

# 3

## Absolute Modal Logic

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## A. Leibnizian World Theory

### 1. Introduction

One of the most fertile ideas in modal logic and metaphysics traces to Leibniz, who proposed the following famous account of necessity.<sup>1</sup>

to be necessary is to be true in/at every possible world

Now, presumably, the sort of things to which the adjectives ‘necessary’ and ‘true’ apply are *propositions*. This leads us to the following more specific formulation.

a *proposition* is necessary if and only if *it* is true in/at every possible world

If we formalize the latter assertion within a standard elementary first-order framework, we obtain the following.

$$(L) \quad \forall x\{Px \rightarrow [Nx \leftrightarrow \forall y(Wy \rightarrow Txy)]\}$$

The associated translation scheme is given as follows:

|       |   |                         |
|-------|---|-------------------------|
| $Px$  | : | $x$ is a proposition    |
| $Nx$  | : | $x$ is necessary        |
| $Wy$  | : | $y$ is a possible world |
| $Txy$ | : | $x$ is true at/in $y$   |

Notice carefully that this approach treats necessity as a *property* of propositions; it is accordingly syntactically represented by a *predicate*. This is what Quine refers to as the "first grade of modal involvement".<sup>2</sup>

### 2. Direct and Indirect Quotation

Although the formula (L) provides a logically perspicuous translation of Leibniz’s principle, it is difficult to use in actual practice. The reason is that we almost never talk about propositions in this completely general and abstract manner. Rather, in ordinary language, proposition-talk arises in connection with *indirect quotation*, which must be carefully distinguished from *direct quotation*.

Consider reporting a debate between two politicians, Smith and Jones. We could report their exchange using direct quotation as follows.

Smith said “Jones, you are an idiot!”  
 Jones said “Smith, you are an idiot!”

In this case, we report the exact words used in the exchange. We might, however, wish to report their claims rather than their exact words. In that case, we would use indirect quotation – for example, as follows.

<sup>1</sup> Leibniz, *Discourse on Metaphysics*, 1686.

<sup>2</sup> Quine, “Three Grades of Modal Involvement”, in *The Ways of Paradox*, pp. 156-174.

Smith said *that* Jones is an idiot.  
 Jones said *that* Smith is an idiot.

A simple logico/grammatical account of these two quotation techniques goes as follows (where we concentrate on what Jones says).

|             |                   |                           |
|-------------|-------------------|---------------------------|
| Jones       | said              | "Smith, you are an idiot" |
| subject     | transitive verb   | direct object             |
| noun phrase | 2-place predicate | noun phrase               |

|             |                   |                        |
|-------------|-------------------|------------------------|
| Jones       | said              | that Smith is an idiot |
| subject     | transitive verb   | direct object          |
| noun phrase | 2-place predicate | noun phrase            |

A partial symbolization of these goes as follows.

$S[J, \text{"Smith, you are an idiot"}]$   
 $S[J, \text{that Smith is an idiot}]$

At this point, it is convenient to symbolize the two quotation-techniques, which we do as follows.

$\lceil S \rceil$  is read: quote- $S$ -unquote or simply: quote- $S$   
 $\langle S \rangle$  is read: that- $S$

Note that these expressions are *subnjectives* [category:  $S \rightarrow N$ ]; each one takes a sentence and delivers a noun phrase.<sup>3</sup> In any event, they enable us to further symbolize the above statements as follows.

$S[J, \lceil \text{Smith, you are an idiot} \rceil]$   
 $S[J, \langle \text{Smith is an idiot} \rangle]$

### 3. Sentences and Propositions

The distinction between direct-quotation and indirect-quotation brings us to a fundamental distinction in philosophy – between *sentences* and *propositions*. Basically, whereas a direct-quote expression names the very sentence within those quotes, the corresponding indirect-quote expression names the proposition expressed by that sentence (what the sentence says).

The basic idea can be summarized by the following schema.

$\langle S \rangle$  is the proposition expressed by the sentence  $\lceil S \rceil$ ;  
 $\lceil S \rceil$  says  $\langle S \rangle$

The following is a simple example.

that\_snow\_is\_white is the proposition expressed by the sentence 'snow is white'  
 'snow is white' says that\_snow\_is\_white

<sup>3</sup> Note, however, that direct-quotation is not categorially well-behaved, since it is completely opaque.

## 4. Reformulating the Basic Leibnizian Principle

Let us now go back and look at the Leibnizian principle. This time, however, rather than quantifying *directly* over propositions using first-order (pro-nominal) variables, let us quantify over them *indirectly* using schematic sentential variables, as follows.<sup>4</sup>

(L)    that( $\mathcal{S}$ ) is necessary  $\leftrightarrow$  that( $\mathcal{S}$ ) is true at every possible world

Here, the gaudy letter ' $\mathcal{S}$ ' is a *schematic variable* ranging over sentences, which means that this schema stands for infinitely-many formulas, one for each substitution for ' $\mathcal{S}$ '. The following are examples of substitutions.

that(snow is white) is necessary  $\leftrightarrow$  that(snow is white) is true at every world  
 that( $2+2=4$ ) is necessary  $\leftrightarrow$  that( $2+2=4$ ) is true at every world

Notice that the insertion of 'that' is grammatically crucial; if we drop 'that' we obtain, for example,

✗     $2+2=4$  is necessary

which is ungrammatical! For ' $\dots$  is necessary' is a predicate and accordingly requires a noun phrase as input. The expression ' $2+2=4$ ' is not a noun phrase, but a sentence. Fortunately, English provides numerous ways of "nominalizing" sentences for proper insertion into noun-phrase position, the "standard" one being the prefix 'that...'.<sup>5</sup>

## 5. The Necessity Connective

So far we have treated necessity as a predicate – what Quine calls the "first grade of modal involvement". We have nevertheless laid the foundation for ascending to the second-grade. In particular, the inclusion of indirect-quotation allows us to perform some neat grammatical tricks. For example, we can take the grammatically complex expression

that( $\mathcal{S}$ ) is necessary,

and remove the schematic variable to produce the "matrix"

that( $\dots$ ) is necessary,

and treat *this* complex expression as a functor in its own right. The resulting functor is a one-place connective (sentential operator; category:  $\mathcal{S} \rightarrow \mathcal{S}$ ); it takes a single sentence and delivers a sentence.

There is nevertheless a certain awkwardness to this expression, so it is customary to perform a further grammatical transformation [extraposition plus 'it' insertion], which yields the following.

it is necessary that ...

Now, the functor we have "discovered", or "manufactured", is sufficiently interesting that we give it a special symbol, the customary symbol being the "box". The following is the official reading.

<sup>4</sup> We could also opt for second-order quantification using pro-sentential variables.

<sup>5</sup> The other methods of sentential-nominalization include using gerunds and infinitives.

( $\Box$ )  $\Box S$  is read: that( $S$ ) is necessary  
 or: it is necessary that  $S$

For example:

$\Box[2+2=4]$  is read: that( $2+2=4$ ) is necessary  
 or: it is necessary that  $2+2=4$

Now, with our new symbol ' $\Box$ ', we can go back and rewrite (L) as follows.

(L)  $\Box S \leftrightarrow$  that( $S$ ) is true at every possible world  
 $\Box S \leftrightarrow \forall x\{\text{world}[x] \rightarrow \text{true}[\text{that}(S), x]\}$

## 6. 'is true at'

Next, let us deal with the predicate ' $\dots$ is true at...'. As it turns out, we can profitably separate this expression into two modules – an 'is true' module, and an 'at' module.

The 'is true' module is very simple, being analyzed using the following fundamental and obvious indirect-quotation principle.

that( $S$ ) is true  $\leftrightarrow S$

This appears even *more* obvious if we apply it-intrusion to obtain the following.

it is true that  $S \leftrightarrow S$

for example:

it is true that snow is white  $\leftrightarrow$  snow is white

Now, simply taking this principle and super-imposing the 'at' module yields the following derivative principle.<sup>6</sup>

$$\left. \begin{array}{l} \text{that}(S) \text{ is true at } i \\ \text{it is true that } S \text{ at } i \\ \text{it is true at } i \text{ that } S \end{array} \right\} \leftrightarrow S \text{ at } i$$

Substituting this back into our earlier expression, we can eliminate the predicate ' $\dots$ is true at...'  
 as follows.

$\Box S \leftrightarrow \forall i\{ \text{world}[i] \rightarrow S \text{ at } i \}$

<sup>6</sup> Note that we employ the variable ' $i$ ', which is intended to be short for 'index', which is further explained in the next section.

## 7. 'at'

What remains is to analyze the word 'at'. The most natural analysis treats 'at' as a special kind of adverbial modifier, whose exact grammatical category is  $[(S \times J) \rightarrow S]$ .  $J$  is a special sub-category of  $N$  called "indices". Indices include spatial indices (e.g., 'at the office') and temporal indices (e.g., 'at two o'clock'), but also include "possible worlds".

