

5

Systems Between K and L

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1. Introduction

In the previous chapters, we have examined two modal systems – System L and System K. Whereas System L is the strongest (non-trivial) normal modal logic, System K is the weakest normal modal logic. In the present chapter, we examine a number of intermediate systems.

2. Axiomatic Characterizations of Modal Systems

Before discussing the rules of the intermediate systems, we consider axiomatic characterizations of modal systems. The following is an axiomatic presentation of the theses of System K.

- (SL) $\vdash \top$ for any SL-thesis (tautology) \top
- (K) $\vdash \Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box\mathcal{A} \rightarrow \Box\mathcal{B})$
- (MP) if $\vdash \mathcal{A}$ and $\vdash \mathcal{A} \rightarrow \mathcal{B}$, then $\vdash \mathcal{B}$
- (Nec) if $\vdash \mathcal{A}$, then $\vdash \Box\mathcal{A}$
- (\Diamond) $\Diamond\mathcal{A} =_{df} \sim\Box\sim\mathcal{A}$

Note that the metalinguistic variables ‘ \mathcal{A} ’ and ‘ \mathcal{B} ’ range over formulas of the object language, and that ‘ \vdash ’ means ‘is a thesis’ (of the system in question, in this case, System K). Thus, the above clauses can be read as follows.

- (SL) Every SL thesis (i.e., tautology) is a thesis.
- (K) Every formula of the following form is a thesis: $\Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box\mathcal{A} \rightarrow \Box\mathcal{B})$
- (MP) If \mathcal{A} is a thesis, and $\mathcal{A} \rightarrow \mathcal{B}$ is a thesis, then \mathcal{B} is a thesis.
- (Nec) If \mathcal{A} is a thesis, then $\Box\mathcal{A}$ is a thesis.
- (\Diamond) $\Diamond\mathcal{A} =_{df} \sim\Box\sim\mathcal{A}$

[Note: in the context of a logic that extends SL, an SL-thesis is a formula of the expanded language that is provable using only the deductive apparatus of SL. So, for example, $(\Box P \vee \sim\Box P)$ and $(\Diamond P \rightarrow \Diamond P)$ are both SL theses.]

System K, so presented, has two primitive thesis forms (axioms): the SL-form, and the K-form, and it has two rules for generating new theses from old ones – (MP), our old friend *modus ponens*, and (Nec), the rule of necessitation.

Stronger modal logic systems have been produced historically by adding further axioms to the above list. The following are some of the better-known examples, beginning with the characteristic K-thesis.

- (k) $\Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box\mathcal{A} \rightarrow \Box\mathcal{B})$
- (d) $\Box\mathcal{A} \rightarrow \Diamond\mathcal{A}$
- (t) $\Box\mathcal{A} \rightarrow \mathcal{A}$
- (b) $\Diamond\Box\mathcal{A} \rightarrow \mathcal{A}$
- (4) $\Box\mathcal{A} \rightarrow \Box\Box\mathcal{A}$
- (g) $\Diamond\Box\mathcal{A} \rightarrow \Box\Diamond\mathcal{A}$
- (Lin) $\Box(\Box\mathcal{A} \rightarrow \mathcal{B}) \vee \Box(\Box\mathcal{B} \rightarrow \mathcal{A})$
- (5) $\Diamond\Box\mathcal{A} \rightarrow \Box\mathcal{A}$

Some of the historically important systems of modal logic are axiomatically characterized by rules (SL), (MP), (Nec), (\Diamond), plus the primitive thesis forms given as follow.

Systems S1-S3	not normal!
System K	characterized by k
System D	characterized by k+d
System T	characterized by k+t
System B	characterized by k+t+b
System S4	characterized by k+t+4
System S4.2	characterized by k+t+4+g
System S4.3	characterized by k+4+Lin
System S5	characterized by k+t+5

Notice that System L is not included in the above list. This is our name for the logical system most closely tied to Leibniz's idea about necessity and possible worlds. As it turns out, System L is equivalent to C. I. Lewis' System S5 in the following sense – an argument form is valid in System L if and only if it is valid in System S5. We will continue to distinguish them, however, because they are semantically different. Indeed, in the next chapter, we combine System L with various relative modal systems into multi-modal systems.

3. Generating Rules from Characteristic Theses

We know how the various historical systems are characterized axiomatically. We know how System K is characterized natural-deductively. The question is how do we characterize these stronger systems using natural deduction. For a very important subclass of systems – the G-systems (terminology to be explained later), which include D, T, B, S4, S5, S4.2 – the answer is fairly simple. In order to produce a rule, we attempt to provide a schematic derivation proving the associated characteristic thesis, and we invent a new rule at the point in the schematic derivation when we need it! Let us do several examples.

d:

(1)	SHOW: $\Box \mathcal{A} \rightarrow \Diamond \mathcal{A}$	/i	CD
(2)	$\Box \mathcal{A}$	/i	As
(3)	SHOW: $\Diamond \mathcal{A}$	/i	4, \Diamond I
(4)	\mathcal{A}	/i+m	2, \Box O(d)
(1)	SHOW: $\Box \mathcal{A} \rightarrow \Diamond \mathcal{A}$	/i	CD
(2)	$\Box \mathcal{A}$	/i	As
(3)	SHOW: $\Diamond \mathcal{A}$	/i	4, \Diamond I(d)
(4)	SHOW: \mathcal{A}	/i+m	DD
(5)	\mathcal{A}	/i+m	2, \Box O

t:

(1)	SHOW: $\Box \mathcal{A} \rightarrow \mathcal{A}$	<i>/i</i>	CD
(2)	$\Box \mathcal{A}$	<i>/i</i>	As
(3)	SHOW: \mathcal{A}	<i>/i</i>	DD
(4)	\mathcal{A}	<i>/i</i>	2, \Box O(t)
(1)	SHOW: $\mathcal{A} \rightarrow \Diamond \mathcal{A}$	<i>/i</i>	CD
(2)	\mathcal{A}	<i>/i</i>	As
(3)	SHOW: $\Diamond \mathcal{A}$	<i>/i</i>	DD
(4)	$\Diamond \mathcal{A}$	<i>/i</i>	2, \Diamond I(t)

b:

(1)	SHOW: $\Diamond \Box \mathcal{A} \rightarrow \mathcal{A}$	<i>/i</i>	CD
(2)	$\Diamond \Box \mathcal{A}$	<i>/i</i>	As
(3)	SHOW: \mathcal{A}	<i>/i</i>	DD
(4)	$\Box \mathcal{A}$	<i>/i+m</i>	2, \Diamond O
(5)	\mathcal{A}	<i>/i</i>	4, \Box O(b)
(1)	SHOW: $\mathcal{A} \rightarrow \Box \Diamond \mathcal{A}$	<i>/i</i>	CD
(2)	\mathcal{A}	<i>/i</i>	As
(3)	SHOW: $\Box \Diamond \mathcal{A}$	<i>/i</i>	\Box D
(4)	SHOW: $\Diamond \mathcal{A}$	<i>/i+m</i>	DD
(5)	$\Diamond \mathcal{A}$	<i>/i+m</i>	2, \Diamond I(b)

4:

(1)	SHOW: $\Box \mathcal{A} \rightarrow \Box \Box \mathcal{A}$	<i>/i</i>	CD
(2)	$\Box \mathcal{A}$	<i>/i</i>	As
(3)	SHOW: $\Box \Box \mathcal{A}$	<i>/i</i>	\Box D
(4)	SHOW: $\Box \mathcal{A}$	<i>/i+m</i>	\Box D
(5)	SHOW: \mathcal{A}	<i>/i+m+n</i>	DD
(6)	\mathcal{A}	<i>/i+m+n</i>	2, \Box O(4)
(1)	SHOW: $\Diamond \Diamond \mathcal{A} \rightarrow \Diamond \mathcal{A}$	<i>/i</i>	CD
(2)	$\Diamond \Diamond \mathcal{A}$	<i>/i</i>	As
(3)	SHOW: $\Diamond \mathcal{A}$	<i>/i</i>	DD
(4)	$\Diamond \mathcal{A}$	<i>/i+m</i>	2, \Diamond O
(5)	\mathcal{A}	<i>/i+m+n</i>	4, \Diamond O
(6)	$\Diamond \mathcal{A}$	<i>/i</i>	5, \Diamond I(4)

