x \Box OFx]$	/0	As
(3)	SHOW: $a = 1xFx$	/0	4,1I
(4)	SHOW: $\forall x \{ Fx \leftrightarrow x = a \}$	/0	UD
(5)	SHOW: $Fb \leftrightarrow b = a$	/0	SL
(6)	SHOW: $\rightarrow$	/0	CD
(7)	$Fb$	/0	As
(8)	SHOW: $b = a$	/0	10,11,QL
(9)	$a = 1x \Box OFx$	/01	2,DO
(10)	$\forall x \{ \Box OFx \leftrightarrow x = a \}$	/01	9,1O
(11)	$\Box Fb$	/01	7,OO
(12)	SHOW: $\leftarrow$	/0	CD
(13)	$b = a$	/0	As
(14)	SHOW: $Fb$	/0	16,17,QL
(15)	$a = 1x \Box OFx$	/01	2,DO
(16)	$\forall x \{ \Box OFx \leftrightarrow x = a \}$	/01	15,1O
(17)	$b = a$	/01	13,=Rep

## #29

(1)	$\Box \Box \exists x (\Box OFx \ \& \ \sim \Box OFx)$	/0	Pr
(2)	$\Diamond \Diamond P$	/0	Pr
(3)	P	/012	2,DO2
(4)	SHOW: $\Box \exists x (\Box OFx \ \& \ \sim Fx)$	/0	ND
(5)	SHOW: $\exists x (\Box OFx \ \& \ \sim Fx)$	/01	9,10,QL
(6)	$\Box \exists x (\Box OFx \ \& \ \sim OFx)$	/01	1,OO
(7)	$\exists x (\Box OFx \ \& \ \sim OFx)$	/012	6,OO
(8)	$\Box OFa \ \& \ \sim OFa$	/012	7,EO
(9)	$\Box Fa$	/01	8a,OO
(10)	$\sim Fa$	/01	8b,OI(-)

## #30

(1)	$\Diamond F[\bigcirc K]$	/0	Pr
(2)	$\exists x[x=K]$	/0	Pr
(3)	SHOW: $(K/x)\Diamond Fx$	/0	Df ( $\tau/v$ )
(4)	SHOW: $\exists x\{x=K \ \& \ \Diamond Fx\}$	/0	6,9,QL
(5)	$F[\bigcirc K]$	/01	1, $\bigcirc O$
(6)	$a=K$	/0	2, $\exists O$
(7)	$a=\bigcirc K$	/01	6, $\bigcirc I$
(8)	$Fa$	/01	5,7,LL
(9)	$\Diamond Fa$	/0	8, $\Diamond I$

## #31

(1)	$\Box\Diamond\Box R[\bigcirc\bigcirc K,\bigcirc K]$	/0	Pr
(2)	$\Box\exists x[x=K]$	/0	Pr
(3)	$\Box\Box\exists x[x=K]$	/0	Pr
(4)	SHOW: $\Box(K/x)\Diamond(K/y)\Box Rxy$	/0	ND
(5)	SHOW: $(K/x)\Box(K/y)\Box Rxy$	/01	Df ( $\tau/v$ )
(6)	SHOW: $\exists x\{x=K \ \& \ \Diamond(K/y)\Box Rxy\}$	/01	8,11,QL
(7)	$\exists x[x=K]$	/01	2, $\Box O$
(8)	$a=K$	/01	7, $\exists O$
(9)	$\Diamond\Box R[\bigcirc\bigcirc K,\bigcirc K]$	/01	1, $\Box O$
(10)	$\Box R[\bigcirc\bigcirc K,\bigcirc K]$	/012	9, $\Diamond O$
(11)	SHOW: $\Diamond(K/y)\Box Ray$	/01	12, $\Diamond I$
(12)	SHOW: $(K/y)\Box Ray$	/012	Df ( $\tau/v$ )
(13)	SHOW: $\exists y\{y=K \ \& \ \Box Ray\}$	/012	15,16,QL
(14)	$\exists x[x=K]$	/012	3, $\Box O2$
(15)	$b=K$	/012	14, $\exists O$
(16)	SHOW: $\Box Rab$	/012	ND
(17)	SHOW: $Rab$	/0123	18-20,LL2
(18)	$R[\bigcirc\bigcirc K,\bigcirc K]$	/0123	10, $\Box O$
(19)	$a=\bigcirc\bigcirc K$	/0123	8, $\bigcirc I2$
(20)	$b=\bigcirc K$	/0123	15, $\bigcirc I$