Given its logical importance, we also give the 'at' functor a special symbol, summarized by the following translation scheme.

$$(at) \quad [S / i] \quad \text{is read:} \quad S \text{ at } i$$

Notice carefully that the brackets are *officially* required. Recall that parentheses are required for any two-place functor written in infix notation; for example, recall that the official formulas for conjunction, disjunction, etc., involve outer parentheses. Of course, just as with two-place connectives, we can drop outer parentheses when they are unnecessary for parsing.

Given our additional syntactic resource, we are now in a position to rewrite the original Leibniz formula as follows.

$$(L) \quad \Box S \leftrightarrow \forall i \{ \text{world}[i] \rightarrow [S/i] \}$$

If it is understood that ' $i$ ' ranges over possible worlds, then we can further simplify this as follows.

$$(L) \quad \Box S \leftrightarrow \forall i [S/i]$$

As usual, ' $S$ ' is a schematic variable, which may be replaced by any sentence; the following are instances.

$$\begin{aligned} \Box P &\leftrightarrow \forall i [P/i] \\ \Box Fa &\leftrightarrow \forall i [Fa/i] \\ \Box Rab &\leftrightarrow \forall i [Rab/i] \\ \Box (Fa \ \& \ Gb) &\leftrightarrow \forall i [(Fa \ \& \ Gb)/i] \\ \Box \forall x Fx &\leftrightarrow \forall i [\forall x Fx/i] \end{aligned}$$

## 8. Iterated (Nested) Modalities

One of the chief differences between first-grade and second-grade modal involvement is that the second-grade permits modal operators to be *iterated* or *nested*. In particular, the following is our official syntactic rule.

$$\text{if } S \text{ is a formula, then so is } \Box S$$

This means that the following are all formulas,

$$\begin{aligned} \Box P \\ \Box \Box P \\ \Box \Box \Box P \\ \text{etc.} \end{aligned}$$

which may be read, respectively:

it is necessary that P  
 it is necessary that it is necessary that P  
 it is necessary that it is necessary that it is necessary that P  
 etc.

For example, consider the proposition  $\langle 2+2=4 \rangle$  (i.e., that  $2+2=4$ ). Suppose that in fact it is necessary that  $2+2=4$ ; i.e.,  $\langle 2+2=4 \rangle$  is necessary. What about *this* fact – the fact that  $\langle 2+2=4 \rangle$  is necessary; is this fact also necessary? Suppose it is necessary; then what about *this* fact – the fact that  $\langle \langle 2+2=4 \rangle$  is necessary  $\rangle$  is necessary; is *this* fact also necessary?

If we accept the Leibnizian scheme, as formulated so far, then answers to these questions can be obtained. First, recall the basic Leibniz-Schema.

$$(L) \quad \Box S \leftrightarrow \forall i[S/i]$$

Substituting ‘P’ and ‘ $\Box P$ ’ respectively for ‘S’ yields the following.

$$(1) \quad \Box P \leftrightarrow \forall i[P/i]$$

$$(2) \quad \Box \Box P \leftrightarrow \forall i[\Box P/i]$$

Supposing substitution works for instances of (L), based on (1), we can substitute ‘ $\forall i[P/i]$ ’ for ‘ $\Box P$ ’ in (2) to obtain:

$$\Box \Box P \leftrightarrow \forall i[\forall i[P/i] / i]$$

## 9. Iterated Indexing

The right constituent of the above biconditional is difficult to parse, so let us approach it in steps. First, notice that the rules of formation are officially formulated so that

if  $S$  is a formula, and  $i$  is an index, then  $[S/i]$  is a formula.

Accordingly, the following are officially all formulas (except for the missing outermost brackets).

|                     |                              |
|---------------------|------------------------------|
| $R / i$             | $R$ at $i$                   |
| $[R / i] / j$       | $R$ at $i$ , at $j$          |
| $[[R / i] / j] / k$ | $R$ at $i$ , at $j$ , at $k$ |
| etc.                |                              |

But what do they mean? Let us do a spatial example; let ‘R’ be ‘it is raining’, and let the indices be ‘Boston’, ‘New York’, ‘Chicago’, respectively, and let us read the indexical-copula ‘/’ as ‘in’ rather than ‘at’. Then the above formulas read:

it is raining **in** Boston  
 it is raining **in** Boston, **in** New York  
 it is raining **in** Boston, **in** New York, **in** Chicago

The latter two sentences seem goofy, if not completely absurd. So what do we do with them. There are at least two approaches to dealing with goofy formulas.

- First Approach: we **exclude** goofy formulas as syntactically ill-formed, by rewriting our official syntactic rules.
- Second Approach: we **include** goofy formulas as syntactically well-formed, but render them semantically harmless.

## 10. An Analogy

An analogy might be useful here. Recall that, according to the standard syntax of first-order logic, we have the following.

if  $\Phi$  is a formula, and  $v$  is a variable, then  $\forall v\Phi$  is a formula.

This means that the following expressions are all officially admitted as syntactically well-formed.

$$\begin{aligned} &\forall zFx \\ &\forall y\forall zFx \\ &\forall x\forall y\forall zFx \end{aligned}$$

Now, these are clearly goofy, but they are rendered semantically harmless, because the following are theorems of first-order logic.

$$\begin{aligned} \forall zFx &\leftrightarrow Fx \\ \forall y\forall zFx &\leftrightarrow Fx \\ \forall x\forall y\forall zFx &\leftrightarrow \forall xFx \end{aligned}$$

Thus, standard first-order logic follows the Second Approach.<sup>7</sup> In other words, it admits goofy formulas as syntactically well-formed, but it renders them semantically harmless by making each goofy formula equivalent to an ungoofy formula.

## 11. Back to Iterated Indices

In dealing with the goofy indexical formulas, we propose likewise to follow the Second Approach. Goofy formulas are syntactically admitted, but they are rendered harmless in the associated semantics. To motivate our procedure, we begin by making the following observation.

suppose it is raining **in** Boston;  
is *this* also true **in** New York?

The answer depends upon what ‘*this*’ refers to. If ‘*this*’ alludes to ‘it is raining’, then the answer is ‘not necessarily’; by ‘not necessarily’, we mean that the following argument is *not* valid.

✗ it is raining **in** Boston  
therefore, it is raining **in** New York

So let us make our question a bit more exact, as follows.

<sup>7</sup> If we wish to pursue the First Approach to standard first-order logic, we can redefine well-formed formulas using the following clause.

(a) If  $\Phi$  is a formula, and  $v$  is a variable that occurs free in  $\Phi$ , then  $\forall v\Phi$  is also a formula.

Notice that this does not produce the goofy formulas listed above.

suppose it is raining **in** Boston;  
 is *this* (i.e., **that** it is raining **in** Boston) also true **in** New York?  
 in other words,  
 is it also true **in** New York that it is raining **in** Boston?

Although the question is odd, the answer is “of course!” This suggests that we take the following to be a valid argument of the logic of ‘in’.

it is raining **in** Boston  
 therefore, it is true **in** New York that it is raining **in** Boston

Similarly, we take the converse to be valid, so the following logical principle holds for the logic of ‘at/in’.

$$[S / i] \leftrightarrow [[S / i] / j]$$

Thus, although multiple-indexing is goofy, it is redundant.

## 12. Quantification over Indices

Next, we consider quantification over indices. Let us continue our meteorological analogy. Suppose I say that it is raining somewhere, which may be written thus.

$\exists i[R / i]$                       it is raining **somewhere**

Does it follow that it is true **in** Boston that it is raining **somewhere**?<sup>8</sup> Supposing truth-in-Boston is not completely bizarre, we must answer “of course!” We also take the converse argument to be valid, and we take the corresponding arguments involving ‘ $\forall$ ’ to be valid. In other words, we take the following to be valid biconditional forms.

$$\begin{aligned} \exists i[S / i] &\leftrightarrow [\exists i[S / i] / j] \\ \forall i[S / i] &\leftrightarrow [\forall i[S / i] / j] \end{aligned}$$

## 13. Back to Iterated Modalities

What we can conclude is that iterating the box-operator is *redundant* in the Leibnizian System. In particular, the following is a theorem of the Leibnizian System.

$$\Box\Box P \leftrightarrow \Box P$$

We prove this in a later section.

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<sup>8</sup> Note carefully how we have to parse this sentence; ‘it is true in Boston that it is raining somewhere’ is not the same as ‘it is raining somewhere in Boston’.