5:

(1)	SHOW: $\Diamond \Box \mathcal{A} \rightarrow \Box \mathcal{A}$	<i>/i</i>	CD
(2)	$\Diamond \Box \mathcal{A}$	<i>/i</i>	As
(3)	SHOW: $\Box \mathcal{A}$	<i>/i</i>	\Box D
(4)	SHOW: \mathcal{A}	<i>/i+m</i>	DD
(5)	$\Box \mathcal{A}$	<i>/i+n</i>	2, \Diamond O
(6)	\mathcal{A}	<i>/i+m</i>	5, \Box O(5)
(1)	SHOW: $\Diamond \mathcal{A} \rightarrow \Box \Diamond \mathcal{A}$	<i>/i</i>	CD
(2)	$\Diamond \mathcal{A}$	<i>/i</i>	As
(3)	SHOW: $\Box \Diamond \mathcal{A}$	<i>/i</i>	\Box D
(4)	SHOW: $\Diamond \mathcal{A}$	<i>/i+m</i>	DD
(5)	\mathcal{A}	<i>/i+n</i>	2, \Diamond O
(6)	$\Diamond \mathcal{A}$	<i>/i+m</i>	5, \Diamond I(5)

4. Rules for the K+ Systems

In the previous section, we saw how one might generate rules associated with various characteristic formulas. We now officially present those rules. We concentrate on what we call K+ systems. Axiomatically, a K+ system is one that can be obtained by taking System K and adding some combination of the thesis forms (d), (t), (b), (4), and (5). The systems so produced are called KD, K4, K45, etc., based on which particular combination is chosen.

Below is the official statement of the rules for the K+ systems, beginning with a restatement of the basic K-rules.

As before, ‘ i ’ denotes sequences of numerals; ‘ m ’, ‘ n ’, denote numerals; $i+m$ is the result of appending numeral m to sequence i .

$\frac{\Box \mathcal{A} \quad /i}{\mathcal{A} \quad /i+m \text{ (old)}}$	$\frac{\mathcal{A} \quad /i+m \text{ (old)}}{\Diamond \mathcal{A} \quad /i}$
$\frac{\Box \mathcal{A} \quad /i}{\mathcal{A} \quad /i+m \text{ (new)}}$	$\frac{\mathcal{A} \quad /i+m \text{ (new)}}{\Diamond \mathcal{A} \quad /i}$
$\frac{\Box \mathcal{A} \quad /i}{\mathcal{A} \quad /i}$	$\frac{\mathcal{A} \quad /i}{\Diamond \mathcal{A} \quad /i}$
$\frac{\Box \mathcal{A} \quad /i+m \text{ (old)}}{\mathcal{A} \quad /i}$	$\frac{\mathcal{A} \quad /i}{\Diamond \mathcal{A} \quad /i+m \text{ (old)}}$
$\frac{\Box \mathcal{A} \quad /i}{\mathcal{A} \quad /i+m+n \text{ (old)}}$	$\frac{\mathcal{A} \quad /i+m+n \text{ (old)}}{\Diamond \mathcal{A} \quad /i}$
$\frac{\Box \mathcal{A} \quad /i+m \text{ (old)}}{\mathcal{A} \quad /i+n \text{ (old)}}$	$\frac{\mathcal{A} \quad /i+m \text{ (old)}}{\Diamond \mathcal{A} \quad /i+n \text{ (old)}}$

NOTE: the rules $\Box D$ and $\Diamond O$ are the same for all K+ systems, and hence carry no suffixes.

5. The K+ Systems

Having listed a number of additional rules that one might add to System K, we now list the associated modal systems one obtains by adding various combinations of rules. These include the historical systems (S4, T, etc.) as well as *neo-systems*, which are characterized "à la carte". In particular, for any combination of rules that includes the k-rules, one obtains a neo-system (K+ system). Some of these neo-systems are trivially equivalent to historical systems; some are less-trivially equivalent, and some are equivalent to each other.

Historical Systems (examples):

K	k rules
D	k, d rules
T	k, t rules
B	k, t, b rules
S4	k, t, 4 rules
S5	k, t, b, 4, 5 rules

Neo-Systems (examples):

KB	k, b rules
K4	k, 4 rules
K5	k, 5 rules
KDB	k, d, b rules
KD4	k, d, 4 rules
KD5	k, d, 5 rules
K45	k, 4, 5 rules
KD45	k, d, 4, 5 rules
KB4	k, b, 4 rules

Given that there are 5 rule-pairs in addition to the K-rules, there are *as many as* 32 K+ systems. As it turns out, however, there are a number of cases in which distinct combinations produce the same logical system, in the sense that they validate precisely the same argument forms. The number of logically distinct systems turns out to be 15.

For example, any K+ system that includes the t-rules automatically includes the d-rules. In order to simulate $\Box O(d)$ in a KT+ system, one merely applies $\Box O(t)$, then $\Diamond I(t)$, then $\Diamond O$.¹

¹ For reasons of economy, we do not officially include the d-rules in systems that employ the t-rules.

Furthermore, the historical systems – S4, S5, B, T – may be characterized in distinct ways. For example, one can prove that historical system S5 is equivalent to KT5, and that historical system B is equivalent to KDB. One can also show that S5 is equivalent to System L.

6. There are At Most 15 K+ Systems

In the previous section, we mentioned that there are 32 *combinatorially-possible* K+ systems, since there are 5 rule-pairs that can be individually added to the K-rules. On the other hand, not all of these systems are logically distinct. In the present section we show how the 32 collapse to 15.

- (t1) Any system that admits the t-rules automatically admits the d-thesis.

Proof: the following is a schematic derivation of the d-thesis in KT+.

(1)	SHOW: $\Box \mathcal{A} \rightarrow \Diamond \mathcal{A}$	/i	CD
(2)	$\Box \mathcal{A}$	/i	As
(3)	SHOW: $\Diamond \mathcal{A}$	/i	DD
(4)	\mathcal{A}	/i	2, $\Box O(t)$
(5)	$\Diamond \mathcal{A}$	/i	4, $\Diamond I(t)$

- (t2) Any system that admits the b-rules and the 4-rules admits the 5-thesis.

Proof: the following is a schematic derivation of the 5-thesis in KB4+.

(1)	SHOW: $\Diamond \Box \mathcal{A} \rightarrow \Box \mathcal{A}$	/i	CD
(2)	$\Diamond \Box \mathcal{A}$	/i	As
(3)	SHOW: $\Box \mathcal{A}$	/i	DD!
(4)	$\Box \mathcal{A}$	/i+a	2, $\Diamond O$
(5)	SHOW: $\Box \Box \mathcal{A}$	/i+a	$\Box D$
(6)	SHOW: $\Box \mathcal{A}$	/i+a+b	$\Box D$
(7)	SHOW: \mathcal{A}	/i+a+b+c	DD
(8)	\mathcal{A}	/i+a+b+c	4, $\Box O(4)$
(9)	$\Box \mathcal{A}$	/i	5, $\Box O(b)$

- (t3) Any system that admits the b-rules and the 5-rules admits the 4-thesis.

Proof: the following is a schematic derivation of the 4-thesis in KB5+.

(1)	SHOW: $\Box \mathcal{A} \rightarrow \Box \Box \mathcal{A}$	/i	CD
(2)	$\Box \mathcal{A}$	/i	As
(3)	SHOW: $\Box \Box \mathcal{A}$	/i	$\Box D$
(4)	SHOW: $\Box \mathcal{A}$	/i+a	$\Box D$
(5)	SHOW: \mathcal{A}	/i+a+b	DD
(6)	$\Diamond \Box \mathcal{A}$	/i+a	2, $\Diamond I(b)$
(7)	$\Box \mathcal{A}$	/i+a+c	6, $\Diamond O$
(8)	\mathcal{A}	/i+a+b	7, $\Box O(5)$

- (t4) Any system that admits the d-rules, b-rules, and 4-rules admits the t-thesis.

Proof: the following is a schematic derivation of the t-thesis in KDB4+.

