6 Modal Predicate Logic

0. Overview

Having seen how modal sentential logic works, we now turn to modal predicate logic. This follows the same progression as introductory symbolic logic; one does sentential logic, followed by predicate logic.

This chapter is divided into three parts. Part A reviews the general ideas of ordinary predicate logic (quantifier logic). Part B develops *Simple* Modal Predicate Logic (MPL), which is obtained by simply combining the apparatus of modal sentential logic (including indexing) with the apparatus of ordinary predicate logic. The system so obtained is fairly simple-minded; in particular, MPL ignores proper nouns, function signs, identity, and descriptions.[1] It also ignores the issue of quantificational domain, which is taken up in Part C, which presents actualist quantificational principles.

A. Ordinary Predicate Logic

1. Introduction

Ordinary predicate logic (PL) is the logical system most students encounter after doing sentential logic. The move from sentential logic to predicate logic involves three major new concepts.

(1) noun phrases
(2) predicates
(3) quantifiers

2. Noun Phrases

The first item represents a new primitive category – which we call "noun phrases", and which we abbreviate by ‘N’. As we propose to use the term, a noun phrase is basically an expression that can serve as a subject or object of a verb. The following are all examples of noun phrases.

<table>
<tr><th>category</th><th colspan="6">examples</th></tr>
<tr><td>proper nouns</td><td>Mozart</td><td>Haydn</td><td>Jupiter</td><td>Saturn</td><td colspan="2">The Eiffel Tower</td></tr>
<tr><td>pronouns</td><td>I</td><td>you</td><td>he</td><td>she</td><td>it</td><td>they</td></tr>
<tr><td>compound noun phrases</td><td colspan="2">Jay's mother</td><td>2+2</td><td colspan="3">the square root of 2</td></tr>
<tr><td>descriptive noun phrases</td><td colspan="3">the man standing next to Bill</td><td colspan="3">the three tallest buildings</td></tr>
<tr><td>nominalized verbs</td><td>running</td><td>to run</td><td>as in</td><td colspan="3">I like to run</td></tr>
<tr><td>direct quotations</td><td colspan="2">“snow is white”</td><td>as in</td><td colspan="3">Jay said “snow is white”</td></tr>
<tr><td>indirect quotations</td><td colspan="2">that snow is white</td><td>as in</td><td colspan="3">Jay said that snow is white</td></tr>
</table>

Noun phrases come with a *number*, which is either singular or plural. In the case of pronouns, this is drilled into us in middle school, if not before. For example, we know that the pronoun ‘we’ is first person *plural*, and we know that the pronoun ‘she’ is third person *singular*. In the case of other such expressions, perhaps the easiest way to decide number is to ask what the appropriate form of the verb ‘to be’ is — if the appropriate verb is ‘is’, then the noun phrase is singular; if the appropriate verb is ‘are’, then the noun phrase is plural; simple as that! And pronouns *mostly* work the same way; I leave the reader to think about what the exceptions are.

[1] These notions are added in a later chapter on *full* quantified modal logic.

In classical predicate logic, only singular noun phrases – officially called *singular terms* – are considered, whereas plural noun phrases are either ignored or finessed. Also, a noun phrase can be grammatically simple or complex,[2] but predicate logic completely ignores their internal structure. For example, from the viewpoint of predicate logic, all of the following are atomic (simple).

Jay's mother
5+7
the woman standing near the window

3. Predicates

In addition to noun phrases, predicate logic concerns predicates. Whereas the category S of sentences, and the category N of noun phrases, are primitive categories, the category of predicates is a derivative (functor) category, just like connectives. The notion of *predicate* is defined as follows, which is followed by the subordinate definition of k-place predicate.

(d1) A *predicate* is an expression with zero or more blanks such that, filling these blanks with noun phrases results in a sentence.

(d2) Where k is any natural number (0, 1, 2, ...), a k-place *predicate* is a predicate with k places (blanks).

Notice that we allow k to be 0. Grammatically, a 0-place predicate is a predicate that takes no grammatical subject. The best examples of subject-less declarative sentences occur in weather reports – for example, ‘it is raining’, ‘it is snowing’, etc. Even though these sentences have an official grammatical subject ‘it’, it is obvious that ‘it’ does not refer to any*thing*. What exactly is "it" in ‘it is raining’?[3] Since there is no "real" subject in such sentences, when we symbolize them in predicate logic (see below), the ‘it’ simply disappears.

Next, if we want to depict the various categories of predicates, we do so as follows.

$N^0 \rightarrow S$	0-place predicates
$N^1 \rightarrow S$	1-place predicates
$N^2 \rightarrow S$	2-place predicates
etc.	

The standard symbolization technique for predicates follows a general pattern, given as follows.

> Predicates are symbolized by upper case Roman letters.
> Singular terms are symbolized by lower case, or "small caps", Roman letters.[4]
> The predicate is written first, followed by its arguments.

[2] Grammatical complexity is related to, but not identical to, lexical complexity. For example, ‘the Eiffel Tower’ consists of three words, but it is regarded as grammatically simple.
[3] Compare this situation to Italian in which, to say that it is raining, one simply says ‘piove’.
[4] We propose to use small caps for proper noun phrases, so for example, ‘ᴋ’ abbreviates ‘Kay’.

Examples

subject	∅	it
predicate	R	is raining
sentence	R	it is tall

subject	J	Jay
predicate	T	is tall
sentence	TJ	Jay is tall

subject	J	Jay
predicate	R	respects
object	K	Kay
sentence	RJK	Jay respects Kay

subject	J	Jay
predicate	S	recommended
direct object	K	Kay
indirect object	L	(to) Elle
sentence	RJKL	Jay recommended Kay to Elle

This is the *minimal* scheme. The *maximal* scheme has the following additional features.

> Maximal punctuation scheme:
> Predicates come with square brackets, and explicitly marked blanks.
> Arguments are separated by commas.

This adjusts the above translations as follows.[5]

… is tall	T[α]	α is tall
… respects …	R[α,β]	α respects β
… recommended … to …	S[α,β,γ]	α sold β to γ

Jay is tall	T[J]
Jay respects Kay	R[J,K]
Jay recommended Kay to Elle	R[J,K,L]

[5] The Greek letters are schematic place holders, which serves as labeled blanks.

4. Quantifiers as Noun Phrases

Predicate Logic involves predicates, but more importantly it involves quantification. In particular, all logical inferences in predicate logic are performed in reference to the two special quantifiers – ∀ (universal) and ∃ (existential). For this reason, predicate logic is also called *quantifier logic*.[6]

To understand quantification, one must begin by recognizing that, in ordinary language, quantifier phrases are noun phrases. For example, the sentences

all humans are mortal
every human is mortal

have the following grammatical form.

Subject	predicate
all humans	are mortal
every human	is mortal

Traditional logic hypothesizes that the *grammatical* form is also the *logical* form. Accordingly, the following all have the same logical form.

Jay is mortal
Kay is mortal
every human is mortal

In particular that form may be represented as follows.

predicate[subject]

We have already seen that predicate logic symbolizes the first two sentences as follows

M[J]
M[K]

If 'every human is mortal' has the same *logical* form, then its translation should be something like:

M[every human]

If we abbreviate 'every' by '∀' and 'human' by 'H', then the symbolization looks thus.

M[∀H]

5. A Problem with the Subject-Predicate Analysis of Quantifier Phrases

Logic starts with grammar, but it does not end there! How do we formally analyze the following sentences?

(1) every human is *not* mortal
(2) every human is *im*mortal
(3) *not* every human is mortal

In particular, do they have the same respective forms as the following.