## 14. The Axioms of World Theory

Having presented the underlying linguistic framework of Leibnizian-World-Theory [code-named WT(L)], we now turn to its *theses* (theorems, claims). These are presented axiomatically. First, the following are axiom schemata. Notice that, except for wt(0), each axiom refers to a particular functor. The index 0 is understood as the *original index*, which can be "here", or "now", or the "actual world".

|                           |   |                   |   |
|---------------------------|---|-------------------|---|
| wt(0)                     | $\mathcal{A}$                                     | $\leftrightarrow$ | $\mathcal{A} / 0$                                       |
| wt( $\sim$ ) <sup>9</sup> | $[\sim \mathcal{A} / i]$                          | $\leftrightarrow$ | $\sim[\mathcal{A}/i]$                                   |
| wt( $\rightarrow$ )       | $[(\mathcal{A} \rightarrow \mathcal{B}) / i]$     | $\leftrightarrow$ | $\{ [\mathcal{A}/i] \rightarrow [\mathcal{B}/i] \}$     |
| wt( $\&$ )                | $[(\mathcal{A} \& \mathcal{B}) / i]$              | $\leftrightarrow$ | $\{ [\mathcal{A}/i] \& [\mathcal{B}/i] \}$              |
| wt( $\vee$ )              | $[(\mathcal{A} \vee \mathcal{B}) / i]$            | $\leftrightarrow$ | $\{ [\mathcal{A}/i] \vee [\mathcal{B}/i] \}$            |
| wt( $\leftrightarrow$ )   | $[(\mathcal{A} \leftrightarrow \mathcal{B}) / i]$ | $\leftrightarrow$ | $\{ [\mathcal{A}/i] \leftrightarrow [\mathcal{B}/i] \}$ |
| wt( $\square$ )           | $[\square \mathcal{A} / i]$                       | $\leftrightarrow$ | $\forall j[\mathcal{A}/j]$                              |
| wt( $\diamond$ )          | $[\diamond \mathcal{A} / i]$                      | $\leftrightarrow$ | $\exists j[\mathcal{A}/j]$                              |
| wt( $\forall$ )           | $[\forall v \mathcal{A} / i]$                     | $\leftrightarrow$ | $\forall v[\mathcal{A}/i]$                              |
| wt( $\exists$ )           | $[\exists v \mathcal{A} / i]$                     | $\leftrightarrow$ | $\exists v[\mathcal{A}/i]$                              |
| wt(/)                     | $[\mathcal{A}/i] / j$                             | $\leftrightarrow$ | $[\mathcal{A} / i]$                                     |

Here, ' $\mathcal{A}$ ', ' $\mathcal{B}$ ' are schematic variables that range over formulas, and ' $i$ ', ' $j$ ' are schematic variables that range over indices. Note that the large biconditional symbol is understood to be the main connective in each axiom; this allows us to drop a number of parentheses that would otherwise clutter up the formulas. Notice also that many axioms could be rewritten with explicit universal quantifiers; for example, wt( $\sim$ ) could just as easily be written as follows.

$$\text{wt}(\sim) \quad \forall i \{ [\sim \mathcal{A}/i] \leftrightarrow \sim[\mathcal{A}/i] \}$$

## 15. Second-Order Considerations

One is naturally tempted to quantify over the schematic sentence variables as well, producing expressions such as the following.

$$\times \quad \forall \mathcal{A} \forall \mathcal{B} \forall i \{ [(\mathcal{A} \& \mathcal{B})/i] \leftrightarrow \{ [\mathcal{A}/i] \& [\mathcal{B}/i] \} \}$$

Such formulas involve quantification into sentence position [pro-sentential quantification], which is strictly forbidden in first-order logic, which only permits pro-nominal quantification. Later, when we discuss second-order logic, this particular constraint will be lifted. However, for the moment, formulas like this one are deemed illegitimate.

<sup>9</sup> Given the official rules of formation, the expression ' $\sim P/i$ ' can only be parsed so that ' $\sim$ ' is narrow, and ' $/i$ ' is wide. If it were the other way around, then brackets would be required around ' $P/i$ ', thus producing ' $\sim[P/i]$ '.

## 16. Theorems of Leibnizian World Theory

In addition to its primitive theses (axioms), every theory has additional theses (theorems), which are the formulas that can be deduced from the axioms. The following are a few examples.

- (t1)  $\Box P \rightarrow P$
- (t2)  $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
- (t3)  $\sim \Diamond P \leftrightarrow \Box \sim P$
- (t4)  $\sim \Box P \leftrightarrow \Diamond \sim P$
- (t5)  $\Box P \rightarrow \Box \Box P$
- (t6)  $\forall i \forall j \{ [\Box P / i] \rightarrow [\Box P / j] \}$
- (t7)  $\forall i \forall j \{ [\Diamond P / i] \rightarrow [\Diamond P / j] \}$

## 17. Examples of Proofs of Theorems

By way of illustration, we examine the proofs of a few theorems. The reader will find an account of the rules of derivation for WT(L) in the appendix on rules of derivation. In this context, it is understood that ‘*i*’, ‘*j*’, etc. are indexical-variables and can/must be replaced by indexical-constants. We use decimal Arabic numerals for this purpose, using ‘0’ in particular for the original/default index.

|  |  |
|--|--|
| <p><b>Example 1:</b></p> <p>(1) SHOW: <math>\Box P \rightarrow P</math> CD<br/> (2) <math>\Box P</math> As<br/> (3) SHOW: <math>P</math> DD<br/> (4) <math>\Box P / 0</math> 2,wt(0)<br/> (5) <math>\forall i[P / i]</math> 4,wt(<math>\Box</math>)<br/> (6) <math>P / 0</math> 5,<math>\forall O</math><br/> (7) <math>P</math> 6,wt(0)</p>   | <p><b>Example 3:</b></p> <p>(1) SHOW: <math>\Box P \rightarrow \Box \Box P</math> CD<br/> (2) <math>\Box P</math> As<br/> (3) SHOW: <math>\Box \Box P</math> wt(<math>\Box</math>)<br/> (4) SHOW: <math>\forall i[\Box P / i]</math> UD<br/> (5) SHOW: <math>\Box P / 1</math> wt(<math>\Box</math>)<br/> (6) SHOW: <math>\forall i[P / i]</math> UD<br/> (7) SHOW: <math>P / 2</math> DD<br/> (8) <math>\Box P / 0</math> 2,wt(0)<br/> (9) <math>\forall i[P / i]</math> 8,wt(<math>\Box</math>)<br/> (10) <math>P / 2</math> 9,<math>\forall O</math></p>  |
| <p><b>Example 2:</b></p> <p>(1) SHOW: <math>\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)</math> CD<br/> (2) <math>\Box(P \rightarrow Q)</math> As<br/> (3) SHOW: <math>\Box P \rightarrow \Box Q</math> CD<br/> (4) <math>\Box P</math> As<br/> (5) SHOW: <math>\Box Q</math> wt(0)<br/> (6) SHOW: <math>\Box Q / 0</math> wt(<math>\Box</math>)<br/> (7) SHOW: <math>\forall i[Q / i]</math> UD<br/> (8) SHOW: <math>Q / 1</math> DD<br/> (9) <math>\Box(P \rightarrow Q) / 0</math> 2,wt(0)<br/> (10) <math>\forall i[(P \rightarrow Q) / i]</math> 9,wt(<math>\Box</math>)<br/> (11) <math>\Box P / 0</math> 4,wt(0)<br/> (12) <math>\forall i[P / i]</math> 11,wt(<math>\Box</math>)<br/> (13) <math>(P \rightarrow Q) / 1</math> 10,<math>\forall O</math><br/> (14) <math>[P / 1] \rightarrow [Q / 1]</math> 13,wt(<math>\rightarrow</math>)<br/> (15) <math>P / 1</math> 12,<math>\forall O</math><br/> (16) <math>Q / 1</math> 14,15,<math>\rightarrow O</math></p> | <p><b>Example 5:</b></p> <p>(1) SHOW: <math>\Diamond \Box P \rightarrow \Box P</math> CD<br/> (2) <math>\Diamond \Box P</math> As<br/> (3) SHOW: <math>\Box P</math> wt(0)<br/> (4) SHOW: <math>\Box P / 0</math> wt(<math>\Box</math>)<br/> (5) SHOW: <math>\forall i[P / i]</math> UD<br/> (6) SHOW: <math>P / 1</math> DD<br/> (7) <math>\Diamond \Box P / 0</math> 2,wt(0)<br/> (8) <math>\exists i[\Box P / i]</math> 7,wt(<math>\Diamond</math>)<br/> (9) <math>\Box P / 2</math> 8,<math>\exists O</math><br/> (10) <math>\forall i[P / i]</math> 9, wt(<math>\Box</math>)<br/> (11) <math>P / 1</math> 10,<math>\forall O</math></p> |

## B. System L

### 1. The Transition to Pure Modal Logic

We now move from world-theory to pure modal logic – in particular, to System L, which is so called because it corresponds most closely to Leibnizian World Theory. In order to make this transition, we must adjust our syntax. In particular, the syntax of pure modal logic does not have indices. Still, indices seem helpful in understanding modal reasoning. For this reason, we adopt a logical system in which there are indices, which we read very much like in world theory ["formula at index"]. **But**, unlike in world theory, in pure modal logic, indices do not attach to formulas *per se*, but only to derivation lines, and they do not admit *explicit* quantification.