(1)	SHOW: $\Box \mathcal{A} \rightarrow \mathcal{A}$	$/i$	CD
(2)	$\Box \mathcal{A}$	$/i$	As
(3)	SHOW: \mathcal{A}	$/i$	DD
(4)	\mathcal{A}	$/i+a$	2, $\Box O(d)$
(5)	$\Diamond \Box \mathcal{A}$	$/i+a$	2, $\Diamond I(b)$
(6)	$\Box \mathcal{A}$	$/i+a+b$	5, $\Diamond O$
(7)	$\Diamond \Box \mathcal{A}$	$/i$	6, $\Diamond I(4)$
(8)	$\Box \mathcal{A}$	$/i+c$	7, $\Diamond O$
(9)	\mathcal{A}	$/i$	8, $\Box O(b)$

(t5) Any system that admits the d-rules, the b-rules, and the 5-rules admits the t-thesis.

Proof. This follows from (t3) and (t4).

(t6) Any system that admits the t-rules and the 5-rules admits the b-thesis

Proof: the following is a schematic derivation of the b-thesis in KT5.

(1)	SHOW: $\Diamond \Box \mathcal{A} \rightarrow \mathcal{A}$	$/i$	CD
(2)	$\Diamond \Box \mathcal{A}$	$/i$	As
(3)	SHOW: \mathcal{A}	$/i$	DD
(4)	$\Box \mathcal{A}$	$/i+a$	2, $\Diamond O$
(5)	SHOW: $\Box \mathcal{A}$	$/i$	$\Box D$
(6)	SHOW: \mathcal{A}	$/i+b$	DD
(7)	\mathcal{A}	$/i+b$	4, $\Box O(5)$
(8)	\mathcal{A}	$/i$	5, $\Box O(t)$

(t7) Any system that admits the t-rules and the 5-rules admits the 4-thesis.

Proof: This follows from (t3) and (t6).

Finally, we argue that there are at most 15 K+ systems. We do this by showing certain equivalences among the 32 nominal systems. We begin by writing some simple equivalences.

TD = T	
B4 = B5	
KT(D)5 = KT(D)45 = KT(D)B4 = KT(D)B5 = KT(D)B45 = KDB45 = KDB4 = KDB5	-12
KTD = KT	-1
KTD4 = KT4	-1
KTDB = KTB	-1
KB4 = KB5 = KB45	-2

Each of these equivalences is based on one of the theorems above. This means that we have 17 duplicate names, which means that there are at most 32-17 logically distinct systems. To argue that there are at least 15 logically distinct systems, we need to prove that certain theses are not admitted by the systems in question. This must wait until we have a technique for proving invalidity.

7. There are Infinitely-Many Modal Systems Between K and L

In the previous section, we examined 5 pairs of rules – d, t, b, 4, 5 – that can be added to our original rules for System K. Because of cases of collapse the 32 abstract combinations produce 15 logically distinct systems.

One might wonder how many other modal logical systems there are that lie between K and L (S5). Well, there are infinitely-many systems, which can be seen as follows.

Consider the following sequence of schematic formulas.

- (4-1) $\Box\mathcal{A} \rightarrow \Box\Box\mathcal{A}$
 (4-2) $\Box\Box\mathcal{A} \rightarrow \Box\Box\Box\mathcal{A}$
 (4-3) $\Box\Box\Box\mathcal{A} \rightarrow \Box\Box\Box\Box\mathcal{A}$
 etc.

Notice that (4-1) is the same as our earlier (4). Now, one can show the following. Suppose one adds any given one of these schematic forms as a primitive thesis form to System K. Then one can prove all the schematic forms below that form, but one cannot prove any schematic form above that form. For example, if one adds (4.3) to the system one can prove (4.4), (4.5), and so forth, but one cannot prove (4.2) or (4.1).

From this it follows that there are infinitely many distinct modal systems between K and L.

8. The G-Systems

Geach has considered a logical system – S4.2 – which is based on the following primitive thesis forms.

- (k) $\Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box\mathcal{A} \rightarrow \Box\mathcal{B})$
 (t) $\Box\mathcal{A} \rightarrow \mathcal{A}$
 (4) $\Box\mathcal{A} \rightarrow \Box\Box\mathcal{A}$
 (g) $\Diamond\Box\mathcal{A} \rightarrow \Box\Diamond\mathcal{A}$

A more general scheme has been devised (Chellas, *Modal Logic*), according to which we exponentiate the modal operators in formula (g), as follows.

$$G(j,k,m,n) \quad \Diamond^j \Box^k \mathcal{A} \rightarrow \Box^m \Diamond^n \mathcal{A}$$

The exponents work in the expected manner.

$$\begin{array}{ll} \Box^0 \mathcal{A} \equiv \mathcal{A} & \Diamond^0 \mathcal{A} \equiv \mathcal{A} \\ \Box^1 \mathcal{A} \equiv \Box \mathcal{A} & \Diamond^1 \mathcal{A} \equiv \Diamond \mathcal{A} \\ \Box^2 \mathcal{A} \equiv \Box \Box \mathcal{A} & \Diamond^2 \mathcal{A} \equiv \Diamond \Diamond \mathcal{A} \\ \text{etc.} & \end{array}$$

This allows us an organizational scheme in which to consider certain modal systems, which we call G-systems. Specifically, a G-system is a modal logic system that contains the K-thesis as well as finitely many theses of the form $G(j,k,m,n)$. As it turns out, nearly every thesis we have considered can be written using the G-notation. The following are examples, including duals in parentheses (some formulas are self-dual).

g	=	G(1,1,1,1)	
d	=	G(0,1,0,1)	
t	=	G(0,1,0,0)	[G(0,0,1,0)]
b	=	G(1,1,0,0)	[G(0,0,1,1)]
4	=	G(0,1,2,0)	[G(2,0,0,1)]
4-2	=	G(0,2,3,0)	[G(3,0,0,2)]
4-3	=	G(0,3,4,0)	[G(4,0,0,3)]
etc.			
5	=	G(1,1,1,0)	[G(1,0,1,1)]

One can also consider other combinations of $G(j,k,m,n)$ formulas. For example, the following is an abstract possibility.

System X1 is specified by the following axioms.

K	$\Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box \mathcal{A} \rightarrow \Box \mathcal{B})$
G(0,1,1,1)	$\Box \mathcal{A} \rightarrow \Box \Diamond \mathcal{A}$

Notice that $G(0,1,1,1)$ is a thesis of any system that contains the t-rules, as seen in the following derivation schema.

(1)	SHOW: $\Box \mathcal{A} \rightarrow \Box \Diamond \mathcal{A}$	/i	CD
(2)	$\Box \mathcal{A}$	/i	As
(3)	SHOW: $\Box \Diamond \mathcal{A}$	/i	$\Box D$
(4)	SHOW: $\Diamond \mathcal{A}$	/i+a	DD
(5)	\mathcal{A}	/i+a	2, $\Box O(k)$
(6)	$\Diamond \mathcal{A}$	/i+a	5, $\Diamond I(t)$

Now, suppose we want to describe System X1 using natural deduction. What rules do we add? In order to discover this, we proceed as we did in an earlier section – we simply write down a schematic derivation and see where a new rule is obviously required! The following is our schematic derivation.

(1)	SHOW: $\Box \mathcal{A} \rightarrow \Box \Diamond \mathcal{A}$	/i	CD
(2)	$\Box \mathcal{A}$	/i	As
(3)	SHOW: $\Box \Diamond \mathcal{A}$	/i	$\Box D$
(4)	SHOW: $\Diamond \mathcal{A}$	/i+a	DD
(5)	\mathcal{A}	/i+a+b	2, $\Box O-x1$
(6)	$\Diamond \mathcal{A}$	/i+a	5, $\Diamond I(k)$

From this derivation, we surmise that the rule we need is:

$\Box O-x1$

$\Box \mathcal{A}$	/i	
—		
\mathcal{A}	/i+a+b	i+a old; b new

Let us do a somewhat more complicated example.

System X2 is specified by the following axioms.

$$\begin{array}{ll} \text{K} & \Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box \mathcal{A} \rightarrow \Box \mathcal{B}) \\ \text{G}(2,1,2,1) & \Diamond \Diamond \Box \mathcal{A} \rightarrow \Box \Box \Diamond \mathcal{A} \end{array}$$

Suppose we want to describe System X2 using natural deduction. To discover the appropriate natural deduction rule, we write down a schematic derivation and see where the new rule is required.