[6] Quantifier logic also includes the logic of function-signs, which we examine in a later chapter.

Jay is *not* mortal
Jay is *im*mortal
not Jay is mortal[7]

The latter are all equivalent, and are straightforwardly symbolized as:

$\sim$M[J]

So, if the quantified sentences have the same forms as their unquantified counterparts, then they are all symbolized as follows.

$\sim$M[$\forall$H]

The problem with this analysis is that the original sentences (1)-(3) are not semantically equivalent, so it would be a *very* bad idea to translate them all as a single logical formula. Clearly, (2) and (3) are not equivalent. What about (1)? Well, it is ambiguous, since two different grammatical constructions lead to the same surface expression. According to one construction, 'not' modifies the verb 'is' and negates the whole sentence. According to the other construction, 'not' modifies the adjective 'mortal' to form a new adjective 'not-mortal'. In other words, sentence (1) is ambiguous in meaning between sentence (2) and sentence (3).

The moral of this story for ordinary users of ordinary English is quite simple — never use sentences like (1), but rather use sentences like (2) or like (3), depending upon which one is intended. The moral of this story for logical analysis is somewhat different. Our initial analysis of the two sentences

every human is *im*mortal
not every human is mortal

produces the very same formula.

$\sim$M[$\forall$H]

This formula is just as loathsome as the unfortunate sentence 'every human is not mortal'; it is ambiguous. For example, on one reading of 'not', the argument form

$\sim$M[$\forall$H]
―――――
$\sim$M[j]

is valid, but on the other reading of 'not', it is *in*valid. Logically speaking, this is a disaster!

6. Quantifier Phrases as Sentential Adverbs

We are now in position to reconstruct the key insight of modern symbolic logic. What distinguishes modern symbolic logic from all logic that preceded it is that quantifier phrases are treated, not as noun phrases, but as sentential adverbs, which are a species of sentential operator (connective). Turning quantifiers into sentential operators does not necessarily make for better grammar, but it does make for better logic, in the sense that it makes the principles of reasoning much easier to formulate.

In order to understand how the transformation takes place, we begin with our earlier example.

every human is mortal

[7] Prefixing 'not' does not generally produce a grammatical English sentence, unless we understand that, *at the beginning of a sentence*, 'not' is officially pronounced "it is not true that".

Next, by a transformation known as *quantifier-movement*,[8] we move the subject phrase from its original grammatical position forward in the sentence to produce the following sentence.

every human is such that he/she is mortal.
or: it is true of every human that he/she is mortal

Next, we propose the following logico/grammatical analysis.

sentential-adverb phrase	every human is such that…	$(\forall H/x)$
	it is true of every human that…	
input sentence	he/she is mortal	$M[x]$
resulting sentence	every human is such that he/she is mortal	$(\forall H/x)\ M[x]$
	it is true of very human that he/she is mortal	

Note the new item – the variable 'x' – which performs a number of inter-related grammatical functions.

(1) it marks the original grammatical location of the quantifier phrase;
(2) it symbolizes the third person singular pronoun 'he/she';
(3) it marks the quantifier phrase 'every human' [$\forall$H] as the pronominal antecedent of the pronoun 'he/she'.

Although linguists prefer to stop their analysis with the formula

$(\forall H/x)\ M[x]$ every human x is such that x is mortal

logicians propose one more transformation, which reduces *specific* universal quantifier phrases

every *human*, every *apple*, every *electron*, etc.,

to a single *generic* universal quantifier:

every *thing*

This is accomplished by the following paraphrase.

every $\mathbb{A}$ is such that it is $\mathbb{B}$

is paraphrased as:

every *thing* is such that, IF it is $\mathbb{A}$, THEN it is $\mathbb{B}$

The generic quantifier expression

every thing is such that ...
it is true of every thing that …

is symbolized by '$\forall$' together with an index variable – for example, '$\forall x$' or '$\forall y$' or '$\forall z$'.

Applying this procedure to our original sentence

[8] This is exactly on a par with the transformation called wh-movement ('who' 'what' 'how' etc.) , which is probably the most famous grammatical transformation, originally discovered by Noam Chomsky.

every human is mortal,

we obtain the following symbolization.

$\forall x(\mathrm{H}x \rightarrow \mathrm{M}x)$

every thing is such that: if it is human, then it is mortal

7. The Corresponding Analysis of 'Some'

A similar story can be told about sentences involving 'some'. Let us start with the following colloquial sentences.

some humans *are* mortal
some human *is* mortal

These have the following grammatical form.

subject	predicate
some humans	are mortal
some human	is mortal

Symbolizing these in *simplified* predicate logic, we obtain the following common formula.

$\mathrm{M}[\exists \mathrm{H}]$

Next, we apply quantifier-movement, which moves the quantifier phrase forward in the sentence to produce the following sentence.[9]

some human is such that he/she is mortal

it is true of at least one human that he/she is mortal

Here, we understand the grammatical form as:

sentential-adverb phrase	some human is such that… it is true of some human that…	$(\exists \mathrm{H}/x)$
input sentence	he/she is mortal	$\mathrm{M}[x]$
resulting sentence	some human is such that he/she is mortal it is true of some human that he/she is mortal	$(\exists \mathrm{H}/x)\ \mathrm{M}[x]$

Finally, we replace the *specific quantifier* 'some *human'* by a *generic quantifier* 'some *thing*', by way of the following paraphrase.

some 𝔸 is such that it is 𝔹
something is such that, it is 𝔸, *and* it is 𝔹
or:
there is something such that, it is 𝔸 *and* it is 𝔹

[9] In keeping with first-order methodology, we ignore the plural form 'some humans' and concentrate on the singular form 'some human'. We discuss the plural form in a later chapter concerning second-order logic.

The quantifier expression 'something is such that...' is symbolized by '∃' together with an index variable – for example: $\exists x$. This yields the following symbolization.

$\exists x(\mathrm{H}x \ \& \ \mathrm{M}x)$

8. Scope

Treating quantifiers as sentential adverbs allows us to construct formulas that are considerably more complex than any formula that is written in subject-verb-object form. It also allows us to construct a powerful, but simple, set of inference rules, as we see later. Equally importantly from a grammatical viewpoint, treating quantifiers as sentential adverbs endows them with a new logico-grammatical feature – *scope*.

Scope, which is an algebraic notion, is a prominent feature of both symbolic logic and symbolic arithmetic. For example, the following arithmetical expressions,

(e1) two plus three times four
(e2) the square root of four plus five
(e3) minus two squared

are all ambiguous.[10] For example, (e1) is ambiguous between the following.

(i1) two plus three … times four
(i2) two plus … three times four

The logical pause represented by '…' tells us which operator is the major operator – i.e., which operator has wide scope.

Now, as everyone knows, the official algebraic method of indicating scope is to use parentheses, which are prominent in both sentential logic and arithmetic. For example, the two informal expressions (i1) and (i2) can be replaced by the following algebraic expressions, respectively.

(a1) $(2 + 3) \times 4$ [= 20]
(a2) $2 + (3 \times 4)$ [= 14]

Similarly, (e2) is ambiguous between the following.

the square root of four … plus five	$sr(4) + 5$	[= 7]
the square root of … four plus five	$sr(4 + 5)$	[= 3]

Finally, (e3) is ambiguous between the following.

minus two … squared	$(-2)^2$	[= 4]
minus … two squared	$-(2^2)$	[= –4]

Sentential examples follow a similar pattern. For example, the following expressions are ambiguous.