### 2. Indices and Indexing

System L – and indeed, every system of modal logic that we examine – is based on an indexing system. Specifically, in derivations, every line includes a *reference point*, or *index*, relative to which the formula is asserted. Reference points need not carry any *particular* semantic or metaphysical significance, although we naturally think of them as possible worlds.

Henceforth, in pure modal logic, all rules will contain reference to indices, although sometimes it will be implicit. For example, we can take every rule of SL and make it indexical, simply by attaching the same index to every line. The following are examples.

|  |   |
|--|---|
| $(\rightarrow O)$ $\frac{\mathcal{A} \rightarrow C \quad \mathcal{A}}{C}$ $i \quad i \quad i$  | $(\vee O)$ $\frac{\mathcal{A} \vee \mathcal{B} \quad \sim \mathcal{A}}{\mathcal{B}}$ $i \quad i \quad i$                              |
| $(CD)$ $\begin{array}{ l} \text{SHOW: } \mathcal{A} \rightarrow C \\ \mathcal{A} \\ \text{SHOW: } C \end{array}$ $i \quad i \quad i$ | $(ID)^{10}$ $\begin{array}{ l} \text{SHOW: } \sim \mathcal{A} \\ \mathcal{A} \\ \text{SHOW: } \times \end{array}$ $i \quad i \quad i$ |

Here we use ‘ $\mathcal{A}$ ’, ‘ $\mathcal{B}$ ’, etc. as metalinguistic variables ranging over formulas of the object language, we and use ‘ $i$ ’, ‘ $j$ ’, etc. as metalinguistic variables ranging over indices.

The syntactic nature of formulas is already clear; they are defined by the rules of formation. The syntactic nature of indices remains to be elaborated. We will consider two syntactic renditions of indices, a specialized one for Absolute Modal Logic (System L), and a more general one, later, for Relative Modal Logic. In Absolute Modal Logic, reference points are formally represented by decimal Arabic numerals. Later, in Relative Modal Logic, reference points are formally represented by finite sequences of numerals. In either case, among these reference points, there is a special reference point, the default, or original, reference point (the "actual world"), which is represented by the numeral ‘0’.

Sometimes the reference point is suppressed, according to two different conventions. According to one convention, if a formula is asserted without an explicit reference point, it generally understood to be asserted relative to the null reference point 0 (the actual world). According to the second convention, if a rule mentions no index point, it is understood that every line refers to the same reference point,

<sup>10</sup> See later for the convention about indexing the contradiction symbol.

whatever it may be. Thus, every rule of elementary logic can be understood as an indexed rule in which every index is the same. Accordingly, we may write Arrow-Out in either of the following ways.

|   |  |
|---|--|
| $  \begin{array}{lcl}  (\rightarrow O) & \mathcal{A} \rightarrow C & /i \\  & \mathcal{A} & /i \\  \hline  & C & /i  \end{array}  $ | $  \begin{array}{l}  \mathcal{A} \rightarrow C \\  \mathcal{A} \\  \hline  C  \end{array}  $ |
|---|--|

### 3. Indexing the Contradiction Symbol

Another situation in which indices may be dropped is in connection with the contradiction symbol ( $\ast$ ), which plays a key role in indirect derivations. Recall that  $\ast$  is governed by the following two rules.

|   |  |
|---|--|
| $  \begin{array}{l}  \ast I \\  \mathcal{A} \\  \sim \mathcal{A} \\  \hline  \ast  \end{array}  $ | $  \begin{array}{l}  \ast O \\  \ast \\  \hline  \mathcal{A}  \end{array}  $ |
|---|--|

In the context of indexed logic, these need to be further clarified, as follows.

|  |  |
|--|--|
| $  \begin{array}{lcl}  \ast I & & \\  \mathcal{A} & & /i \\  \sim \mathcal{A} & & /i \\  \hline  \ast & & /j  \end{array}  $ | $  \begin{array}{lcl}  \ast O & & \\  \ast & & /i \\  \hline  \mathcal{A} & & /j  \end{array}  $ |
|--|--|

Notice that the following is a special case of  $\ast O$ , which we call  $\ast R$  ( $\ast$ -repetition).

|   |
|---|
| $  \begin{array}{lcl}  \ast R & & \\  \ast & & /i \\  \hline  \ast & & /j  \end{array}  $ |
|---|

We describe this by saying that contradictions are *absolute*; generally, a formula is said to be *absolute* if it is true everywhere or nowhere.

Since  $\ast$  is absolute, it seems that we don't really need to index it at all. However, we may feel more comfortable indexing every line, in which case we opt to index every  $\ast$ -line by the "wildcard" character  $\ast$ .<sup>11</sup> See examples below.

<sup>11</sup> Familiar from computer operating systems like Unix and Dos.

## 4. Rules for $\Box$

In the present section, we examine the rules for the necessity operator.

### 1. Box-Elimination (Box-Out)

The easiest rule to apply is  $\Box O$ , which is given as follows.

|          |                      |      |
|----------|----------------------|------|
| $\Box O$ | $\Box \mathcal{A}$   | $/i$ |
|          | $\hline \mathcal{A}$ | $/j$ |

Here,  $i$  and  $j$  are arbitrary indices. In other words, if one has  $\Box \mathcal{A}$  at any index, then one can deduce  $\mathcal{A}$  at any index. In other words,  $\Box O$  reflects the Leibnizian idea that, if  $\Box \mathcal{A}$  is true (at  $i$ ) then  $\mathcal{A}$  is true everywhere.

### 1. Necessity Derivation

The reverse (introduction) rule is necessity derivation (ND or  $\Box D$ ).

|                          |                     |  |
|--------------------------|---------------------|--|
| $\Box D$ (ND)            |                     |  |
| SHOW: $\Box \mathcal{A}$ | $/i$                |  |
| SHOW: $\mathcal{A}$      | $/k$ ( <b>new</b> ) |  |
|                          |                     |  |

Here,  $i$  is any index, and  $k$  is any new index. The following is our *initial* official definition.

An index counts as **old** precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as **new**.

This is formally similar to the use of new constants in elementary quantifier logic. This is not surprising since we are trying to capture the idea that  $\Box \mathcal{A}$  means  $\mathcal{A}$  *everywhere*.

## 2. A Derivation Example

In order to illustrate the derivation technique, especially the index technique, let us do a complete example. Consider the following argument form.

$$\Box(P \rightarrow Q) ; \Box P / \Box Q$$

In order to prove it valid, we start by writing down the premises as well as the obvious show-line, all indexed by the default index 0.

|     |                         |    |    |
|-----|-------------------------|----|----|
| (1) | $\Box(P \rightarrow Q)$ | /0 | Pr |
| (2) | $\Box P$                | /0 | Pr |
| (3) | SHOW: $\Box Q$          | /0 |    |

At this point, we have to show a  $\Box$ -statement, so we use the strategic rule designed explicitly for such statements, which is  $\Box D$  (ND). That means that we write down an immediately subordinate line, as follows.

|     |                         |    |          |
|-----|-------------------------|----|----------|
| (1) | $\Box(P \rightarrow Q)$ | /0 | Pr       |
| (2) | $\Box P$                | /0 | Pr       |
| (3) | SHOW: $\Box Q$          | /0 | $\Box D$ |
| (4) | SHOW: Q                 | /1 |          |

Here, notice that we have indexed line (4) by a new index – 1 – in accordance with the requirements of  $\Box D$ . Of course, we could use any decimal Arabic numeral except 0, but wanting to be systematic, we choose 1.

Now, we have ‘SHOW: Q’, which is atomic, so we must use a generic strategy – either DD or ID. It happens that we can do it by DD, so that is how we proceed.

|     |                         |    |                      |
|-----|-------------------------|----|----------------------|
| (1) | $\Box(P \rightarrow Q)$ | /0 | Pr                   |
| (2) | $\Box P$                | /0 | Pr                   |
| (3) | SHOW: $\Box Q$          | /0 | $\Box D$             |
| (4) | SHOW: Q                 | /1 | DD                   |
| (5) | $P \rightarrow Q$       | /1 | 1, $\Box O$          |
| (6) | P                       | /1 | 2, $\Box O$          |
| (7) | Q                       | /1 | 5,6, $\rightarrow O$ |

Line (5) follows from line (1) by  $\Box O$ . Notice that we could attach any index to line (5), but ‘1’ [and only ‘1’!] is useful. The same thing applies to line (6), which follows from line (2) by  $\Box O$ . Finally, on line (7), we apply an SL inference rule to lines (5) and (6).