(1)	SHOW: $\Diamond \Diamond \Box \mathcal{A} \rightarrow \Box \Box \Diamond \mathcal{A}$	/i	CD
(2)	$\Diamond \Diamond \Box \mathcal{A}$	/i	CD
(3)	SHOW: $\Box \Box \Diamond \mathcal{A}$	/i	\Box D
(4)	SHOW: $\Box \Diamond \mathcal{A}$	/i+a	\Box D
(5)	SHOW: $\Diamond \mathcal{A}$	/i+a+b	\Diamond D
(6)	$\Diamond \Box \mathcal{A}$	/i+c	2, \Diamond O
(7)	$\Box \mathcal{A}$	/i+c+d	6, \Diamond O
(8)	\mathcal{A}	/i+a+b+e	7, \Box O-x2
(9)	$\Diamond \mathcal{A}$	/i+a+b	8, \Diamond I(k)

From this derivation, we surmise that the rule we need is:

\Box O-x2

$$\frac{\Box \mathcal{A} \quad /i+a+b \quad \text{old}}{\mathcal{A} \quad /i+c+d+e \quad i+c+d \text{ old}; e \text{ new}}$$

9. Some Non-G Systems

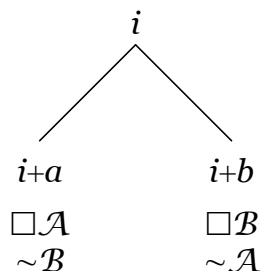
As mentioned in the last section, most of the systems we consider are G-systems. Not all modal systems are G-systems, however. We have already seen one – S4.3, which is based on the following axioms.

$$\begin{array}{ll} \text{(K)} & \Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box \mathcal{A} \rightarrow \Box \mathcal{B}) \\ \text{(T)} & \Box \mathcal{A} \rightarrow \mathcal{A} \\ \text{(4)} & \Box \mathcal{A} \rightarrow \Box \Box \mathcal{A} \\ \text{(Lin)} & \Box(\Box \mathcal{A} \rightarrow \mathcal{B}) \vee \Box(\Box \mathcal{B} \rightarrow \mathcal{A}) \end{array}$$

The problem is (Lin), which is short for ‘linear’, which is decidedly not G-like. Can we produce a corresponding rule in the natural deduction system? Well, let’s try our usual trick.

(1)	SHOW: $\Box(\Box \mathcal{A} \rightarrow \mathcal{B}) \vee \Box(\Box \mathcal{B} \rightarrow \mathcal{A})$	/i	\vee D
(2)	$\sim \Box(\Box \mathcal{A} \rightarrow \mathcal{B})$	/i	As
(3)	$\sim \Box(\Box \mathcal{B} \rightarrow \mathcal{A})$	/i	As
(4)	SHOW: \ast	/*	
(5)	$\sim(\Box \mathcal{A} \rightarrow \mathcal{B})$	/i+a	2, \sim \Box O
(6)	$\sim(\Box \mathcal{B} \rightarrow \mathcal{A})$	/i+b	3, \sim \Box O
(7)	$\Box \mathcal{A} \ \& \ \sim \mathcal{B}$	/i+a	5, SL
(8)	$\Box \mathcal{B} \ \& \ \sim \mathcal{A}$	/i+b	6, SL
(9)	\ast	/*	7,8, Rule XX

The basic idea is that we have the following situation.



If $i+a$ "sees" $i+b$, then we get \mathcal{A} at $i+b$, and hence a contradiction. Similarly, if $i+b$ "sees" $i+a$, then we get \mathcal{B} at $i+a$, and hence a contradiction. In System 4.3, we have that $i+a$ sees $i+b$ or vice versa, which is to say that the indices lie along a line. But the only way to convey this in the natural deduction system is to take Rule XX as an additional rule. The problem is that it is not a $\square O$ rule. So it is unlike all the rules we have so far introduced.

The following is an even stranger example of a non-G formula; in fact, it is an anti-G formula!

$$(Z) \quad \square \diamond \mathcal{A} \rightarrow \diamond \square \mathcal{A}$$

This is not a thesis of L, so no system that contains it lies between K and L. Still, one might consider adding it as a thesis to System K, in which case we get System KZ, which is arguably a strange system. By way of orientation, the following is a loose reading in deontic logic, which might be useful to consider because it does not satisfy all the principles of System L.

if it is required that \mathcal{A} be permitted, then it is permitted that \mathcal{A} be required

I leave the reader to contemplate what ethical system might produce such a result.

10. Appendix 1 – World Theory for K+ Systems

1. Introduction

WT(K) is the *minimal* world theory; System K is the *minimal* modal logic. Many interesting modal arguments/formulas are valid in K. On the other hand, a number of plausible modal arguments/formulas are not valid in K, including the following.

$$\begin{aligned}\Box P &\rightarrow \Diamond P \\ \Box P &\rightarrow P\end{aligned}$$

Given the correspondence between System K and WT(K), the K-validity of these two formulas corresponds respectively to the whether the following are theorems of WT(K).

$$\begin{aligned}\forall j\{i \prec j \rightarrow \mathcal{A}/j\} &\rightarrow \exists j\{i \prec j \ \& \ \mathcal{A}/j\} \\ \forall j\{i \prec j \rightarrow \mathcal{A}/j\} &\rightarrow \mathcal{A}/i\end{aligned}$$

However, neither is a theorem WT(K).

Consider the first one, which may be read as follows.

if \mathcal{A} is true at every i -accessible world,
then \mathcal{A} is true at least one i -accessible world.

In order to prove this, we need the following additional axiom, which is not provable in WT(K).

$$(r1) \quad \forall i \exists j [i \prec j]$$

which reads

for any world i , there is at least one i -accessible world;
[every world sees/reaches at least one world].

Now, consider the second formula, which may be read as follows.

if \mathcal{A} is true at every i -accessible world,
then \mathcal{A} is true at i

In order to prove this, we need the following additional axiom, which is not provable in WT(K).

$$(r2) \quad \forall i [i \prec i]$$

which reads

for any world i , i is i -accessible;
[every world sees/reaches itself].

The upshot of the above reasoning is that, in order to obtain the validity of certain formulas (arguments), we must strengthen Minimal World Theory – i.e., WT(K) – by adding further axioms. Generally, this is accomplished by placing restrictions on the accessibility relation \prec

2. Kinds of Relations: Ways One Might Restrict \prec

We already have seen two plausible restrictions. The first one, (r1), says, in effect, that every world "reaches" at least one world. The other one, (r2), says that every world "reaches" itself. Notice that the latter entails the former. A slightly weaker condition than (r2) is the following.

$$(r3) \quad \forall x \forall y \{x \prec y \rightarrow x \prec x\}$$

This says that if a world reaches any world it reaches itself. Notice that (r1)+(r3) entails (r2). A restriction somewhat stronger than (r3) may be stated in any of the following ways, all equivalent.

$$(r4.1) \quad \forall x \forall y \{ (x \prec y \vee y \prec x) \rightarrow x \prec x \}$$

$$(r4.2) \quad \forall x \{ \exists y \{ x \prec y \vee y \prec x \} \rightarrow x \prec x \}$$

$$(r4.1) \quad \forall x \forall y \{ x \prec y \rightarrow x \prec x \} \ \& \ \forall x \forall y \{ y \prec x \rightarrow x \prec x \}$$

These conditions variously assert that, if x reaches anything or anything reaches x , then x reaches itself.

It is customary to categorize relations according to what restrictions they satisfy. The following are some of the more prominent examples. The first three correspond to restrictions (r1), (r4), and (r2), respectively.