[10] These are examples of harmful ambiguity, which are distinguished from expressions like 'two plus three plus four', which are ambiguous, but not harmfully so. Also note that numerous conventions can be adopted according to which expressions without needed parentheses have *preferred* parses. For example, some calculators will parse the input string '2 + 3 × 4 =' by adding 2 and 3, and then multiplying the result by 4. Other calculators treat × as out-scoping + *by default*, and accordingly add 2 to the result of multiplying 3 by 4.

not $\mathcal{A}$ and $\mathcal{B}$	
not $\mathcal{A}$ … and $\mathcal{B}$	$\sim\mathcal{A}$ & $\mathcal{B}$
not … $\mathcal{A}$ and $\mathcal{B}$	$\sim(\mathcal{A}$ & $\mathcal{B})$
$\mathcal{A}$ and $\mathcal{B}$ or $\mathcal{C}$	
$\mathcal{A}$ and $\mathcal{B}$ … or $\mathcal{C}$	$(\mathcal{A}$ & $\mathcal{B}) \vee \mathcal{C}$
$\mathcal{A}$ and … $\mathcal{B}$ or $\mathcal{C}$	$\mathcal{A}$ & $(\mathcal{B} \vee \mathcal{C})$

9. Quantifier Scope

Let us now return to the formula

$\sim$M[$\forall$H]

which corresponds roughly to the sentence

every human is *not* mortal

When we proceed to transform the quantifier phrase 'every human' into a sentential adverb, we notice immediately that there are two equally plausible ways it can be moved forward in the formula. It can be moved in front of the 'M', or it can be moved in front of the '$\sim$'m which results in the following two formulas, respectively.

(m1) $\sim$ ($\forall$H/x) M[x]
(m2) ($\forall$H/x) $\sim$ M[x]

In the first case, the quantifier is given *narrow scope* relative to the negation operator; in the second case, the quantifier is given wide *scope relative* to the negation operator.

The difference between these two becomes more obvious perhaps when we transform the specific quantifier 'every H' into a generic quantifier, as follows.

(g1) $\sim\forall x(\mathrm{H}x \rightarrow \mathrm{M}x)$
(g2) $\forall x(\mathrm{H}x \rightarrow \sim\mathrm{M}x)$

These of course correspond to our original unambiguous sentences.

(s1) *not* every human is mortal
(2) every human is *im*mortal

10. Appendix – The Official Syntax of Ordinary Predicate Logic

1. Vocabulary

1. Predicate letters

0. 0-place: A-Z, A_1-Z_1, A_2-Z_2, etc.
1. 1-place: ^1A-^1Z, $^1A_1$-$^1Z_1$, $^1A_2$-$^1Z_2$, etc.
2. 2-place: ^2A-^2Z, $^2A_1$-$^2Z_1$, $^2A_2$-$^2Z_2$, etc.

etc.

2. Individual Variables

t-z, t_1-z_1, etc.

3. Individual Constants

a-s, a_1-s_1, etc.

4. SL Connective Symbols

&, $\vee$, $\rightarrow$, $\leftrightarrow$, ~

5. Quantifiers

$\forall$, $\exists$

6. Punctuation symbols

(,), [,], etc.

2. Rules of Formation

1. Singular-Terms

1. Every individual variable is a singular-term.
2. Every individual constant is a singular-term.
3. Nothing else is a singular-term.

2. Atomic Formulas

1. If $\mathbb{P}$ is an n-place predicate, and $\tau_1,...,\tau_n$ are singular-terms, then $\mathbb{P}[\tau_1,...,\tau_2]$ is an atomic formula.
2. Nothing else is an atomic formula.

3. Formulas

1. Every atomic formula is a formula.
2. If Φ is a formula, then so is: $\sim\Phi$
3. If Φ_1 and Φ_2 are formulas, then so are:
   (a) $(\Phi_1 \,\&\, \Phi_2)$
   (b) $(\Phi_1 \vee \Phi_2)$
   (c) $(\Phi_1 \rightarrow \Phi_2)$
   (d) $(\Phi_1 \leftrightarrow \Phi_2)$
4. If Φ is a formula and ν is a variable, then the following are formulas:
   (a) $\forall\nu\Phi$
   (b) $\exists\nu\Phi$
5. Nothing else is a formula.

11. Appendix – The Rules of Derivation for Ordinary Predicate Logic

Notational Conventions
In the following, Φ is any formula, ν is any variable. Also, for any expression ε, Φ[ε/ν] is the formula that results when every free occurrence of ν in Φ is replaced by ε.

Universal-Out (∀O)	Existential-In (∃I)
∀νΦ ——— Φ[*a*/ν] *a* is any constant	Φ[*o*/ν] ——— ∃νΦ *a* is any constant

Universal-Derivation (UD)	Existential-Out (∃O)
~~SHOW~~: ∀νΦ │ ~~SHOW~~: Φ[*n*/ν] │ │ *n* is any ***new*** constant	∃νΦ ——— Φ[*n*/ν] *n* is any ***new*** constant

Quantifier Negation	
∼∀νΦ ═══ ∃ν∼Φ	∼∃νΦ ═══ ∀ν∼Φ

Tilde-Universal-Out (∼∀O)	Tilde-Existential-Out (∼∃O)
∼∀νΦ ——— ∼Φ[*n*/ν] *n* is any ***new*** constant.	∼∃νΦ ——— ∼Φ[*a*/ν] *a* is any constant.

Definition of 'old' and 'new'
A constant counts as ***old*** precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as ***new***.

B. Simple Modal Predicate Logic

1. Introduction

Simple Modal Predicate Logic is obtained by simply combining the principles of predicate logic with the principles of sentential modal logic. Syntactically, this is very easy to accomplish; we simply add the following clause to the official syntax of predicate logic.

(c1) the symbols '□' and '◇' are one-place connectives; in other words, if Φ is a formula, then so is □Φ and ◇Φ.[11]

2. Basic Rules for Simple Modal Predicate Logic

The derivation rules for Simple Modal Predicate Logic are also very each to write down.

1. Sentential Rules – Ordinary SL Rules plus Indexing

As before, the SL rules are the same as ordinary SL, except that they are indexed.[12]

2. Modal Operator Rules (Round up the Usual Suspects!)

For every sentential modal system σ, there is a corresponding modal predicate logic, denoted MPL(σ). For example, MPL(K) admits all K-rules, MPL(K4) admits all K4-rules, etc.

3. Quantifier Rules – Ordinary Rules plus Indexing

The quantifier rules are the same as ordinary predicate logic, except that they are indexed. For example, the universal-out rule is officially written as follows.

∀O

∀νΦ	/*i*
Φ[*a*/ν]	/*i*

a is any constant.

[11] Also, we can optionally add other modal operators, including '⥽', and '⥱'.
[12] Recall that the contradiction symbol '※' is either not indexed at all, or indexed by the wild card symbol '∗'.

3. Examples of Derivations in Modal Predicate Logic

In order to illustrate the derivation rules, we look at a few examples. We begin with an equivalence that is well-known in the history of modal logic – the **Barcan formula**.[13]

(1) $\Box\forall x Fx \leftrightarrow \forall x\Box Fx$

In order to show this formula is logically true in MPL(k), it is sufficient to derive each constituent from the other.

(1)	$\Box\forall x Fx$	/0	As
(2)	~~SHOW~~: $\forall x\Box Fx$	/0	UD
(3)	~~SHOW~~: $\Box Fa$	/0	$\Box$D
(4)	~~SHOW~~: Fa	/01	DD
(5)	$\forall x Fx$	/01	1,$\Box$O(k)
(6)	Fa	/01	5,$\forall$O

(1)	$\forall x\Box Fx$	/0	As
(2)	~~SHOW~~: $\Box\forall x Fx$	/0	$\Box$D
(3)	~~SHOW~~: $\forall x Fx$	/01	UD
(4)	~~SHOW~~: Fa	/01	DD
(5)	$\Box Fa$	/0	1,$\forall$O
(6)	Fa	/01	5,$\Box$O(k)

This shows how $\Box$ and $\forall$ interact – they commute. By duality, the same thing can be said about the relation between $\Diamond$ and $\exists$.