Notice that lines (5)-(7) all have the same index. The following, for example, is *not* a valid inference.

|                   |    |     |
|-------------------|----|-----|
| $P \rightarrow Q$ | /0 |     |
| P                 | /1 |     |
| —                 |    |     |
| Q                 | /1 | *** |

This is because the premises refer to different indices.

In order to complete our derivation, we proceed back up the page. First, we cancel line (4) in accordance with DD, which yields the following.

|     |                         |    |                      |
|-----|-------------------------|----|----------------------|
| (1) | $\Box(P \rightarrow Q)$ | /0 | Pr                   |
| (2) | $\Box P$                | /0 | Pr                   |
| (3) | SHOW: $\Box Q$          | /0 | $\Box D$             |
| (4) | SHOW: Q                 | /1 | DD                   |
| (5) | $P \rightarrow Q$       | /1 | 1, $\Box O$          |
| (6) | P                       | /1 | 2, $\Box O$          |
| (7) | Q                       | /1 | 5,6, $\rightarrow O$ |

Then we cancel line (3) in accordance with  $\Box D$ , which yields the following.

|     |                         |    |                      |
|-----|-------------------------|----|----------------------|
| (1) | $\Box(P \rightarrow Q)$ | /0 | Pr                   |
| (2) | $\Box P$                | /0 | Pr                   |
| (3) | SHOW: $\Box Q$          | /0 | $\Box D$             |
| (4) | SHOW: $Q$               | /1 | DD                   |
| (5) | $P \rightarrow Q$       | /1 | 1, $\Box O$          |
| (6) | $P$                     | /1 | 2, $\Box O$          |
| (7) | $Q$                     | /1 | 5,6, $\rightarrow O$ |

## 5. Rules for $\Diamond$

Next, we turn to the possibility operator, which has its own pair of rules.

### 1. Diamond-Introduction (Diamond-In)

First, we take the  $\Box O$  rule and turn it upside down to obtain its dual –  $\Diamond I$ .

|              |  |                  |
|--------------|--|------------------|
| $\Diamond I$ | $\frac{\mathcal{A}}{\Diamond \mathcal{A}}$ | $/i$<br><br>$/j$ |
|--------------|--|------------------|

Here,  $i$  and  $j$  are arbitrary indices. In other words, if one has  $\mathcal{A}$  at any index, then one can deduce  $\Diamond \mathcal{A}$  at any index.

### 2. Diamond-Elimination (Diamond-Out)

The Box-Introduction rule  $\Box D$  can also be turned upside down to produce the following.

|              |  |                                |
|--------------|--|--------------------------------|
| $\Diamond O$ | $\frac{\Diamond \mathcal{A}}{\mathcal{A}}$ | $/i$<br><br>$/j \text{ (new)}$ |
|--------------|--|--------------------------------|

Like  $\exists O$  in quantifier logic, this is an *assumption rule*, and not a true inference rule. It basically says that if one has  $\Diamond \mathcal{A}$  at  $i$ , then  $\mathcal{A}$  is true somewhere; so one is entitled to give that somewhere a name, and any name will do so long as it is new.

### 3. A Derivation Example

Let us do an example. Consider the following argument form, quite analogous to our earlier example.

|   |                         |    |                       |
|---|-------------------------|----|-----------------------|
| $\Box(P \rightarrow Q) ; \Diamond P / \Diamond Q$ |                         |    |                       |
| (1)   | $\Box(P \rightarrow Q)$ | /0 | Pr                    |
| (2)   | $\Diamond P$            | /0 | Pr                    |
| (3)   | SHOW: $\Diamond Q$      | /0 | DD                    |
| (4)   | $P$                     | /1 | $2, \Diamond O$       |
| (5)   | $P \rightarrow Q$       | /1 | $1, \Box O$           |
| (6)   | $Q$                     | /1 | $4, 5, \rightarrow O$ |
| (7)   | $\Diamond Q$            | /0 | $6, \Diamond I$       |

Notice the following points.

|     |                   |    |   |
|-----|-------------------|----|---|
| (4) | $P$               | /1 | 1 is new as required by $\Diamond O$                  |
| (5) | $P \rightarrow Q$ | /1 | 1 is old, but that is ok to use with $\Box O$         |
| (6) | $Q$               | /1 | follows by SL, since (4) and (5) are indexed the same |
| (7) | $\Diamond Q$      | /0 | follows from (6) by $\Diamond I$ .                    |

### 6. Official Modal Negation Rules

In the current section, we look at the rules that pertain to negations of modal formulas. First our official modal negation rule is given as follows.

|   |   |
|---|---|
| MN  |   |
| $\frac{\sim \Box \mathcal{A}}{\hline \hline \Diamond \sim \mathcal{A}}$ | $\frac{\sim \Diamond \mathcal{A}}{\hline \hline \Box \sim \mathcal{A}}$ |

These are both bi-directional rules. Notice that these rules do not have *explicit* indices, but only implicit indices. This follows a general notational convention – whenever all the lines in a rule pertain to the same index, the indices are dropped altogether in the statement of that rule. For example, in this connection, except for the  $\times$ -rules, every line in every SL rule pertains to the same index, so these rules do not require explicit indices in their statement.

The following is an example that uses one of these new rules.

|      |                         |    |                      |
|------|-------------------------|----|----------------------|
| (1)  | $\Box(P \rightarrow Q)$ | /0 | Pr                   |
| (2)  | $\sim \Diamond Q$       | /0 | Pr                   |
| (3)  | SHOW: $\sim \Diamond P$ | /0 | ID                   |
| (4)  | $\Diamond P$            | /0 | As                   |
| (5)  | SHOW: $\ast$            | /* | DD                   |
| (6)  | P                       | /1 | 4, $\Diamond O$      |
| (7)  | P $\rightarrow$ Q       | /1 | 1, $\Box O$          |
| (8)  | Q                       | /1 | 6,7, $\rightarrow O$ |
| (9)  | $\Box \sim Q$           | /0 | 2, MN                |
| (10) | $\sim Q$                | /1 | 9, $\Box O$          |
| (11) | $\ast$                  | /* | 8,10, $\ast I$       |

### 7. Short-Cut Modal Negation Rules

It is often convenient to skip steps that seem fairly obvious. For this reason we adopt short-cut rules. The first such rules concern negations of modal formulas. They are given as follows.

|  |  |
|--|--|
| $\sim \Box O$  | $\sim \Diamond O$  |
| $\frac{\sim \Box \mathcal{A} \quad /i}{\sim \mathcal{A} \quad /k \text{ (new)}}$ | $\frac{\sim \Diamond \mathcal{A} \quad /i}{\sim \mathcal{A} \quad /j}$ |
| $\sim \Box O = MN + \Diamond O;$<br>thus, $k$ must be new.                       | $\sim \Diamond O = MN + \Box O;$<br>thus $j$ can be any index.         |

### 8. Strict Conditional and Biconditional Rules

That leaves just the strict conditional and strict biconditional connectives. Both rules are bi-directional.

|   |   |
|---|---|
| Def $\prec$   | Def =   |
| $\frac{\mathcal{A} \prec \mathcal{B}}{\Box(\mathcal{A} \rightarrow \mathcal{B})}$ | $\frac{\mathcal{A} = \mathcal{B}}{\Box(\mathcal{A} \leftrightarrow \mathcal{B})}$ |

Notice that, like most SL rules, and like the official modal negation rules, these rules do not explicitly mention indices. It is understood that the index must be the same for both input and output.

### 9. Short-Cut Rules

There are a large number of short-cut rules that we frequently use. For a list, please consult the Appendix "Rules of Derivation".

## C. Counter-Models in System L

### 1. Introduction

In Part B, we have examined a derivation system for System L. With this method, if an argument form is valid in System L, then we can construct a derivation of its conclusion from its premises; similarly, if an argument form is invalid, then we cannot construct such a derivation. This means that, thus far, we have a method of demonstrating validity, but we do not have a method of demonstrating invalidity. Note carefully that the mere failure on our part to construct a derivation does not mean that no such derivation can be constructed.

We accordingly need a companion technique for demonstrating the invalidity of arguments in System L.

### 2. Invalidity in Truth-Value Semantics

Before considering the specific case of System L, we examine some general ideas about invalidity, which are presented by way of formal definitions.