Serial	$\text{Ser}[R] =_{df} \forall x \exists y [xRy]$
Reflexive	$\text{Ref}[R] =_{df} \forall x \{ \exists y (xRy \vee yRx) \rightarrow xRx \}$
Completely Reflexive	$\text{C-Ref}[R] =_{df} \forall x [xRx]$
Irreflexive	$\text{Irref}[R] =_{df} \sim \exists x [xRx]$
Transitive	$\text{Tran}[R] =_{df} \forall xyz \{ xRy \ \& \ yRz \rightarrow xRz \}$
Symmetric	$\text{Sym}[R] =_{df} \forall xy \{ xRy \rightarrow yRx \}$
Euclidean	$\text{Euc}[R] =_{df} \forall xyz \{ xRy \ \& \ xRz \rightarrow yRz \}$
Incestual	$\text{Incest}[R] =_{df} \forall xyz \{ xRy \ \& \ xRz \rightarrow \exists w (yRw \ \& \ zRw) \}$
Asymmetric	$\text{Asym}[R] =_{df} \forall xy \{ xRy \rightarrow \sim [yRx] \}$
Anti-Symmetric	$\text{Anti}[R] =_{df} \forall xy \{ xRy \ \& \ yRx \rightarrow x=y \}$
Universal	$\text{Univ}[R] =_{df} \forall xy [xRy]$
Identity	$\text{Iden}[R] =_{df} \forall xy \{ xRy \rightarrow x=y \}$
Functional	$\text{Fun}[R] =_{df} \forall xyz \{ xRy \ \& \ xRz \rightarrow y=z \}$
Inverse-Functional	$\text{I-Fun}[R] =_{df} \forall xyz \{ xRz \ \& \ yRz \rightarrow x=y \}$
Strongly Connected	$\text{SCon}[R] =_{df} \forall xy [xRy \vee yRx]$
Weakly Connected	$\text{WCon}[R] =_{df} \forall xy \{ x \neq y \rightarrow xRy \vee yRx \}$
Non-Convergent	$\text{N-Con}[R] =_{df} \forall xyz \{ xRz \ \& \ yRz \ \& \ x \neq y \rightarrow xRy \vee yRx \}$
Non-Divergent	$\text{N-Div}[R] =_{df} \forall xyz \{ xRy \ \& \ xRz \ \& \ y \neq z \rightarrow yRz \vee zRy \}$
Equivalence Relation	$\text{Equ}[R] =_{df} \text{CRef}[R] \ \& \ \text{Tran}[R] \ \& \ \text{Sym}[R]$
Quasi-Ordering	$\text{QO}[R] =_{df} \text{CRef}[R] \ \& \ \text{Tran}[R]$
Partial Ordering	$\text{PO}[R] =_{df} \text{QO}[R] \ \& \ \text{Anti}[R]$
Reflexive Linear Ordering	$\text{RLO}[R] =_{df} \text{PO}[R] \ \& \ \text{SCon}[R]$
Irreflexive Linear Ordering	$\text{ILO}[R] =_{df} \text{Asym}[R] \ \& \ \text{Tran}[R] \ \& \ \text{W-Con}[R]$
Tree-Ordering	$\text{Tree}[R] =_{df} \text{PO}[R] \ \& \ \text{N-Conv}[R]$
Root-Ordering	$\text{Root}[R] =_{df} \text{PO}[R] \ \& \ \text{N-Div}[R]$
Upper-Bounded	$\text{U-Bounded}[R] =_{df} \forall xy \{ \exists z [xRz] \ \& \ \exists z [yRz] \rightarrow \exists z (xRz \ \& \ yRz) \}$
Lower-Bounded	$\text{L-Bounded}[R] =_{df} \forall xy \{ \exists z [zRx] \ \& \ \exists z [zRy] \rightarrow \exists z (zRx \ \& \ zRy) \}$
Bounded	$\text{Bounded}[R] =_{df} \text{U-Bounded}[R] \ \& \ \text{L-Bounded}[R]$
Dense	$\text{Dense}[R] =_{df} \forall xz \{ xRz \ \& \ x \neq z \rightarrow \exists y (x \neq y \ \& \ y \neq z \ \& \ xRy \ \& \ yRz) \}$

3. The Relation between \prec and \Box

What is interesting is that, by placing various restrictions on the accessibility relation, we are able to ensure that certain type-0 formulas are theorems of the resulting strengthened world theory

For example, consider the following formulas.

- (k) $\Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box \mathcal{A} \rightarrow \Box \mathcal{B})$
- (d) $\Box \mathcal{A} \rightarrow \Diamond \mathcal{A}$
- (t) $\Box \mathcal{A} \rightarrow \mathcal{A}$
- (b) $\mathcal{A} \rightarrow \Box \Diamond \mathcal{A}$
- (4) $\Box \mathcal{A} \rightarrow \Box \Box \mathcal{A}$
- (g) $\Diamond \Box \mathcal{A} \rightarrow \Box \Diamond \mathcal{A}$
- (L) $\Box(\Box \mathcal{A} \rightarrow \mathcal{B}) \vee \Box(\Box \mathcal{B} \rightarrow \mathcal{A})$
- (5) $\Diamond \Box \mathcal{A} \rightarrow \Box \mathcal{A}$

Each of these formulas corresponds to an associated restriction on the accessibility relation \prec , as follows.

restrict(k)	null
restrict(d)	\prec is serial
restrict(t)	\prec is c-reflexive
restrict(b)	\prec is symmetric
restrict(4)	\prec is transitive
restrict(g)	\prec is incestual
restrict(L)	\prec is a reflexive and non-divergent
restrict(5)	\prec is Euclidean

To be more explicit we have the following theorem schema of WT(K).

- (th) $\text{restrict}(\Phi) \rightarrow \text{formula}(\Phi)$

Here, ‘ Φ ’ represents any of the labels: K, D, T, B, 4, G, L, 5. For example, we have:

- (th) $\text{restrict}(\mathbf{K}) \rightarrow \text{formula}(\mathbf{K})$
- .
- .
- (th) $\text{restrict}(\mathbf{5}) \rightarrow \text{formula}(\mathbf{5})$

Plugging in, we obtain:

- (K) $\Box(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\Box \mathcal{A} \rightarrow \Box \mathcal{B})$
- (D) $\text{Ser}[\prec] \rightarrow \{\Box \mathcal{A} \rightarrow \Diamond \mathcal{A}\}$
- (T) $\text{C-Ref}[\prec] \rightarrow \{\Box \mathcal{A} \rightarrow \mathcal{A}\}$

- (B) $\text{Sym}[\prec] \rightarrow \{\mathcal{A} \rightarrow \Box \Diamond \mathcal{A}\}$
- (4) $\text{Trans}[\prec] \rightarrow \{\Box \mathcal{A} \rightarrow \Box \Box \mathcal{A}\}$
- (G) $\text{Incest}[\prec] \rightarrow \{\Diamond \Box \mathcal{A} \rightarrow \Box \Diamond \mathcal{A}\}$
- (L) $\text{Linear}[\prec] \rightarrow \{\Box(\Box \mathcal{A} \rightarrow \mathcal{B}) \vee \Box(\Box \mathcal{B} \rightarrow \mathcal{A})\}$
- (5) $\text{Euc}[\prec] \rightarrow \{\Diamond \Box \mathcal{A} \rightarrow \Box \mathcal{A}\}$

Of course, as they stand, these are not completely satisfying, because they are *conditionals*. What we would like are the corresponding *bi-conditionals* for these theorems. These are simply unavailable at the object level. To obtain anything like the converses to the above theorems, we need either to ascend to the *meta-language*, or we need to ascend to a *second-order* formulation of World Theory.

In the first case, we would seek meta-theorems of the following sort.

- (t8) Let \mathcal{T} be any extension of WT(K). Suppose formula(Φ) is a theorem of \mathcal{T} ; then $\text{restrict}(\Phi)$ is a theorem of \mathcal{T} .

These can be proved, but the proofs take us well outside the realm of *elementary* modal logic, which is the chief topic of this text.

In order to stay within the realm of elementary logic, we opt instead to ascend to second-order logic. In that case, we will obtain object theorems such as the following.

- (t9) $\text{Euc}[\prec] \leftrightarrow \forall \mathcal{A}(\Diamond \Box \mathcal{A} \rightarrow \mathcal{A})$

These ideas are discussed further in a later chapter.

11. Appendix 2 – Counter-Models in K+ Systems

1. Introduction

As with earlier systems, we now turn to the issue of invalidity. In particular, we examine how to demonstrate invalidity for a number of systems that extend System K.

2. Frames and Valuations

Just as we did with System K, we begin with the definition of K-model.

- (d1) A **K-model** is, by definition, a structure $\langle \Omega, <, \omega_0, \langle p_1, p_2, \dots \rangle \rangle$, where Ω is a non-empty set, $<$ is a binary relation on Ω , ω_0 is a privileged element of Ω , and $\langle p_1, p_2, \dots \rangle$ is a sequence of subsets of Ω .

As before, Ω is the set of indices (reference points, possible worlds), $<$ is the accessibility relation, ω_0 is the "actual" world, and $\langle p_1, p_2, \dots \rangle$ are the "propositions" that interpret the atomic formulas $\langle A_1, A_2, \dots \rangle$.