## #32

(1)	$\Box\exists x[x=K]$	/0	Pr
(2)	$\Box\Diamond[K=\bigcirc K]$	/0	Pr
(3)	SHOW: $\Box(K/x)\Diamond[K=x]$	/0	ND
(4)	SHOW: $(K/x)\Diamond[K=x]$	/01	Df ( $\tau/v$ )
(5)	SHOW: $\exists x\{x=K \ \& \ \Diamond[K=x]\}$	/01	7,12,QL
(6)	$\exists x[x=K]$	/01	1, $\Box O$
(7)	$a=K$	/01	6, $\exists O$
(8)	$\Diamond[K=\bigcirc K]$	/01	2, $\Box O$
(9)	$K=\bigcirc K$	/012	8, $\Diamond O$
(10)	$a=\bigcirc K$	/012	7, $\bigcirc I$
(11)	$K=a$	/012	9,10,LL
(12)	$\Diamond[K=a]$	/01	11, $\Diamond I$

## #33

(1)	SHOW: $\forall x\{\Box\Diamond[x=\bigcirc x\bigcirc Fx] \rightarrow \Box[x=\bigcirc xFx]\}$	/0	UCD
(2)	$\Box\Diamond[a=\bigcirc x\bigcirc Fx]$	/0	As
(3)	SHOW: $\Box[a=\bigcirc xFx]$	/0	ND
(4)	SHOW: $a=\bigcirc xFx$	/01	5, $\bigcirc I$
(5)	SHOW: $\forall x\{Fx \leftrightarrow x=a\}$	/01	UD
(6)	$\Diamond[a=\bigcirc x\bigcirc Fx]$	/01	2, $\Box O$
(7)	$a=\bigcirc x\bigcirc Fx$	/012	5, $\Diamond O$
(8)	$\forall x\{\bigcirc Fx \leftrightarrow x=a\}$	/012	7, $\bigcirc O$
(9)	SHOW: $Fb \leftrightarrow b=a$	/01	10,15,SL
(10)	SHOW: $\rightarrow$	/01	CD
(11)	$Fb$	/01	As
(12)	SHOW: $b=a$	/01	14,=Rep
(13)	$\bigcirc Fb$	/012	11, $\bigcirc I$
(14)	$b=a$	/012	8,13,QL
(15)	SHOW: $\leftarrow$	/01	CD
(16)	$b=a$	/01	As
(17)	SHOW: $Fb$	/01	19, $\bigcirc O$
(18)	$b=a$	/012	16,=Rep
(19)	$\bigcirc Fb$	/012	8,18,QL