- (d1) Let  $\mathcal{L}$  be a formal language. Then a **valuation** on  $\mathcal{L}$  is, by definition, a function that assigns a truth-value to every formula of  $\mathcal{L}$ .
- (d2) Let  $\mathcal{L}$  be a formal language. Then a **truth-value semantics** for  $\mathcal{L}$  is, by definition, a set  $\mathbb{V}$  of valuations on  $\mathcal{L}$ . The set  $\mathbb{V}$  is the set of *admissible valuations* for that semantics.
- (d3) Let  $\mathcal{L}$  and  $\mathbb{V}$  be as before. Let  $\langle P_1, \dots, P_m / C \rangle$  be an argument in  $\mathcal{L}$ . Then  $\langle P_1, \dots, P_m / C \rangle$  is **valid** relative to  $\mathbb{V}$  iff:

there is no  $v$  in  $\mathbb{V}$  such that  $v(P_1)=v(P_2)=\dots=v(P_m)=T$ , and  $v(C)=F$ .

- (d4) Let  $\mathcal{L}$ ,  $\mathbb{V}$ , and  $\langle P_1, \dots, P_m / C \rangle$  be as before. Let  $v$  be a valuation in  $\mathbb{V}$ . Then  $v$  is said to be a **counter-model** to  $\langle P_1, \dots, P_m / C \rangle$  iff:

$v(P_1)=T, \dots, v(P_m)=T$ , and  $v(C)=F$ .

Given these definitions, we have the following principle, which describes the *method* of counter-models:

In order to demonstrate that an argument  $\mathbb{A}$  is invalid,  
it is sufficient to produce/exhibit a counter-model to  $\mathbb{A}$

### 3. Valuations in Ordinary SL

Let us now consider how ordinary SL arguments are shown to be invalid. Recall that in ordinary SL, one can show that an argument form is invalid by using truth tables. In this technique, one constructs a truth table, and one exhibits a case (line) in which the premises are all true but the conclusion is false.

Behind this pencil-and-paper techniques lies a formal mathematical definition, given as follows.

(d5) Let  $\mathcal{L}$  be the customary language of ordinary SL. Let  $\nu$  be a valuation on  $\mathcal{L}$ . Then  $\nu$  is **admissible** (for the usual truth-functional semantics) if and only if  $\nu$  satisfies the following conditions.

- (1)  $\nu(\sim\phi) = T$  iff  $\nu(\phi) = F$
- (2)  $\nu(\phi\&\psi) = T$  iff  $\nu(\phi) = T$  and  $\nu(\psi) = T$
- (3)  $\nu(\phi\vee\psi) = T$  iff  $\nu(\phi) = T$  and/or  $\nu(\psi) = T$
- (4)  $\nu(\phi\rightarrow\psi) = T$  iff  $\nu(\phi) \leq \nu(\psi)$ <sup>12</sup>
- (5)  $\nu(\phi\leftrightarrow\psi) = T$  iff  $\nu(\phi) = \nu(\psi)$

How, does one "produce" a counter-model to argument  $\mathbb{A}$ ? By exhibiting an assignment of truth-values to the atomic formulas in  $\mathbb{A}$ , which when extended to all formulas in  $\mathbb{A}$  makes all the premises of  $\mathbb{A}$  true, but makes the conclusion of  $\mathbb{A}$  false. One way to do this, but not the only way, is to construct a truth-table for  $\mathbb{A}$ .

It is left as an exercise for the reader to see how the abstract definition of validity relates to the more mundane matter of doing truth tables. The difference between the abstract truth-functional semantics and doing truth-tables is similar to the difference between mathematical division, which is an abstract 3-place relation, and the manner in which humans *do* division. This in turn is similar to the difference between cakes and *how we make* cakes, or the difference between places and *how we find* places.

#### 4. Valuations in Indexed Sentential Logic

Having given a formal account of the semantics of ordinary SL, we now consider Indexed Sentential Logic. As mentioned earlier, whereas in ordinary SL, a sentence is true or false *simpliciter*, in indexed SL, a sentence is true or false *at* a reference point (or index). Just as we did with ordinary SL, we can formally describe the semantics of Indexed SL, which we do as follows.

(d6) Let  $\mathcal{L}$  and  $S$  be as before. Let  $I$  be a non-empty set (of indices). Let  $\nu$  be a function from  $S \times I$  into  $\{T, F\}$ . Then  $\nu$  is said to be an **admissible  $I$ -valuation** on  $\mathcal{L}$  iff it satisfies the following conditions for every element  $i$  of  $I$  [where we write ' $\nu(\phi/i)$ ' for ' $\nu(\phi, i)$ '].

- (1)  $\nu(\sim\phi / i) = T$  iff  $\nu(\phi/i) = F$
- (2)  $\nu(\phi\&\psi / i) = T$  iff  $\nu(\phi/i) = T$  and  $\nu(\psi/i) = T$
- (3)  $\nu(\phi\vee\psi / i) = T$  iff  $\nu(\phi/i) = T$  and/or  $\nu(\psi/i) = T$
- (4)  $\nu(\phi\rightarrow\psi / i) = T$  iff  $\nu(\phi/i) \leq \nu(\psi/i)$
- (5)  $\nu(\phi\leftrightarrow\psi / i) = T$  iff  $\nu(\phi/i) = \nu(\psi/i)$

(d7) Let  $\mathcal{L}$ ,  $S$ , and  $I$  be as before. Let  $\nu$  be a valuation on  $\mathcal{L}$ . Then  $\nu$  is said to be an **admissible valuation** on  $\mathcal{L}$  iff:

there is an admissible  $I$ -indexed valuation  $w$ , and index  $i$ , such that  
for any formula  $\phi$  in  $S$ ,  $\nu(\phi) = w(\phi/i)$ .

These conditions can be made somewhat more intuitive by rewriting them in accordance with the following definitions.

(d8)  $\phi$  is true at  $i$  according to  $\nu$   $\stackrel{\text{def}}{=} \nu(\phi/i) = T$

(d9)  $\phi$  is false at  $i$  according to  $\nu$   $\stackrel{\text{def}}{=} \nu(\phi/i) = F$

<sup>12</sup> Here  $\leq$  is the partial-order relation defined so that:

$F \leq F; F \leq T; T \leq T; T \not\leq F.$

If the valuation  $\nu$  is understood, then we drop the expression ‘according to  $\nu$ ’. This allows us to employ the following abbreviations.

$$\begin{aligned} T/i &=_{df} \text{ true at } i \text{ according to } \nu \\ F/i &=_{df} \text{ false at } i \text{ according to } \nu \end{aligned}$$

rewrite the semantic conditions as follows. Note also that we abbreviate, since we don’t refer to truth-values, we use ‘T’ instead of ‘True’, and we use ‘F’ instead of ‘False’

$$\begin{aligned} \sim \mathcal{A} \text{ is } T/i & \text{ iff } \mathcal{A} \text{ is } F/i \\ \mathcal{A} \&\mathcal{B} \text{ is } T/i & \text{ iff } \mathcal{A} \text{ is } T/i \text{ and } \mathcal{B} \text{ is } T/i \\ \mathcal{A} \vee \mathcal{B} \text{ is } T/i & \text{ iff } \mathcal{A} \text{ is } T/i \text{ and/or } \mathcal{B} \text{ is } T/i \\ \mathcal{A} \rightarrow \mathcal{B} \text{ is } T/i & \text{ iff } \mathcal{A} \text{ is } F/i \text{ and/or } \mathcal{B} \text{ is } T/i \\ \mathcal{A} \leftrightarrow \mathcal{B} \text{ is } T/i & \text{ iff } \text{ either } \mathcal{A} \text{ and } \mathcal{B} \text{ are } T/i, \text{ or } \mathcal{A} \text{ and } \mathcal{B} \text{ are } F/i \end{aligned}$$

Notice that these closely parallel the axioms of world theory that pertain to the truth-functional connectives. To show the parallel, one first notes that ‘ $\mathcal{A}$  is  $F/i$ ’ is equivalent to ‘it is not the case that  $\mathcal{A}$  is  $T/i$ ’.

## 5. Valuations in System L

Finally, we turn to System L. To obtain the semantics for System L, we begin with the semantics of ISL, and append to it clauses pertaining to the modal operators. This is done as follows.