Recall from Appendix 1 that, associated with each axiom schema, there is a corresponding restriction on the accessibility relation $<$. These are given as follows.

restrict(D)	$<$ is serial
restrict(T)	$<$ is c-reflexive
restrict(B)	$<$ is symmetric
restrict(4)	$<$ is transitive
restrict(G)	$<$ is incestual
restrict(L)	$<$ is a reflexive and non-divergent
restrict(5)	$<$ is Euclidean

Using these restrictions, we can also define various species of K-frames as follows.

- (d2) Let $\langle \Omega, <, \omega_0, \langle p_1, p_2, \dots \rangle \rangle$ be a K-model. Let α, β, \dots be any of $\{D, T, B, 4, G, L, 5\}$. Then $\langle \Omega, <, \omega_0, \langle p_1, p_2, \dots \rangle \rangle$ is said to be a

K α -model iff $<$ satisfies restrict(α)
K $\alpha\beta$ -model iff $<$ satisfies restrict(α) and restrict(β)
 etc.

For example, a K45-model is a K-model $\langle \Omega, < \rangle$ in which $<$ is transitive and Euclidean.

- (d3) Let $\langle \Omega, <, \omega_0, \langle p_1, p_2, \dots \rangle \rangle$ be a K-model. Then $\langle \Omega, <, \omega_0, \langle p_1, p_2, \dots \rangle \rangle$ is said to be a
- T-model iff $<$ is reflexive
 B-model iff $<$ is reflexive and symmetric
 S4-model iff $<$ is reflexive and transitive
 S4.2-model iff $<$ is reflexive and incestual
 S4.3-model iff $<$ is linear
 S5-model iff $<$ is an equivalence relation

The alert reader may notice that there is a certain amount of duplication between the last two definitions. For example, KT-models and T-models are identical; similarly, KTB-frames and B-frames are identical. The duplication has to do with the difference between systematic nomenclature (the first definition) and historical nomenclature (the second definition).

Recall the definition of K-admissible valuation, and the derivative notion of K-counter-model. This can now be generalized to K5-admissible valuation, S4-admissible valuation, etc. In what follows, Σ is any of the above mentioned kinds of frames – K4, K45, S5, etc.

- (d4) Let \mathcal{L} and S be as before. Let v be a valuation on \mathcal{L} . Then v is a Σ -*admissible valuation* if and only if there is a Σ -model that gives rise to v .
- (d5) Let \mathcal{L} and S be as before. Let $\langle P_1, \dots, P_m / C \rangle$ be an argument in \mathcal{L} . Then a Σ -*counter-model* to $\langle P_1, \dots, P_m / C \rangle$ is any Σ -admissible valuation v such that $v(P_1) = T, \dots, v(P_m) = T$, and $v(C) = F$.

3. Summary of Rules for Constructing Counter-Models in K+ Systems

1. The Usual Truth-Functional Rules

2. The Same General Modal Rules as System K

3. Rules concerning accessibility

- k: i sees $i+m$; note that, in K, a world need not see any world, even itself.
- d: accessibility is *serial*: every world sees at least one world; however, it need not see itself.
- t: accessibility is *completely reflexive*: every world sees itself.
- b: accessibility is *symmetric*: $\forall ij$, if i sees j , then j sees i .
- 4: accessibility is *transitive*: $\forall ijk$, if i sees j , and j sees k , then i also sees k .
- 5: accessibility is *Euclidean*: $\forall ijk$, if i sees both j and k , then j and k see each other.
- g: accessibility is *incestual*: $\forall ijk$, if i sees both j and k , then there is an m such that both j and k see m .

4. Kinds of Models

K models satisfy 1, 2a, 2b, plus k.

D models also satisfy rule d.

T models also satisfy rule t.

B models also satisfy rules t and b.

S4 models also satisfy rules t and 4.

S4.2 models also satisfy rules t, 4, and g.

S5 models also satisfy rules t and 5.

K5 models also satisfy rule 5.

K4 models also satisfy rule 4.

K45 models also satisfy rules 4,5.

etc.

12. Exercises

1. Derivations in K+ Systems

Directions: For each of the following, construct a derivation of the conclusion from the premises (if any) in the designated system. In case of formulas separated by ‘//’, derive each formula from the other in the designated system.

A. SYSTEM KD (historical system D)

1. $/ \Box P \rightarrow \Diamond P$
2. $/ \sim \Diamond P \rightarrow \Diamond \sim P$
3. $/ \sim \Box(P \ \& \ \sim P)$
4. $/ \sim \Box(P \leftrightarrow \sim P)$
5. $/ \Diamond(P \rightarrow P)$
6. $/ \sim \Box(\Box P \ \& \ \Box(P \rightarrow \sim P))$
7. $/ \Diamond P \vee \Diamond \sim P$
8. $/ (\Diamond P \vee \Diamond Q) \vee \Diamond(\sim P \ \& \ \sim Q)$
9. $/ \sim(\Box P \ \& \ \Box \sim P)$
10. $/ \Box \Box P \rightarrow \Diamond \Diamond P$
11. $/ \Box \Box P \rightarrow \Diamond \Box P$
12. $/ \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Diamond Q)$

B. SYSTEM KT (historical system T)

13. $/ \Box P \rightarrow P$
14. $/ P \rightarrow \Diamond P$
15. $/ \Diamond(P \rightarrow \Box Q) \rightarrow (\Box P \rightarrow \Diamond Q)$
16. $/ \Diamond \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Diamond Q)$
17. $/ \Box(P \rightarrow \Box Q) \rightarrow (\Diamond P \rightarrow \Diamond Q)$
18. $/ \Diamond(P \rightarrow \Box Q) \rightarrow (\Box P \rightarrow \Diamond Q)$

C. SYSTEM KTB (historical system B)

19. $/ P \rightarrow \Box \Diamond P$
20. $/ \Diamond \Box P \rightarrow P$
21. $/ \Diamond \Box P \rightarrow \Box \Diamond P$
22. $\Box \Diamond \Box P // \Box P$
23. $\Diamond \Box \Diamond P // \Diamond P$
24. $/ \Diamond \Box \Diamond \Box P \rightarrow \Diamond \Box P$
25. $/ \Box \Diamond P \rightarrow \Box \Diamond \Box \Diamond \Box \Diamond P$

D. SYSTEM KT4 (historical system S4)

26. $/ \Box P \rightarrow \Box \Box P$
27. $/ \Box P \rightarrow \Box \Box \Box P$
28. $/ \Diamond \Diamond P \rightarrow \Diamond P$
29. $/ \Diamond \Diamond \Diamond P \rightarrow \Diamond P$
30. $/ \Diamond \Box \Diamond \Box P \rightarrow \Diamond \Box P$
31. $/ \Box P \rightarrow \Box \Diamond \Box P$
32. $\Box \Diamond P // \Box \Diamond \Box \Diamond P$
33. $\Diamond \Box P // \Diamond \Box \Diamond \Box P$
34. $/ (\Box P \vee \Box Q) \rightarrow \Box(\Box P \vee \Box Q)$
35. $/ (\Diamond P \ \& \ \Box Q) \rightarrow \Diamond(P \ \& \ \Box Q)$

E. SYSTEM KTB45 (historical system S5)

36. $\Box P \vee \Box Q / \Box(\Box P \vee \Box Q)$
37. $\Box(\Box P \rightarrow Q) / \Box(\Box P \rightarrow \Box Q)$
38. $\Diamond P ; \Box Q / \Diamond(P \ \& \ \Box Q)$
39. $\Box(P \vee \Box Q) // \Box P \vee \Box Q$
40. $\Box(P \vee \Diamond Q) // \Box P \vee \Diamond Q$
41. $\Diamond(P \ \& \ \Box Q) // \Diamond P \ \& \ \Box Q$
42. $\Diamond(P \ \& \ \Diamond Q) // \Diamond P \ \& \ \Diamond Q$
43. $\Box(P \rightarrow \Box Q) // \Diamond P \rightarrow \Box Q$
44. $\Box(P \rightarrow \Diamond Q) // \Diamond P \rightarrow \Diamond Q$
45. $\Diamond(P \rightarrow \Box Q) // \Box P \rightarrow \Box Q$
46. $\Diamond(P \rightarrow \Diamond Q) // \Box P \rightarrow \Diamond Q$
47. $\Box(P \leftrightarrow \Box Q) / \Box P \leftrightarrow \Box Q$
48. $\Box(P \leftrightarrow \Diamond Q) / \Box P \leftrightarrow \Diamond Q$
49. $\Diamond P \leftrightarrow \Box Q / \Diamond(P \leftrightarrow \Box Q)$
50. $\Diamond P \leftrightarrow \Diamond Q / \Diamond(P \leftrightarrow \Diamond Q)$