(2) $\Diamond\exists x Fx \leftrightarrow \exists x\Diamond Fx$

Once again, in order to show this formula is logically true in MPL(k), it is sufficient to derive each constituent from the other.

(1)	$\Diamond\exists x Fx$	/0	As
(2)	~~SHOW~~: $\exists x\Diamond Fx$	/0	DD
(3)	$\exists x Fx$	/01	1,$\Diamond$O
(4)	Fa	/01	3,$\exists$O
(5)	$\Diamond Fa$	/0	4,$\Diamond$I(k)
(6)	$\exists x\Diamond Fx$	/0	5,$\exists$I

(1)	$\exists x\Diamond Fx$	/0	As
(2)	~~SHOW~~: $\Diamond\exists x Fx$	/0	DD
(3)	$\Diamond Fa$	/0	1,$\exists$O
(4)	Fa	/01	3,$\Diamond$O
(5)	$\exists x Fx$	/01	4,$\exists$I
(6)	$\Diamond\exists x Fx$	/0	5,$\Diamond$I(k)

$\Box$ interacts nicely with $\forall$, and $\Diamond$ interacts nicely with $\exists$. What about the cross-relations. We start with the relation between $\exists$ and $\Box$. Consider the following exchange.

Q: why are you cleaning up the litter on the street?
A: well, someone's got to!

[13] Ruth Barcan (later Ruth Marcus), "A Functional Calculus of First Order Based on Strict Implication", *Journal of Symbolic Logic* (1946), 11:1-16.

The following is a natural paraphrase of the answer.

someone *must* clean up the litter

Now, the sense of 'must' may not be obvious; is it legal, ethical, metaphysical, logical? What sort of world is being ruled out? For the sake of concreteness, let us say that 'must' it is being used legally – 'must' means 'it is legally required that'.

Still there is another question worth asking – what are the relative scopes of 'someone' and 'must'. Depending upon which of the two operators gets wide scope, we have two different readings of 'someone must clean up the litter', given as follows.[14]

(t1) $\exists x\Box Cx$
(t2) $\Box\exists xCx$

In the first reading, ∃ has wide scope; this means that there is a particular individual who must be C. For example, there is a *particular* individual who is legally required to clean up the litter. For example, suppose that a person, Jones, has been convicted of littering, and the judge has sentenced Jones to perform community service – in particular, to clean up litter on the street. Therefore, since Jones is legally required to clean up litter, by existential-introduction (∃I), someone is required to clean up litter.

On the second reading, □ has wide scope. In this case, although it is required that someone clean up the street, no particular person is so required. So long as someone does it, it doesn't matter who, the law is satisfied.

Another example of the difference between wide and narrow scope occurs in the following example. Imagine a TV ad that tells you that you can order anytime by phone; the ad might even say:

don't worry about calling; someone is always on duty here to take your call.

Now, 'always' is an example of a box-modality, so the two readings of 'someone is always on duty here to take your call' are:

(r1) $\exists x\Box Dx$
(r2) $\Box\exists xDx$

The first one is read more or less literally:

there is someone such that, it is always true that he/she is on duty

In other words, there is some poor guy who is always on duty; he is waiting at the phone 24 hours a day!

The more plausible reading is the second one, which says more or less literally:

it is always true that there is someone who is on duty

In other words, no matter when you call, there will be someone on duty to answer your call, although it need not be the same some person every time.

Now, we consider the logical relation between the two readings; this is summarized by the following.

(3) $\exists x\Box Fx \rightarrow \Box\exists xFx$

[14] There are actually two *more* readings, which we discuss in the chapter on actuality.

(1)	~~SHOW~~: $\exists x \Box Fx \rightarrow \Box \exists x Fx$	/0	CD
(2)	$\exists x \Box Fx$	/0	As
(3)	~~SHOW~~: $\Box \exists x Fx$	/0	□D
(4)	~~SHOW~~: $\exists x Fx$	/01	DD
(5)	$\Box Fa$	/0	2,∃O
(6)	Fa	/01	5,□O(k)
(7)	$\exists x Fx$	/01	6,∃I

Note that the converse is not true. The fact that there is always someone on duty does not logically imply that there is some poor slob who is always on duty.

As one might expect, there is a dual problem involving $\Diamond$ and $\forall$. Consider the sentence.

everyone can be a winner!!

Here, let us suppose there is a contest, and 'everyone' means 'every contestant'. Nevertheless, the sentence is ambiguous. Does it mean that it is possible that every single contestant gets a prize? That is a lot of prizes! Or, does it mean that the contest is fair; no matter who you are, you have a non-zero chance of winning the contest. These two readings may be formulated as follows.

(r1)	$\Diamond \forall x Wx$	there is a chance that everyone wins
(r2)	$\forall x \Diamond Wx$	everyone has a chance of winning

As it turns out, the logical relation is given as follows.

(4) $\Diamond \forall x Fx \rightarrow \forall x \Diamond Fx$

(1)	~~SHOW~~: $\Diamond \forall x Fx \rightarrow \forall x \Diamond Fx$	/0	CD
(2)	$\Diamond \forall x Fx$	/0	As
(3)	~~SHOW~~: $\forall x \Diamond Fx$	/0	UD
(4)	~~SHOW~~: $\Diamond Fa$	/0	DD
(5)	$\forall x Fx$	/01	2,◇O
(6)	Fa	/01	5,∀O
(7)	$\Diamond Fa$	/0	6,◇I(k)

C. Actualist Modal Predicate Logic

1. Introduction

In Part B, we examined the logical system that results when one simply combines the principles of modal sentential logic with the principles of non-modal quantifier logic. This process was done is a largely innocent and un-self-conscious manner.

Now, we wish to reconsider what we have created. In particular, we wish to examine more carefully the quantifier principles implicit in Simple Modal Predicate Logic. In particular – what exactly do, or should, '$\exists x$' and '$\forall x$' mean in the context of modal logic? For example, what is the domain of quantification?

Consider the following expressions.

$\exists x\mathrm{F}x \quad /i$
$\forall x\mathrm{F}x \quad /i$

These read roughly:

there is something which is F ... at i
everything is F... at i

The issue is how do we understand the adverbial modifier 'at i'. We know how the modifier 'at i' interacts with the standard SL connectives, which is summarized in the following principles of World Theory.

wt(~)	$[\sim\mathrm{P} / i]$	$\leftrightarrow$	$\sim[\mathrm{P} / i]$
wt(&)	$[(\mathrm{P}\&\mathrm{Q}) / i]$	$\leftrightarrow$.	$[\mathrm{P} / i]\ \&\ [\mathrm{Q} / i]$
wt(∨)	$[(\mathrm{P}\vee\mathrm{Q}) / i]$	$\leftrightarrow$.	$[\mathrm{P} / i] \vee [\mathrm{Q} / i]$
wt(→)	$[(\mathrm{P}\rightarrow\mathrm{Q}) / i]$	$\leftrightarrow$.	$[\mathrm{P}/i] \rightarrow [\mathrm{Q}/i]$
wt(↔)	$[(\mathrm{P}\leftrightarrow\mathrm{Q}) / i]$	$\leftrightarrow$.	$[\mathrm{P}/i] \leftrightarrow [\mathrm{Q}/i]$

What we need are the corresponding principles for the quantifiers.