- (d10) Let  $\mathcal{L}$  be the language of modal sentential logic; let  $S$  be the associated set of formulas of  $\mathcal{L}$ . Let  $I$  be a non-empty set of indices. Then an ***I*-indexed valuation** on  $\mathcal{L}$  is by definition a function from  $S \times I$  into  $\{T, F\}$  subject to the following conditions for every element  $i$  of  $I$ .

$$\begin{aligned} (1) \quad \nu(\sim\phi / i) &= T \quad \text{iff} \quad \nu(\phi/i) = F \\ (2) \quad \nu(\phi \&\psi / i) &= T \quad \text{iff} \quad \nu(\phi/i) = T \text{ and } \nu(\psi/i) = T \\ (3) \quad \nu(\phi \vee \psi / i) &= T \quad \text{iff} \quad \nu(\phi/i) = T \text{ and/or } \nu(\psi/i) = T \\ (4) \quad \nu(\phi \rightarrow \psi / i) &= T \quad \text{iff} \quad \nu(\phi/i) \leq \nu(\psi/i) \\ (5) \quad \nu(\phi \leftrightarrow \psi / i) &= T \quad \text{iff} \quad \nu(\phi/i) = \nu(\psi/i) \\ (6) \quad \nu(\Box\phi / i) &= T \quad \text{iff} \quad \nu(\phi/j) = T, \text{ for every } j \text{ in } I \\ (7) \quad \nu(\Diamond\phi / i) &= T \quad \text{iff} \quad \nu(\phi/j) = T, \text{ for at least one } j \text{ in } I \\ (8) \quad \nu(\phi \prec \psi / i) &= T \quad \text{iff} \quad \nu(\phi/j) \leq \nu(\psi/j), \text{ for every } j \text{ in } I \\ (9) \quad \nu(\phi = \psi / i) &= T \quad \text{iff} \quad \nu(\phi/j) = \nu(\psi/j), \text{ for every } j \text{ in } I \end{aligned}$$

- (d11) Let  $\mathcal{L}$  and  $S$  be as before. Let  $\langle P_1, \dots, P_m / C \rangle$  be an argument in  $\mathcal{L}$ . Then a ***counter-model*** to  $\langle P_1, \dots, P_m / C \rangle$  is any indexed valuation  $\nu$  on  $\mathcal{L}$  and index  $i$  such that  $\nu(P_1/i) = T, \dots, \nu(P_m/i) = T$ , and  $\nu(C/i) = F$ .

## 6. Counter-Models in System L

Just as there is a concrete (paper-and-pencil) method of finding counter-models in ordinary SL, there is also a concrete method of finding counter-models in System L.

As an example, consider the following argument.

$$\Box(P \vee Q) / (\Box P \vee \Box Q)$$

This is an invalid argument form, so we want to construct a counter-model to it, which means that we need to find an indexed-valuation that makes the premise true and the conclusion false.

We do this by construction. First, we start our indexed truth table at index 0, by assigning T to the premise and F to the conclusion, which gives us the following situation.

$$\Box ( P \vee Q ) / ( \Box P \vee \Box Q )$$

|    |   |  |  |  |  |  |  |  |  |   |  |  |  |
|----|---|--|--|--|--|--|--|--|--|---|--|--|--|
| 0: | T |  |  |  |  |  |  |  |  | F |  |  |  |
|----|---|--|--|--|--|--|--|--|--|---|--|--|--|

Next, we apply the truth conditions to index 0 wherever we can. First, since  $\Box(P \vee Q)$  is T/0,  $P \vee Q$  is T/0. Second, since  $\Box P \vee \Box Q$  is F/0,  $\Box P$  is F/0, and  $\Box Q$  is F/0. This yields the following.

$$\Box ( P \vee Q ) / ( \Box P \vee \Box Q )$$

|    |   |  |   |  |  |  |  |   |  |   |   |  |  |
|----|---|--|---|--|--|--|--|---|--|---|---|--|--|
| 0: | T |  | T |  |  |  |  | F |  | F | F |  |  |
|----|---|--|---|--|--|--|--|---|--|---|---|--|--|

Next, since  $\Box P$  is F/0, P must be F at *some* index. To achieve this, we "create" a new index, labeled '1', at which P is F. Similarly,  $\Box Q$  is F/0, so Q must be F at some index. For this, we create a new index, labeled '2', at which Q is F. This yields the following. [Notice: if a formula occurs twice on a line, its truth value is written twice.]

$$\Box ( P \vee Q ) / ( \Box P \vee \Box Q )$$

|    |   |   |   |   |  |  |  |   |  |   |   |   |  |
|----|---|---|---|---|--|--|--|---|--|---|---|---|--|
| 0: | T |   | T |   |  |  |  | F |  | F | F |   |  |
| 1: |   | F |   |   |  |  |  | F |  |   |   |   |  |
| 2: |   |   |   | F |  |  |  |   |  |   |   | F |  |

But since  $\Box(P \vee Q)$  is T/0,  $P \vee Q$  must be T/1 and T/2. Applying this to the above diagram, we obtain:

$$\Box ( P \vee Q ) / ( \Box P \vee \Box Q )$$

|    |   |   |   |   |  |  |  |   |  |   |   |   |  |
|----|---|---|---|---|--|--|--|---|--|---|---|---|--|
| 0: | T |   | T |   |  |  |  | F |  | F | F |   |  |
| 1: |   | F | T |   |  |  |  | F |  |   |   |   |  |
| 2: |   |   | T | F |  |  |  |   |  |   |   | F |  |

Next, applying the truth-function for  $\vee$  yields the following.

$$\Box ( P \vee Q ) / ( \Box P \vee \Box Q )$$

|    |   |  |   |   |   |  |  |  |   |   |   |   |  |   |
|----|---|--|---|---|---|--|--|--|---|---|---|---|--|---|
| 0: | T |  |   | T |   |  |  |  | F |   | F | F |  |   |
| 1: |   |  | F | T | T |  |  |  |   | F |   |   |  | T |
| 2: |   |  | T | T | F |  |  |  |   | T |   |   |  | F |

Finally, we note that the table can be completed in a manner that is consistent with the semantic rules. One way to do this (but not the only way) is as follows.

$$\Box ( P \vee Q ) / ( \Box P \vee \Box Q )$$

|    |   |  |   |   |   |  |  |  |   |   |   |   |   |   |
|----|---|--|---|---|---|--|--|--|---|---|---|---|---|---|
| 0: | T |  | T | T | F |  |  |  | F | T | F | F | F |   |
| 1: | T |  | F | T | T |  |  |  | F | F | F | F | F | T |
| 2: | T |  | T | T | F |  |  |  | F | T | F | F | F |   |

By way of concluding this section, we observe that the above truth-table can be replaced by a smaller one that consolidates indices 0 and 2. This produces the following more compact counter-model.

$$\Box ( P \vee Q ) / ( \Box P \vee \Box Q )$$

|    |   |  |   |   |   |  |  |  |   |   |   |   |   |   |
|----|---|--|---|---|---|--|--|--|---|---|---|---|---|---|
| 0: | T |  | T | T | F |  |  |  | F | T | F | F | F |   |
| 1: | T |  | F | T | T |  |  |  | F | F | F | F | F | T |

## 7. Pictorial Presentation of Rules for Constructing Counter-Models in L

### 1. The Usual Truth-Functional Rules of Indexed SL

|  |     |   |
|--|-----|---|
| $\sim \mathcal{A}$ is T/ <i>i</i>                        | iff | $\mathcal{A}$ is F/ <i>i</i>  |
| $\mathcal{A} \& \mathcal{B}$ is T/ <i>i</i>              | iff | $\mathcal{A}$ is T/ <i>i</i> and $\mathcal{B}$ is T/ <i>i</i>   |
| $\mathcal{A} \vee \mathcal{B}$ is T/ <i>i</i>            | iff | $\mathcal{A}$ is T/ <i>i</i> and/or $\mathcal{B}$ is T/ <i>i</i>  |
| $\mathcal{A} \rightarrow \mathcal{B}$ is T/ <i>i</i>     | iff | $\mathcal{A}$ is F/ <i>i</i> and/or $\mathcal{B}$ is T/ <i>i</i>  |
| $\mathcal{A} \leftrightarrow \mathcal{B}$ is T/ <i>i</i> | iff | either $\mathcal{A}$ and $\mathcal{B}$ are T/ <i>i</i> , or $\mathcal{A}$ and $\mathcal{B}$ are F/ <i>i</i> |

### 2. Modal Rules

|   |   |
|---|---|
| <p><math>\diamond \text{O}</math></p> <p>If <math>\diamond \mathcal{A}</math> is T at index <i>i</i>, then <math>\mathcal{A}</math> is T at some (new!) index <i>i+n</i>.</p> <div style="text-align: center; margin: 10px 0;"> </div>    | <p><math>\square \text{O}</math></p> <p>If <math>\square \mathcal{A}</math> is T at index <i>i</i>, then <math>\mathcal{A}</math> is T at every index accessible to <i>i</i>.</p> <div style="text-align: center; margin: 10px 0;"> </div>        |
| <p><math>\sim \square \text{O}</math></p> <p>If <math>\square \mathcal{A}</math> is F at index <i>i</i>, then <math>\mathcal{A}</math> is F at some (new!) index <i>i+n</i>.</p> <div style="text-align: center; margin: 10px 0;"> </div> | <p><math>\sim \diamond \text{O}</math></p> <p>If <math>\diamond \mathcal{A}</math> is F at index <i>i</i>, then <math>\mathcal{A}</math> is F at every index accessible to <i>i</i>.</p> <div style="text-align: center; margin: 10px 0;"> </div> |
| <p><math>\longrightarrow \blacktriangleright</math> indicates "creating" a "new" accessible world.</p>  | <p><math>\longrightarrow \blacktriangleright</math> indicates discharging a rule at every existing ("old") accessible world.</p>  |