F. OTHER K+ SYSTEMS

51. K5: $/ \Box(P \rightarrow Q) \vee \Box(Q \rightarrow \Diamond P)$
52. K5: $/ \Diamond P \rightarrow \Diamond(\Box Q \rightarrow Q)$
53. KDB4: $/ \Box P \rightarrow P$
54. KT5: $/ \Box P \rightarrow \Box \Box P$
55. KB5: $/ \Box P \rightarrow \Box \Box P$
56. KT5: $/ P \rightarrow \Box \Diamond P$
57. KB4: $/ \Diamond \Box P \rightarrow \Box P$
58. KB: $/ \Diamond \Box P \rightarrow \Box \Diamond P$
59. K5: $/ \Diamond \Box P \rightarrow \Box \Diamond P$
60. KB: $/ \Box \Diamond \Box P \rightarrow \Box P$

2. World Theory Exercises

Directions: For each of the argument forms in the previous section, construct a derivation of the conclusion from the premises (if any) in $WT(\sigma)$, where σ is the designated system.

3. Counter-Model Exercises

Directions (the following refer to the earlier derivation problems):

For each D-problem, construct a counter-model or derivation in K.

For each T-problem, construct a counter-model or derivation in D.

For each B-problem, and for each S4-problem, construct a counter-model or derivation in T.

Also, for each B-problem, *either* construct a derivation in S4, *or* construct a counter-model in S4.

Also, for each S4-problem, *either* construct a derivation in B, *or* construct a counter-model in B.

For each S5-problem, construct a counter-model or derivation in S4, *and* construct a counter-model or derivation in B.

13. Answers to Selected Exercises

1. Derivations

#2 (D)

(1)	SHOW: $\sim \Diamond P \rightarrow \Diamond \sim P$	/0	CD
(2)	$\sim \Diamond P$	/0	As
(3)	SHOW: $\Diamond \sim P$	/0	DD
(4)	$\Box \sim P$	/0	2, MN
(5)	$\sim P$	/01	4, $\Box O(d)$
(6)	$\Diamond \sim P$	/0	5, $\Diamond I(k)$

#12 (D)

(1)	SHOW: $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Diamond Q)$	/0	CD
(2)	$\Box(P \rightarrow Q)$	/0	As
(3)	SHOW: $\Box P \rightarrow \Diamond Q$	/0	CD
(4)	$\Box P$	/0	As
(5)	SHOW: $\Diamond Q$	/0	6, $\Diamond I(d)$
(6)	SHOW: Q	/01	DD
(7)	$P \rightarrow Q$	/01	2, $\Box O$
(8)	P	/01	4, $\Box O$
(9)	Q	/01	7,8,SL

#16 (T)

(1)	SHOW: $\Diamond \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Diamond Q)$	/0	CD
(2)	$\Diamond \Box(P \rightarrow Q)$	/0	As
(3)	SHOW: $(\Box P \rightarrow \Diamond Q)$	/0	CD
(4)	$\Box P$	/0	As
(5)	SHOW: $\Diamond Q$	/0	DD
(6)	$\Box(P \rightarrow Q)$	/01	2, $\Diamond O$
(7)	$P \rightarrow Q$	/01	6, $\Box O(t)$
(8)	P	/01	4, $\Box O(k)$
(9)	Q	/01	7,8,SL
(10)	$\Diamond Q$	/0	9, $\Diamond I(k)$

#18	(T)			
(1)	SHOW: $\diamond(P \rightarrow \Box Q) \rightarrow (\Box P \rightarrow \diamond Q)$	/0	CD	
(2)	$\diamond(P \rightarrow \Box Q)$	/0	As	
(3)	SHOW: $\Box P \rightarrow \diamond Q$	/0	CD	
(4)	$\Box P$	/0	As	
(5)	SHOW: $\diamond Q$	/0	DD	
(6)	$P \rightarrow \Box Q$	/01	2, $\diamond O$	
(7)	P	/01	4, $\Box O(k)$	
(8)	$\Box Q$	/01	6,7,SL	
(9)	Q	/01	8, $\Box O(t)$	
(10)	$\diamond Q$	/0	9, $\diamond I(k)$	
#24	(B)			
(1)	SHOW: $\diamond \Box \diamond \Box P \rightarrow \diamond \Box P$	/0	Pr	
(2)	$\diamond \Box \diamond \Box P$	/0	As	
(3)	SHOW: $\diamond \Box P$	/0	DD	
(4)	$\Box \diamond \Box P$	/01	2, $\diamond O$	
(5)	$\diamond \Box P$	/0	4, $\Box O(b)$	
#30	(KT4)			
(1)	SHOW: $\diamond \Box \diamond \Box P \rightarrow \diamond \Box P$	/0	Pr	
(2)	$\diamond \Box \diamond \Box P$	/0	As	
(3)	SHOW: $\diamond \Box P$	/0	DD	
(4)	$\Box \diamond \Box P$	/01	2, $\diamond O$	
(5)	$\diamond \Box P$	/01	4, $\Box O(t)$	
(6)	$\Box P$	/012	5, $\diamond O$	
(7)	$\diamond \Box P$	/0	6, $\diamond I(4)$	
#33	(KT4)			
(1)	$\diamond \Box P$	/0	Pr	
(2)	SHOW: $\diamond \Box \diamond \Box P$	/0	ID	
(3)	$\sim \diamond \Box \diamond \Box P$	/0	As	
(4)	SHOW: \ast	/*	9,10,SL	
(5)	$\Box P$	/01	1, $\diamond O$	
(6)	$\sim \Box \diamond \Box P$	/01	3, $\sim \diamond O(k)$	
(7)	$\sim \diamond \Box P$	/012	6, $\sim \Box O$	
(8)	$\sim \Box P$	/012	7, $\sim \diamond O(t)$	
(9)	$\sim P$	/0123	8, $\sim \Box O$	
(10)	P	/0123	5, $\Box O(4)$	
#51	(K5)			
(1)	SHOW: $\Box(P \rightarrow Q) \vee \Box(Q \rightarrow \diamond P)$	/0	$\vee ID$	
(2)	$\sim \Box(P \rightarrow Q)$	/0	As	
(3)	$\sim \Box(Q \rightarrow \diamond P)$	/0	As	
(4)	SHOW: \ast	/*	7,9,SL	
(5)	$\sim(P \rightarrow Q)$	/01	2, $\sim \Box O$	
(6)	$\sim(Q \rightarrow \diamond P)$	/02	3, $\sim \Box O$	
(7)	P	/01	5,SL	
(8)	$\sim \diamond P$	/02	6,SL	
(9)	$\sim P$	/01	7,8, $\Box O(5)$	

#52 (K5)

(1)	SHOW: $\diamond P \rightarrow \diamond(\Box Q \rightarrow Q)$	/0	CD
(2)	$\diamond P$	/0	As
(3)	SHOW: $\diamond(\Box Q \rightarrow Q)$	/0	DD
(4)	P	/01	2, $\diamond O$
(5)	SHOW: $\Box Q \rightarrow Q$	/01	CD
(6)	$\Box Q$	/01	As
(7)	SHOW: Q	/01	DD
(8)	Q	/01	6, $\Box O(5)$
(9)	$\diamond(\Box Q \rightarrow Q)$	/0	5, $\diamond I(k)$

#53 (KDB4)

(1)	SHOW: $\Box P \rightarrow P$	/0	CD
(2)	$\Box P$	/0	As
(3)	SHOW: P	/0	ID
(4)	$\sim P$	/0	As
(5)	SHOW: \ast	/*	8,9,SL
(6)	P	/01	2, $\Box O(d)$
(7)	$\diamond \sim P$	/01	4, $\diamond I(b)$
(8)	$\sim P$	/012	7, $\diamond O$
(9)	P	/012	2, $\Box O(4)$

#54 (KT5)