(∃?)	$[\exists\nu\Phi / i]$	$\leftrightarrow$	??
(∀?)	$[\forall\nu\Phi / i]$	$\leftrightarrow$	??

Let us concentrate on the first one. There seem to be at least two readings of the 'at i' modifier. In the following, the original sentence is given, followed by two parsings.

(0)	there is something which is F... at i	$[\exists x\mathrm{F}x/i]$
(1)	there is some (generic) thing ...... which is F ... at i	$\exists x[\mathrm{F}x/i]$
(2)	there is some thing at i ...... which is F ... at i	$\exists[x/i][\mathrm{F}x/i]$

Also, the clause

(c) which is F ... at i,

must be parsed. The most plausible reading is:

(r) which is F-at-i,

where 'F-at-i' is a relativized predicate. Specifically, for any given world i, and predicate F, there are the things that are F at i.

Now, back to the original readings. Reading (1) is plausible; indeed, it is the parsing that is implicit in Simple Modal Predicate Logic, as presented in Part A. The second parsing also seems plausible, so long as we can clarify what it means to be a "thing-at-i".

This is probably best understood by reference to tense (time) logic, in which case the indices are temporal. Temporal indices may be moments, or instants, or intervals, or whatever, depending on your semantic preferences. For example, in theoretical physics, the temporal index is instantaneous time (the "time" of mathematical analysis); yet, instantaneous functions are all defined in terms of limits of interval ("average") functions.[15]

In any case, in tense logic it is pretty clear what it means to be a thing-at-i. To be a thing-at-i is to be a thing that *exists at* i (or *during* i, if i is an interval). This is understood in terms of birth, life, and death, broadly understood. Specifically, the underlying intuition is that an entity (e.g., the earth) is "born" at some point in time, after which it "lives" for awhile, until it "dies" at some later point in time.[16] During (inside) this life-interval, the object exists; outside this life-interval, the object does not exist. Where i is a temporal index (a time), to be a thing-at-i is to be a thing whose life-interval includes i.

2. Actualism versus Possibilism

This leads us to an important metaphysical question: is there a difference between existence *simpliciter* and existence-at-a-time. Concerning this question, there are two prominent viewpoints, called *actualism* and *possibilism.*

According to actualism, to exist is to be *actual*. Or, in the specific case of time, temporal actualism claims that to exist is to be *present* (i.e., existing *now*).[17] Accordingly, actualist quantifiers range over actual/present objects; in particular, '$\exists x\text{F}x$' means 'some *actual*/*present* thing is F'.

The actualist viewpoint divides the world simply into the actual and the non-actual. The non-actual includes Aristotle (and all other dead persons), Santa Claus (and all other fictional objects), and the round square (and all other impossible objects).

Possibilism divides the world differently. First, it divides the world into the possible and the non-possible. Second, it divides the world into the actual and the non-actual. Note, of course, that the latter division is *indexical*; what is actual depends upon when/where you are speaking. Technically speaking, for any given index i, there are the things that exist-at-i. These are overlapping categories: every actual thing is possible.

Next, the possibilist viewpoint maintains that quantifiers range over possible objects; '$\exists x\text{F}x$' means 'some *possible* thing is F'. The temporal version of possibilism claims that '$\exists x\text{F}x$' means 'some *past, present, or future* thing is F'.

Since the possibilist viewpoint distinguishes between (possible) existence and actuality, it must accordingly distinguish between the sentences 'some (possible) thing is F' and 'some actual (present) thing is F'. This is accomplished by introducing an additional logical predicate, called *actual existence*, which we propose to symbolize by the ligature 'Æ'. This allows us the following translations.

$\exists x\text{F}x$	some possible thing is F

[15] For example, the derivative function is defined as a limit of interval-change functions.
[16] Let us ignore the very tricky idea of re-birth
[17] Note: the French cognate 'actuel' translates as 'present'.

$\exists x(\text{Æ}x \;\&\; \mathrm{F}x)$ some actual thing is F

The indexical (world-theoretic) renderings of these formulas are as follows.

$[\exists x\mathrm{F}x \,/\, i]$
$\equiv$ $\exists x[\mathrm{F}x \,/\, i]$
$\equiv$ there is a possible thing, which is F-at-i

$[\exists x(\text{Æ}x \;\&\; \mathrm{F}x)/i]$
$\equiv$ $\exists x\{[\text{Æ}x/i] \;\&\; [\mathrm{F}x/i]\}$
$\equiv$ there is a possible thing, which is actual-at-i, and which is F-at-i

3. Reducing Actualist Modal Logic to Possibilist Modal Logic

The logical principles of actualist modal logic can be simulated, or mimicked, within possibilist modal logic by introducing the following definitions, which are the possibilist versions of actualist quantifiers.

(d1) $\forall'\nu\Phi \;=_{\text{df}}\; \forall\nu(\text{Æ}[\nu] \rightarrow \Phi)$
(d2) $\exists'\nu\Phi \;=_{\text{df}}\; \exists\nu(\text{Æ}[\nu] \;\&\; \Phi)$

In the following, the rules of Actualist Quantified Modal Logic are listed in one column, and in the other column are listed the corresponding argument forms as rendered in Possibilist Quantified Modal Logic. Note, that the indices are the same for every line, and are accordingly dropped.

	Actualist Rule	**Possibilist Counterpart**
$\forall'$O	$\forall'\nu\Phi$ $\text{Æ}[c]$ ――― $\Phi[c/\nu]$	$\forall\nu(\text{Æ}[\nu] \rightarrow \Phi)$ $\text{Æ}[c]$ ―――――― $\Phi[c/\nu]$
U′D	~~SHOW~~: $\forall'\nu\Phi$ U′D │ $\text{Æ}[n]$ │ ~~SHOW~~: $\Phi[n/\nu]$ │ │	~~SHOW~~: $\forall\nu(\text{Æ}[\nu] \rightarrow \Phi)$ UD │ ~~SHOW~~: $\text{Æ}[n] \rightarrow \Phi[n/\nu]$ CD │ │ $\text{Æ}[n]$ As │ │ ~~SHOW~~: $\Phi[n/\nu]$ │ │ │
$\exists'$I	$\Phi[c/\nu]$ $\text{Æ}[c]$ ――― $\exists'\nu\Phi$	$\Phi[c/\nu]$ $\text{Æ}[c]$ ―――――― $\exists\nu(\text{Æ}[\nu] \;\&\; \Phi)$
$\exists'$O	$\exists'\nu\Phi$ ―――――― $\text{Æ}[n] \;\&\; \Phi[n/\nu]$	$\exists\nu(\text{Æ}[\nu] \;\&\; \Phi)$ ―――――― $\text{Æ}[n] \;\&\; \Phi[n/\nu]$
Here, Φ is any formula, ν is any variable, c is any constant, and n is any new constant. Also, where ε is any expression, $\Phi[\varepsilon/\nu]$ is the formula that results when ε replaces every occurrence of ν that is free in Φ.		

Notice that ∃′O is simply a special case of ∃O, and U′D is a special case of UD. ∀′O is not a special case of ∀O, nor is ∃′I a special case of ∃I. They are nevertheless derivable rules, the proofs of which are left as an exercise.

4. Rules for Modal Predicate Logic with Actual Existence

To obtain system MPLA(σ), one takes the rules of MPL(σ) and adds the following.

Def ∀′	Def ∃′
∀′νΦ ══════ ∀ν(Æ[ν] → Φ)	∃′νΦ ══════ ∃ν(Æ[ν] & Φ)

Note: As usual, '══' indicates a dual-direction rule, which can be used as an in-rule, an out-rule, and a show-rule.