## #34

(1)	SHOW: $\forall x\{\Diamond\Diamond[x=\bigcirc x\bigcirc Fx] \rightarrow \Diamond[x=\bigcirc xFx]\}$	/0	UCD
(2)	$\Diamond\Diamond[a=\bigcirc x\bigcirc Fx]$	/0	As
(3)	SHOW: $\Diamond[a=\bigcirc xFx]$	/0	ND
(4)	$\Diamond[a=\bigcirc x\bigcirc Fx]$	/01	2, $\Diamond O$
(5)	$a=\bigcirc x\bigcirc Fx$	/012	4, $\Diamond O$
(6)	$\forall x\{\bigcirc Fx \leftrightarrow x=a\}$	/012	5, $\bigcirc O$
(7)	SHOW: $a=\bigcirc xFx$	/01	6, $\bigcirc I$
(8)	SHOW: $\forall x\{Fx \leftrightarrow x=a\}$	/01	UD
(9)	SHOW: $Fb \leftrightarrow b=a$	/01	10,15,SL
(10)	SHOW: $\rightarrow$	/01	CD
(11)	$Fb$	/01	As
(12)	SHOW: $b=a$	/01	14,=Rep
(13)	$\bigcirc Fb$	/012	11, $\bigcirc I$
(14)	$b=a$	/012	6,13,QL
(15)	SHOW: $\leftarrow$	/01	CD
(16)	$b=a$	/01	As
(17)	SHOW: $Fb$	/01	19, $\bigcirc O$
(18)	$b=a$	/012	16,=Rep
(19)	$\bigcirc Fb$	/012	6,18,QL

## #35

(1)	$\Box\exists x[x = 1xFx]$	/0	Pr
(2)	$\Box\Box[1x\Box Fx = 1xFx]$	/0	Pr
(3)	SHOW: $\Box\exists x\Box[x = 1xFx]$	/0	ND
(4)	SHOW: $\exists x\Box[x = 1xFx]$	/01	7,QL
(5)	$\exists x[x = 1xFx]$	/01	1, $\Box$ O
(6)	$a = 1xFx$	/01	5, $\exists$ O
(7)	SHOW: $\Box[a = 1xFx]$	/01	ND
(8)	SHOW: $a = 1xFx$	/012	9-11,LL2
(9)	$1x\Box Fx = 1xFx$	/012	2, $\Box$ O2
(10)	$a = \Box 1xFx$	/012	6, $\Box$ I
(11)	$a = 1xFx$	/01	10, $\Box$ O
(12)	$\forall x\{Fx \leftrightarrow x=a\}$	/01	11, $\Box$ O
(13)	SHOW: $a = 1x\Box Fx$	/012	14, $\Box$ I
(14)	SHOW: $\forall x\{\Box Fx \leftrightarrow x=a\}$	/012	UD
(15)	SHOW: $\Box Fb \leftrightarrow b=a$	/012	16,21SL
(16)	SHOW: $\rightarrow$	/012	CD
(17)	$\Box Fb$	/012	As
(18)	SHOW: $b=a$	/012	20,=Rep
(19)	$Fb$	/01	17, $\Box$ O
(20)	$b=a$	/01	19,20,QL
(21)	SHOW: $\leftarrow$	/012	CD
(22)	$b=a$	/012	As
(23)	SHOW: $\Box Fb$	/012	25, $\Box$ I
(24)	$b=a$	/01	22,=Rep
(25)	$Fb$	/01	12,24,QL

## #36

(1)	$\Box\forall x(Fx \rightarrow \Box Fx)$	/0	Pr
(2)	$\Box\Box\exists x(\Box Fx \& Gx)$	/0	Pr
(3)	SHOW: $\Box\Box\exists x\{Fx \& Gx\}$	/0	ND2
(4)	SHOW: $\exists x\{Fx \& Gx\}$	/012	7b,10,QL
(5)	$\forall x\{Fx \rightarrow \Box Fx\}$	/01	1, $\Box$ O
(6)	$\exists x(\Box Fx \& Gx)$	/012	2, $\Box$ O2
(7)	$\Box Fa \& Ga$	/012	6, $\exists$ O
(8)	$Fa$	/01	7a, $\Box$ O
(9)	$\Box Fa$	/01	5,8,QL
(10)	$Fa$	/012	9, $\Box$ O