(1)	SHOW: $\Box P \rightarrow \Box \Box P$	/0	CD
(2)	$\Box P$	/0	As
(3)	SHOW: $\Box \Box P$	/0	ND
(4)	SHOW: $\Box P$	/01	ND
(5)	SHOW: P	/012	DD
(6)	$\diamond \Box P$	/0	2, $\diamond I(t)$
(7)	$\Box P$	/03	6, $\diamond O$
(8)	$\diamond \Box P$	/01	7, $\diamond I(5)$
(9)	$\Box P$	/014	8, $\diamond O$
(10)	P	/012	9, $\Box O(5)$

#56 (KT5)

(1)	SHOW: $P \rightarrow \Box \diamond P$	/0	CD
(2)	P	/0	As
(3)	SHOW: $\Box \diamond P$	/0	ND
(4)	SHOW: $\diamond P$	/01	ID
(5)	$\sim \diamond P$	/01	As
(6)	SHOW: \ast	/*	8,9,SL
(7)	$\diamond P$	/0	2, $\diamond I(t)$
(8)	P	/02	7, $\diamond O$
(9)	$\sim P$	/02	5, $\sim \diamond O(5)$

#57 (KB4)

(1)	SHOW: $\diamond \Box P \rightarrow \Box P$	/0	CD
(2)	$\diamond \Box P$	/0	As
(3)	SHOW: $\Box P$	/0	DD
(4)	$\Box P$	/02	2, $\diamond O$
(5)	SHOW: $\Box \Box P$	/02	ND
(6)	SHOW: $\Box P$	/023	ND
(7)	SHOW: P	/0234	DD
(8)	P	/0234	4, $\Box O(4)$
(9)	$\Box P$	/0	5, $\Box O(b)$

#59 (K5)

(1)	SHOW: $\diamond \Box P \rightarrow \Box \diamond P$	/0	CD
(2)	$\diamond \Box P$	/0	As
(3)	SHOW: $\Box \diamond P$	/0	ND
(4)	SHOW: $\diamond P$	/01	ID
(5)	$\sim \diamond P$	/01	As
(6)	SHOW: \ast	/*	10,11,SL
(7)	$\Box P$	/02	2, $\diamond O$
(8)	$\diamond \Box P$	/01	7, $\diamond I(5)$
(9)	$\Box P$	/013	8, $\diamond O$
(10)	$\sim P$	/013	5, $\sim \diamond O$
(11)	P	/013	9, $\Box O(5)$

2. World Theory Exercises**#53 (KDB4)**

(1)	SHOW: $\Box P \rightarrow P$	CD
(2)	$\Box P$	As
(3)	SHOW: P	wt(0)
(4)	SHOW: $P / 0$	DD
(5)	$\Box P / 0$	2, wt(0)
(6)	$\forall i \{ 0 < i \rightarrow [P/i] \}$	5, wt(\Box)
(7)	$\forall i \exists j [i < j]$	wt(d) [=restrict(d)]
(8)	$0 < 1$	7, $\forall \exists O$
(9)	$\forall i \forall j \{ i < j \rightarrow j < i \}$	wt(b) [=restrict(b)]
(10)	$1 < 0$	8,9,QL
(11)	$\forall ijk \{ i < j \ \& \ j < k \rightarrow i < k \}$	wt(4) [=restrict(4)]
(12)	$0 < 0$	8,10,11,QL
(13)	$P / 0$	6,12,QL

#54 (KT5)

(1)	SHOW: $\Box P \rightarrow \Box \Box P$	CD
(2)	$\Box P$	As
(3)	SHOW: $\Box \Box P$	wt(0)
(4)	SHOW: $\Box \Box P / 0$	wt(\Box)
(5)	SHOW: $\forall i\{ 0 < i \rightarrow [\Box P / i] \}$	UCD
(6)	$0 < 1$	As
(7)	SHOW: $\Box P / 1$	wt(\Box)
(8)	SHOW: $\forall i\{ 1 < i \rightarrow [P / i] \}$	UCD
(9)	$1 < 2$	As
(10)	SHOW: $P / 2$	DD
(11)	$\forall i[i < i]$	wt(t)
(12)	$0 < 0$	11,QL
(13)	$\forall ijk\{ i < j \ \& \ i < k \rightarrow j < k \}$	wt(5)
(14)	$1 < 0$	6,12,13,QL
(15)	$0 < 2$	9,14,13,QL
(16)	$\Box P / 0$	2,wt(0)
(17)	$\forall i\{ 0 < i \rightarrow [P / i] \}$	16,wt(\Box)
(18)	$P / 2$	15,17,QL

3. Counter-Model Exercises

In what follows, it is left to the reader to fill in some of these truth tables. Note carefully: if $(i < j)$ is not listed explicitly, then presume $\text{not}(i < j)$.

<p>A T-model that falsifies: $\Diamond \Box \Diamond \Box \rightarrow \Diamond \Box P$</p> <p>$0 < 0, 01$ $01 < 01, 012$ $012 < 012$</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td></td> <td>\Diamond</td> <td>\Box</td> <td>\Diamond</td> <td>\Box</td> <td>P</td> <td>\rightarrow</td> <td>\Diamond</td> <td>\Box</td> <td>P</td> </tr> <tr> <td>0</td> <td>T</td> <td>F</td> <td>F</td> <td>F</td> <td>F</td> <td>F</td> <td>F</td> <td>F</td> <td>F</td> </tr> <tr> <td>01</td> <td></td> <td>T</td> <td>T</td> <td>F</td> <td>F</td> <td></td> <td></td> <td></td> <td></td> </tr> <tr> <td>012</td> <td></td> <td></td> <td></td> <td>T</td> <td>T</td> <td></td> <td></td> <td></td> <td></td> </tr> </table>		\Diamond	\Box	\Diamond	\Box	P	\rightarrow	\Diamond	\Box	P	0	T	F	F	F	F	F	F	F	F	01		T	T	F	F					012				T	T					<p>A D-model that falsifies: $\Diamond \Box (P \rightarrow Q) \rightarrow (\Box P \rightarrow \Diamond Q)$</p> <p>$0 < 01$ $01 < 012$ $012 < 012$</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td></td> <td>\Diamond</td> <td>\Box</td> <td>$(P \rightarrow Q)$</td> <td>\rightarrow</td> <td>$(\Box P \rightarrow \Diamond Q)$</td> </tr> <tr> <td>0</td> <td>T</td> <td>F</td> <td></td> <td></td> <td>F</td> </tr> <tr> <td>01</td> <td></td> <td>T</td> <td>T</td> <td>F</td> <td>F</td> </tr> <tr> <td>012</td> <td></td> <td></td> <td>T</td> <td></td> <td></td> </tr> </table>		\Diamond	\Box	$(P \rightarrow Q)$	\rightarrow	$(\Box P \rightarrow \Diamond Q)$	0	T	F			F	01		T	T	F	F	012			T		
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<p>A K-model that falsifies: $\sim \Diamond P \rightarrow \Diamond \sim P$</p> <p>$0 < \text{nothing}$</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td></td> <td>\sim</td> <td>\Diamond</td> <td>P</td> <td>\rightarrow</td> <td>\Diamond</td> <td>\sim</td> <td>P</td> </tr> <tr> <td>0</td> <td>T</td> <td>F</td> <td></td> <td>F</td> <td>F</td> <td></td> <td></td> </tr> </table>		\sim	\Diamond	P	\rightarrow	\Diamond	\sim	P	0	T	F		F	F			<p>A B-model that falsifies: $(\Diamond P \ \& \ \Box Q) \rightarrow \Diamond (P \ \& \ \Box Q)$</p> <p>$0 < 0, 01$ $01 < 01, 012, 0$ $012 < 012, 01$</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td></td> <td>$(\Diamond P \ \& \ \Box Q)$</td> <td>$\rightarrow$</td> <td>$\Diamond (P \ \& \ \Box Q)$</td> </tr> <tr> <td>0</td> <td>T</td> <td>F</td> <td>F</td> </tr> <tr> <td>01</td> <td></td> <td>T</td> <td></td> </tr> <tr> <td>012</td> <td></td> <td></td> <td>F</td> </tr> </table>		$(\Diamond P \ \& \ \Box Q)$	\rightarrow	$\Diamond (P \ \& \ \Box Q)$	0	T	F	F	01		T		012			F																																
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