Later, when we consider general first-order modal logic, we will add two more rules pertaining to the Æ predicate.

D. Exercises

1. Simple Modal Predicate Logic

Directions: for each of the following argument forms, construct a formal derivation of the conclusion from the premises (if any) in MPL(k). In cases in which two formulas are separated by '//', derive each formula from the other.

1. $/\ \forall x\Box Fx \leftrightarrow \Box\forall xFx$
2. $/\ \exists x\Diamond Fx \leftrightarrow \Diamond\exists xFx$
3. $/\ \exists x\Box Fx \rightarrow \Box\exists xFx$
4. $/\ \Diamond\forall xFx \rightarrow \forall x\Diamond Fx$
5. $\forall x\Box(Fx \rightarrow Gx)\ ;\ \Box\forall xFx\ /\ \Box\forall xGx$
6. $\Box\forall x(Fx \rightarrow Gx)\ ;\ \forall x\Box Fx\ /\ \forall x\Box Gx$
7. $\forall x\Box(Fx \rightarrow Gx)\ ;\ \Diamond\forall xFx\ /\ \Diamond\forall xGx$
8. $\Box\forall x(Fx \rightarrow Gx)\ ;\ \exists x\Box Fx\ /\ \exists x\Box Gx$
9. $\forall x\Box(Fx \rightarrow Gx)\ ;\ \Box\exists xFx\ /\ \Box\exists xGx$
10. $\Diamond\forall x(Fx \rightarrow Gx)\ ;\ \exists x\Box Fx\ /\ \Diamond\exists xGx$
11. $\Diamond\forall x(Fx \rightarrow Gx)\ ;\ \Box\exists xFx\ /\ \Diamond\exists xGx$
12. $\Box\forall x(Fx \rightarrow P)\ ;\ \Box\exists xFx\ /\ \Box P$
13. $\forall x\Box(Fx \rightarrow P)\ ;\ \exists x\Box Fx\ /\ \Box P$
14. $\Diamond Q;\ \Box\exists x(Fx \rightarrow P)\ ;\ \Box\forall xFx\ /\ \Diamond P$
15. $\forall x\Box(Fx \rightarrow Gx)\ /\ \Box(\forall xFx \rightarrow \forall xGx)$
16. $\forall x\Box(Fx \rightarrow Gx)\ /\ \Box(\exists xFx \rightarrow \exists xGx)$
17. $\forall x(\Box Fx \vee \Box Gx)\ /\ \Box\forall x(Fx \vee Gx)$
18. $\Diamond\forall xFx\ ;\ \Diamond\forall xGx\ /\ \forall x(\Diamond Fx\ \&\ \Diamond Gx)$
19. $\exists x(\Box Fx \vee \Box Gx)\ /\ \Box\exists xFx \vee \Box\exists xGx$
20. $\Diamond(\exists xFx\ \&\ \exists xGx)\ /\ \exists x\Diamond Fx\ \&\ \exists x\Diamond Gx$
21. $\exists x\Diamond(Fx\ \&\ Gx)\ /\ \Diamond(\exists xFx\ \&\ \exists xGx)$
22. $\Diamond\forall x(Fx \rightarrow Gx)\ ;\ \Box\exists xFx$
    $/\ \Diamond\exists x(Fx\ \&\ Gx)$
23. $\Diamond\exists xFx\ ;\ \Box\forall x(Fx \rightarrow Gx)$
    $/\ \Diamond\exists x(Fx\ \&\ Gx)$
24. $\Diamond\forall x(Fx \rightarrow \Box Gx)\ ;\ \exists x\Box(Fx\ \&\ \Diamond Hx)$
    $/\ \Diamond\exists x\Diamond(Gx\ \&\ Hx)$
25. $\exists x\Box(Fx \rightarrow Gx)\ ;\ \Diamond\forall xFx\ /\ \exists x\Diamond Gx$
26. $\forall x(\Diamond Fx \rightarrow \Box Gx)\ ;\ \Diamond{\sim}\exists xGx$
    $/\ \Box{\sim}\exists xFx$
27. $\forall x\forall y\Box Rxy\ //\ \Box\forall x\forall yRxy$
28. $\exists x\exists y\Diamond Rxy\ //\ \Diamond\exists x\exists yRxy$
29. $\Box\exists x\forall yRxy\ /\ \forall x\Box\exists yRyx$
30. $\exists x\Box\forall yRxy\ /\ \forall x\exists y\Box Ryx$
31. $\exists x\exists y\Box Rxy\ /\ \Box\exists x\exists yRxy$
32. $\forall x\exists y\Box Rxy\ /\ \Box\forall x\exists yRxy$
33. $\exists x(\Box Fx\ \&\ \Diamond\forall y(Gy \rightarrow Rxy))$
    $/\ \forall x(\Box Gx \rightarrow \exists y(\Box Fy\ \&\ \Diamond Ryx))$

2. Actualist Modal Predicate Logic

1. Comparing Actualist MPL and Simple (Possibilist) MPL

Consider the derivation problems in Section 1. Consider replacing every quantifier expression by its actualist counterpart.

(1) Use the strongest modal sentential logic (i.e., System L) as the SL underpinning. Which of the arguments remain valid? Construct counterexamples for the ones that are not valid.

(2) Use the weakest (normal) sentential logic (i.e., System K) as the SL underpinning. Which of the arguments remain valid? Construct counterexamples for the ones that are not valid.

Directions for Parts 3.2–3.4: for each of the following argument forms, construct a formal derivation of the conclusion from the premises (if any) in the above system, using the sentential rules listed in parentheses.

2. Pure (Actualist) Quantifiers (system K)

1. ∀′*x*□(F*x* → Æ*x*) **;** ∃′*x*◇F*x* / ◇∃′*x*F*x*
2. ∀′*x*□Æ*x* **;** ∃′*x*◇F*x* / ◇∃′*x*F*x*
3. ∀′*x*□(F*x* → Æ*x*) **;** ∃′*x*□F*x* / □∃′*x*F*x*
4. ∀′*x*□Æ*x* **;** ∃′*x*□F*x* / □∃′*x*F*x*
5. ∀′*x*□Æ*x* **;** □∀′*x*F*x* / ∀′*x*□F*x*

3. Pure (Actualist) Quantifiers (system KB)

6. □∀′*x*□Æ*x* **;** ◇∃′*x*F*x* / ∃′*x*◇F*x*
7. □∀′*x*(F*x* → □Æ*x*) **;** ◇∃′*x*F*x* / ∃′*x*◇F*x*
8. □∀′*x*□Æ*x* **;** ∀′*x*□F*x* / □∀′*x*F*x*
9. □∀′*x*□Æ*x* **;** ◇∃′*x*F*x* / ∃′*x*◇F*x*
10. □∀′*x*□Æ*x* **;** ∀′*x*□F*x* / □∀′*x*F*x*

4. Mixed (Possibilist and Actualist) Quantifiers (system K)

11. □∀*x*(F*x* → Æ*x*) **;** ∃′*x*◇F*x* / ◇∃′*x*F*x*
12. □∀*x*(F*x* → Æ*x*) **;** ∃′*x*□F*x* / □∃′*x*F*x*
13. ∀*x*(◇F*x* → Æ*x*) **;** ◇∃′*x*F*x* / ∃′*x*◇F*x*
14. ∀*x*(◇Æ*x* → Æ*x*) **;** ◇∃′*x*F*x* / ∃′*x*◇F*x*
15. ∀*x*(◇Æ*x* → Æ*x*) **;** ∀′*x*□F*x* / □∀′*x*F*x*
16. ∀*x*(◇Æ*x* → □F*x*) / □∀′*x*F*x*