## #37

(1)	$\Box F[\Box K]$	/0	Pr
(2)	$\exists x[x=K]$	/0	Pr
(3)	SHOW: $(K/x)\Box Fx$	/0	Df ( $\tau/\nu$ )
(4)	SHOW: $\exists x\{x=K \& \Box Fx\}$	/0	5,6,QL
(5)	$a=K$	/0	2, $\exists$ O
(6)	SHOW: $\Box Fa$	/0	ND
(7)	SHOW: $Fa$	/01	8,9,LL
(8)	$F[\Box K]$	/01	1, $\Box$ O
(9)	$a=\Box K$	/01	5, $\Box$ I

## #38

(1)	$\exists x\Diamond Fx$	/0	Pr
(2)	$\Box\exists x\forall y(Fy \rightarrow y=x)$	/0	Pr
(3)	SHOW: $\Diamond\exists x\Box[x = \Box 1xFx]$	/0	22, $\Diamond$ I
(4)	$\Diamond Fa$	/0	1, $\exists$ O
(5)	$Fa$	/01	4, $\Diamond$ O
(6)	$\exists x\forall y(Fy \rightarrow y=x)$	/01	2, $\Box$ O
(7)	$\forall y(Fy \rightarrow y=b)$	/01	6, $\exists$ O
(8)	$a=b$	/01	5,7,QL
(9)	SHOW: $\Box[a = \Box 1xFx]$	/01	ND
(10)	SHOW: $a = \Box 1xFx$	/012	11, $\Box$ I
(11)	SHOW: $a = 1xFx$	/01	12,I
(12)	SHOW: $\forall x\{Fx \leftrightarrow x=a\}$	/01	UD
(13)	SHOW: $Fc \leftrightarrow c=a$	/01	SL
(14)	SHOW: $\rightarrow$	/01	CD
(15)	$Fc$	/01	As
(16)	SHOW: $c=a$	/01	8,17,LL
(17)	$c=b$	/01	7,15,QL
(18)	SHOW: $\leftarrow$	01	CD
(19)	$c=b$	/01	As
(20)	SHOW: $Fc$	/01	5,21,LL
(21)	$a=c$	/01	8,19,LL
(22)	$\exists x\Box[x = \Box 1xFx]$	/01	9,QL

## #39

(1)	$\Box\exists x(\Box Fx \& Gx)$	/0	Pr
(2)	$\exists x[x = 1xFx]$	/0	Pr
(3)	SHOW: $\exists x\Box Gx$	/0	6,QL
(4)	$a = 1xFx$	/0	2, $\exists$ O
(5)	$\forall x\{Fx \leftrightarrow x=a\}$	/0	4, $\Box$ O
(6)	SHOW: $\Box Ga$	/0	ND
(7)	SHOW: $Ga$	/01	9b,12,LL
(8)	$\exists x(\Box Fx \& Gx)$	/01	1, $\Box$ O
(9)	$\Box Fb \& Gb$	/01	8, $\exists$ O
(10)	$Fb$	/0	9a, $\Box$ O
(11)	$b=a$	/0	5,10,QL
(12)	$b=a$	/01	11,=Rep

## #40

(1)	$\Box\exists x[x = 1xFx]$	/0	Pr
(2)	$\Box\Diamond[1x\Box Fx = 1xFx]$	/0	Pr
(3)	SHOW: $\Box\exists x\Diamond[x = 1xFx]$	/0	ND
(4)	SHOW: $\exists x\Diamond[x = 1xFx]$	/01	12, $\exists$ I
(5)	$\exists x[x = 1xFx]$	/01	1, $\Box$ O
(6)	$a = 1xFx$	/01	5, $\exists$ O
(7)	$\Diamond[1x\Box Fx = 1xFx]$	/01	2, $\Box$ O
(8)	$1x\Box Fx = 1xFx$	/012	7, $\Diamond$ O
(9)	$a = \Box 1xFx$	/012	6, $\Box$ I
(10)	$a = 1x\Box Fx$	/012	9+ET
(11)	$a = 1xFx$	/012	8,10,EQ=
(12)	$\Diamond[a = 1xFx]$	/01	11, $\Diamond$ I