3. Answers to Selected Exercises

1. Simple Modal Predicate Logic

#5

(1)	∀*x*□(F*x* → G*x*)	/0	Pr
(2)	□∀*x*F*x*	/0	Pr
(3)	~~SHOW~~: □∀*x*G*x*	/0	ND
(4)	~~SHOW~~: ∀*x*G*x*	/01	UD
(5)	~~SHOW~~: G*a*	/01	7,9,SL
(6)	│ □(F*a* → G*a*)	/0	1,∀O
(7)	│ F*a* → G*a*	/01	6,□O
(8)	│ ∀*x*F*x*	/01	2,□O
(9)	│ F*a*	/01	8,∀O

#8

(1)	□∀*x*(F*x* → G*x*)	/0	Pr
(2)	∃*x*□F*x*	/0	Pr
(3)	~~SHOW~~: ∃*x*□G*x*	/0	5,∃I
(4)	│ □F*a*	/0	2,∃O
(5)	│ ~~SHOW~~: □G*a*	/0	ND
(6)	│ ~~SHOW~~: G*a*	/01	DD
(7)	│ │ ∀*x*(F*x* → G*x*)	/01	1,□O(k)
(8)	│ │ F*a*	/01	4,□O(k)
(9)	│ │ G*a*	/01	7,8,QL

#14

(1)	◇Q	/0	Pr
(2)	☐∃*x*(F*x* → P)	/0	Pr
(3)	☐∀*x*F*x*	/0	Pr
(4)	~~SHOW~~: ◇P	/0	DD
(5)	Q	/01	1,◇O
(6)	∃*x*(F*x* → P)	/01	2,☐O(k)
(7)	∀*x*F*x*	/01	3,☐O(k)
(8)	F*a* → P	/01	6,∃O
(9)	F*a*	/01	7,∀O
(10)	P	/01	8,9,SL
(11)	◇P	/0	10,◇I(k)

#19

(1)	∃*x*(☐F*x* ∨ ☐G*x*)	/0	Pr
(2)	~~SHOW~~: ☐∃*x*F*x* ∨ ☐∃*x*G*x*	/0	∨ID
(3)	~☐∃*x*F*x* & ~☐∃*x*G*x*	/0	As
(4)	~~SHOW~~: ※	/✳	7,8-15, SC
(5)	~∃*x*F*x*	/01	3a,~☐O
(6)	~∃*x*G*x*	/02	3b,~☐O
(7)	☐F*a* ∨ ☐G*a*	/0	1,∃O
(8)	c1: ☐F*a*	/0	As
(9)	F*a*	/01	8,☐O(k)
(10)	~F*a*	/01	5,~∃O
(11)	※	/✳	9,10
(12)	c2: ☐G*a*	/0	As
(13)	G*a*	/02	12,☐O(k)
(14)	~G*a*	/02	6,~∃O
(15)	※	/✳	13,14,SL

#22

(1)	◇∀*x*(F*x* → G*x*)	/0	Pr
(2)	☐∃*x*F*x*	/0	Pr
(3)	~~SHOW~~: ◇∃*x*(F*x* & G*x*)	/0	DD
(4)	∀*x*(F*x* → G*x*)	/01	1,◇O
(5)	∃*x*F*x*	/01	2.☐O(k)
(6)	F*a*	/01	5,∃O
(7)	G*a*	/01	4,6,QL
(8)	F*a* & G*a*	/01	6,7,SL
(9)	∃*x*(F*x* & G*x*)	/01	8,∃I
(10)	◇∃*x*(F*x* & G*x*)	/0	9,◇I(k)

#24

(1)	$\Diamond\forall x(Fx \rightarrow \Box Gx)$	/0	Pr
(2)	$\exists x\Box(Fx \,\&\, \Diamond Hx)$	/0	Pr
(3)	~~SHOW~~: $\Diamond\exists x\Diamond(Gx \,\&\, Hx)$	/0	ID
(4)	$\sim\Diamond\exists x\Diamond(Gx \,\&\, Hx)$	/0	As
(5)	~~SHOW~~: ※	/∗	12-13,SL
(6)	$\forall x(Fx \rightarrow \Box Gx)$	/01	1,◇O
(7)	$\Box(Fa \,\&\, \Diamond Ha)$	/0	2,∃O
(8)	$\sim\exists x\Diamond(Gx \,\&\, Hx)$	/01	4,~◇O(k)
(9)	$\sim\Diamond(Ga \,\&\, Ha)$	/01	8,QL
(10)	$Fa \,\&\, \Diamond Ha$	/01	7,□O(k)
(11)	$\Box Ga$	/01	4,10a,QL
(12)	Ha	/012	10b,◇O
(13)	Ga	/012	11,□O(k)
(14)	$\sim(Ga \,\&\, Ha)$	/012	9,~◇O(k)

#26

(1)	$\forall x(\Diamond Fx \rightarrow \Box Gx)$	/0	Pr
(2)	$\Diamond\sim\exists x Gx$	/0	Pr
(3)	~~SHOW~~: $\Box\sim\exists x Fx$	/0	ND
(4)	~~SHOW~~: $\sim\exists x Fx$	/01	ID
(5)	$\exists x Fx$	/01	As
(6)	~~SHOW~~: ※	/∗	11,12,SL
(7)	Fa	/01	5,∃O
(8)	$\Diamond Fa$	/0	7,◇I(k)
(9)	$\Box Ga$	/0	1,8,QL
(10)	$\sim\exists x Gx$	/02	2,◇O
(11)	$\sim Ga$	/02	10,QL
(12)	Ga	/02	9,□O(k)

#30

(1)	$\exists x\Box\forall y Rxy$	/0	Pr
(2)	~~SHOW~~: $\forall x\exists y\Box Ryx$	/0	UD
(3)	~~SHOW~~: $\exists y\Box Rya$	/0	8,QL
(4)	$\Box\forall y Rby$	/0	1,∃O
(5)	~~SHOW~~: $\Box Rba$	/0	ND
(6)	~~SHOW~~: Rba	/01	DD
(7)	$\forall y Rby$	/01	4,□O(k)
(8)	Rba	/01	7,∀O

#33

(1)	$\exists x(\Box Fx \,\&\, \Diamond\forall y(Gy \rightarrow Rxy))$	/0	Pr
(2)	~~SHOW~~: $\forall x(\Box Gx \rightarrow \exists y(\Box Fy \,\&\, \Diamond Ryx))$	/0	UCD
(3)	$\Box Ga$	/0	As
(4)	~~SHOW~~: $\exists y(\Box Fy \,\&\, \Diamond Rya)$	/0	DD
(5)	$\Box Fb \,\&\, \Diamond\forall y(Gy \rightarrow Rby)$	/0	1,∃O
(6)	$\forall y(Gy \rightarrow Rby)$	/01	5b,◇O
(7)	Ga	/01	3,□O
(8)	Rba	/01	6,7,QL
(9)	$\Diamond Rba$	/0	8,◇I(k)
(10)	$\Box Fb \,\&\, \Diamond Rba$	/0	5a,9,SL
(11)	$\exists y(\Box Fy \,\&\, \Diamond Rya)$	/0	10,∃I

2. Actualist Modal Predicate Logic

(From Part 3.1)

#1. ∀′*x*□F*x* // □∀′*x*F*x*

(a) an attempted derivation, which suggests a counter-model:

(1)	∀′*x*□F*x*	/0	Pr
(2)	SHOW: □∀′*x*F*x*	/0	ND
(3)	SHOW: ∀′*x*F*x*	/01	Def ∀′
(4)	SHOW: ∀*x*(Æ*x* → F*x*)	/01	UCD
(5)	Æ*a*	/01	As
(6)	SHOW: F*a*	/01	ID
(7)	~F*a*	/01	As
(8)	SHOW: ※	/∗	?
(9)	∀*x*(Æ*x* → □F*x*)	/0	1,Def ∀′
(10)	Æ*a* → □F*a*	/0	7,∀O
(11)	~□F*a*	/0	7,□O(–)(k)
(12)	~Æ*a*	/0	10,11,SL
(13)	(5) and (12) are not contradictory!		

(b) an attempted derivation, which suggests a counter-model:

(1)	□∀′*x*F*x*	/0	Pr
(2)	SHOW: ∀′*x*□F*x*	/0	Def ∀′
(3)	SHOW: ∀*x*(Æ*x* → □F*x*)	/0	UCD
(4)	Æ*a*	/0	As
(5)	SHOW: □F*a*	/0	ND
(6)	SHOW: F*a*	/01	ID
(7)	~F*a*	/01	As
(8)	SHOW: ※	/∗	?
(9)	∀′*x*F*x*	/01	1,□O(k)
(10)	∀*x*(Æ*x* → F*x*)	/01	9,Def ∀′
(11)	Æ*a* → F*a*	/01	10,∀O
(12)	~Æ*a*	/01	7,11,SL
(13)	(4) and (12) are not contradictory!		

#2. ∃′*x*◇F*x* // ◇∃′*x*F*x*

(a) an attempted derivation, which suggests a counter-model:

(1)	∃′*x*◇F*x*	/0	Pr
(2)	SHOW: ◇∃′*x*F*x*	/0	ID
(3)	~◇∃′*x*F*x*	/0	As
(4)	SHOW: ※	/∗	?
(5)	∃*x*(Æ*x* & ◇F*x*)	/0	Def ∃′
(6)	Æ*a* & ◇F*a*	/0	5,∃O
(7)	F*a*	/01	6b,◇O
(8)	~∃′*x*F*x*	/01	3,~◇O(k)
(9)	~∃*x*(Æ*x* & F*x*)	/01	8,Def ∃′(–)
(10)	~(Æ*a* & F*a*)	/01	9,~∃O
(11)	~Æ*a*	/01	7,10,SL
(12)	(6a) and (11) are not contradictory!		

(b) an attempted derivation, which suggests a counter-model:

(1)	◇∃′*x*F*x*	/0	Pr
(2)	SHOW: ∃′*x*◇F*x*	/0	Def ∃′
(3)	SHOW: ∃*x*(Æ*x* & ◇F*x*)	/0	ID
(4)	~∃*x*(Æ*x* & ◇F*x*)	/0	As
(5)	SHOW: ※	/∗	?
(6)	∃′*x*F*x*	/01	1,◇O
(7)	∃*x*(Æ*x* & F*x*)	/01	6,Def ∃′
(8)	Æ*a* & F*a*	/01	7,∃O
(9)	~(Æ*a* & ◇F*a*)	/0	4,~∃O
(10)	◇F*a*	/0	8b,◇I(k)
(11)	~Æ*a*	/0	9,10,SL
(12)	(8a) and (11) are not contradictory!		

#3. ∃′*x*☐F*x* / ☐∃′*x*F*x*

an attempted derivation, which suggests a counter-model:

(1)	∃′*x*☐F*x*	/0	Pr
(2)	SHOW: ☐∃′*x*F*x*	/0	ND
(3)	SHOW: ∃′*x*F*x*	/01	Def ∃′
(4)	SHOW: ∃*x*(Æ*x* & F*x*)	/01	ID
(5)	~∃*x*(Æ*x* & F*x*)	/01	As
(6)	SHOW: ※	/∗	?
(7)	∃*x*(Æ*x* & ☐F*x*)	/0	1,Def ∃′
(8)	Æ*a* & ☐F*a*	/0	7,∃O
(9)	F*a*	/01	8b,☐O(k)
(10)	~(Æ*a* & F*a*)	/01	5,~∃O
(11)	~Æ*a*	/01	9,10,SL
(12)	(8a) and (11) are not contradictory!		

#4. ◇∀′*x*F*x* / ∀′*x*◇F

an attempted derivation, which suggests a counter-model:

(1)	◇∀′*x*F*x*	/0	As
(2)	SHOW: ∀′*x*◇F*x*	/0	Def ∀′
(3)	SHOW: ∀*x*(Æ*x* → ◇F*x*)	/0	UCD
(4)	Æ*a*	/0	As
(5)	SHOW: ◇F*a*	/0	ID
(6)	~◇F*a*	/0	As
(7)	SHOW: ※	/∗	?
(8)	∀′*x*F*x*	/01	1,◇O
(9)	∀*x*(Æ*x* → F*x*)	/01	8,Def ∀′
(10)	~F*a*	/01	6,~◇O(k)
(11)	Æ*a* → F*a*	/01	9,∀O
(12)	~Æ*a*	/01	10,11,SL
(13)	(4) and (12) are not contradictory!		

(From Parts 3.2–3.4)

#1

(1)	∀′x□(Fx → Æx)	/0	Pr
(2)	∃′x◇Fx	/0	Pr
(3)	~~SHOW~~: ◇∃′xFx	/0	DD
(4)	∀x(Æx → □(Fx → Æx))	/0	1,Def ∀′
(5)	∃x(Æx & ◇Fx)	/0	2,Def ∃′
(6)	Æa & ◇Fa	/0	5,∃O
(7)	□(Fa → Æa)	/0	4,6a,QL
(8)	Fa	/01	6b,◇O
(9)	Fa → Æa	/01	7,□O(k)
(10)	Æa & Fa	/01	8,9,SL
(11)	∃x(Æx & Fx)	/01	10,∃I
(12)	∃′xFx	/01	11, Def ∃′
(13)	◇∃′xFx	/0	12,◇I(k)

#8

(1)	□∀′x□Æx	/0	Pr
(2)	∀′x□Fx	/0	Pr
(3)	~~SHOW~~: □∀′xFx	/0	ND
(4)	~~SHOW~~: ∀′xFx	/01	Def ∀′
(5)	~~SHOW~~: ∀x(Æx → Fx)	/01	UCD
(6)	Æa	/01	As
(7)	~~SHOW~~: Fa	/01	DD
(8)	∀′x□Æx	/01	1,□O(k)
(9)	∀x(Æx → □Æx)	/01	8,Def ∀′
(10)	□Æx	/01	6,9,QL
(11)	Æa	/0	10,□O(b)
(12)	∀x(Æx → □Fx)	/0	2,Def ∀′
(13)	□Fa	/0	11,12,QL
(14)	Fa	/01	13,□O(k)

#15

(1)	∀x(◇Æx → Æx)	/0	Pr
(2)	∀′x□Fx	/0	Pr
(3)	~~SHOW~~: □∀′xFx	/0	ND
(4)	~~SHOW~~: ∀′xFx	/01	Def ∀′
(5)	~~SHOW~~: ∀x(Æx → Fx)	/01	UCD
(6)	Æa	/01	As
(7)	~~SHOW~~: Fa	/01	DD
(8)	◇Æa → Æa	/0	1,∀O
(9)	◇Æa	/0	6,◇I(k)
(10)	Æa	/0	8,9,SL
(11)	∀x(Æx → □Fx)	/0	2,Def ∀′
(12)	□Fa	/0	10,11,QL
(13)	Fa	/01	12,□O(k)