### 3. Strict-Arrow Rules

By way of simplifying the semantic scheme, we propose to treat the strict-conditional and strict-biconditional as purely definitional. So, in order to evaluate a formula involving either of these connectives, one first resolves each occurrence of the connective according to the following definitions.

$$\mathcal{A} \prec \mathcal{B} \quad =_{df} \quad \square(\mathcal{A} \rightarrow \mathcal{B})$$

$$\mathcal{A} = \mathcal{B} \quad =_{df} \quad \square(\mathcal{A} \leftrightarrow \mathcal{B})$$

Example:

Original formula:  $P \prec (Q \prec R)$

Resolution:  $\square(P \rightarrow \square(Q \rightarrow R))$

## 8. The Relation Between Derivations and Counter-Models in System L

The astute reader will have noticed that the rules for constructing counter-models, as presented in the previous section, are precisely the same as the rules for constructing derivations. This is not entirely coincidental. In the present section, we show how counter-models parallel derivations. For example, consider our earlier example.

$$\Box(P \vee Q) / \Box P \vee \Box Q$$

Let us now construct a *partial* derivation of the conclusion from the premises. Notice that we go immediately into an indirection derivation. This corresponds to the first step in constructing a counter-model.

|      |                            |    |                 |
|------|----------------------------|----|-----------------|
| (1)  | $\Box(P \vee Q)$           | /0 | Pr              |
| (2)  | $\sim: \Box P \vee \Box Q$ | /0 | ID              |
| (3)  | $\sim(\Box P \vee \Box Q)$ | /0 | As              |
| (4)  | $\sim: \ast$               | /* |                 |
| (5)  | $\sim\Box P$               | /0 | 3,SL            |
| (6)  | $\sim\Box Q$               | /0 | 3,SL            |
| (7)  | $P \vee Q$                 | /0 | 1, $\Box O$     |
| (8)  | $\sim P$                   | /1 | 5, $\sim\Box O$ |
| (9)  | $\sim Q$                   | /2 | 6, $\sim\Box O$ |
| (10) | $P \vee Q$                 | /1 | 1, $\Box O$     |
| (11) | $P \vee Q$                 | /2 | 1, $\Box O$     |
| (12) | $P$                        | /1 | 8,10,SL         |
| (13) | $Q$                        | /2 | 9,11,SL         |

At this point, we have discharged all the rules that pertain to the sub-formulas of the argument. Since we have not reached a contradiction, we surmise that the argument is not valid.

Indeed, the partial derivation above provides a blueprint for constructing a counter-model to the argument. Specifically, according to the blueprint, there are three indices – 0, 1, 2; furthermore,

|                    |         |
|--------------------|---------|
| $P \vee Q$ is T/0; | line 7  |
| $P$ is T/1;        | line 12 |
| $Q$ is F/1;        | line 9  |
| $P$ is F/2;        | line 8  |
| $Q$ is T/2;        | line 13 |

any assignment of truth values to  $P$  and  $Q$  at 0 consistent with this completes the counter-model.

## D. Exercises

### 1. Derivations in System L

**Directions:** for each of the following, construct a formal derivation of the conclusion (marked by '/') from the premises (if any) in System L. In problems in which two formulas are separated by '//', construct a derivation of each formula from the other.

1.  $/ \Box P \prec \Box(P \vee Q)$
2.  $/ P \prec \Diamond P$
3.  $/ \Box P \prec P$
4.  $/ \Box P \prec \Diamond P$
5.  $/ P \prec \Box \Diamond P$
6.  $/ \Diamond \Box P \prec P$
7.  $/ \Diamond \Box P \prec \Box \Diamond P$
8.  $/ \Box P = \Box \Box P$
9.  $/ \Diamond P = \Diamond \Diamond P$
10.  $/ \Box P = \Diamond \Box P$
11.  $/ \Diamond P = \Box \Diamond P$
12.  $/ \Box P = \Box \Diamond \Box P$
13.  $/ \Diamond P = \Diamond \Box \Diamond P$
14.  $/ \sim \Box P = \Diamond \sim P$
15.  $/ \sim \Diamond P = \Box \sim P$
16.  $/ \Box \sim P \prec \sim \Box P$
17.  $/ \sim \Diamond P \prec \Diamond \sim P$
18.  $/ \sim \Diamond(P \& \sim P)$
19.  $/ \Box(P \vee \sim P)$
20.  $P \prec Q // \sim \Diamond(P \& \sim Q)$
21.  $P \prec Q // \Box(P \prec Q)$
22.  $P \prec Q // \Diamond(P \prec Q)$
23.  $\Box(P \rightarrow Q); \Box P / \Box Q$
24.  $/ \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$
25.  $\Box(P \leftrightarrow Q) / \Box P \leftrightarrow \Box Q$
26.  $\Box(P \& Q) // \Box P \& \Box Q$
27.  $\Box P \vee \Box Q / \Box(P \vee Q)$
28.  $\Box(P \rightarrow Q); \Diamond P / \Diamond Q$
29.  $/ \Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q)$
30.  $\Diamond(P \vee Q) // \Diamond P \vee \Diamond Q$
31.  $\Diamond(P \& Q) / \Diamond P \& \Diamond Q$
32.  $\Diamond P \& \Box Q / \Diamond(P \& Q)$
33.  $\Box(P \vee Q) / \Box P \vee \Box Q$
34.  $P \prec \sim P // \sim \Diamond P$
35.  $\sim P \prec P // \Box P$
36.  $\sim \Diamond P / P \prec Q$
37.  $\Box Q / P \prec Q$
38.  $P; \sim Q / \sim(P \prec Q)$
39.  $P = Q / \Box P \prec \Box Q$
40.  $P = Q / \Diamond P \prec \Diamond Q$
41.  $\Box(P \vee Q); \sim P / Q$
42.  $\Box(P \vee Q); \sim \Diamond P / \Box Q$
43.  $\Box P; \Box Q / P = Q$
44.  $\sim \Diamond P; \sim \Diamond Q / P = Q$
45.  $P \prec Q; \Diamond \sim Q / \sim \Box P$
46.  $\Diamond P \prec \Box Q / P \prec Q$
47.  $P \prec \Diamond Q / \Diamond P \prec \Diamond Q$
48.  $P = \Diamond Q / \Diamond P = \Diamond Q$
49.  $P = \Diamond Q / \Box P = \Diamond Q$
50.  $P = \Box Q / \Diamond P = \Box Q$
51.  $P \prec Q; \Diamond Q \prec R / P \prec \Box R$
52.  $\sim(P \prec Q) / \Diamond P \& \Diamond \sim Q$
53.  $\sim(P = Q) / \Diamond P \vee \Diamond Q$
54.  $P \prec Q; P \prec \sim Q / \sim \Diamond P$
55.  $P \prec Q; \sim P \prec Q / \Box Q$
56.  $\Diamond(P \rightarrow Q); \Box P / \Diamond Q$
57.  $/ \Diamond(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Diamond Q)$
58.  $\Box \Diamond P; \Diamond \Box(P \rightarrow Q) / \Diamond \Diamond Q$
59.  $\Diamond(P \rightarrow \Box Q); \Box P / \Diamond Q$
60.  $\Diamond \Box P / \Diamond(P \vee Q)$
61.  $\Box P; \Diamond(P \prec Q) / \Diamond Q$
62.  $\Box P \vee \Box Q / \Box(\Box P \vee \Box Q)$
63.  $\Box(\Box P \rightarrow Q) / \Box P \prec \Box Q$
64.  $\Diamond P; \Box Q / \Diamond(P \& \Box Q)$
65.  $\Box(P \vee \Box Q) // \Box P \vee \Box Q$
66.  $\Box(P \vee \Diamond Q) // \Box P \vee \Diamond Q$
67.  $\Diamond(P \& \Box Q) // \Diamond P \& \Box Q$
68.  $\Diamond(P \& \Diamond Q) // \Diamond P \& \Diamond Q$
69.  $\Box(P \rightarrow \Box Q) // \Diamond P \rightarrow \Box Q$
70.  $\Box(P \rightarrow \Diamond Q) // \Diamond P \rightarrow \Diamond Q$
71.  $\Diamond(P \rightarrow \Box Q) // \Box P \rightarrow \Box Q$
72.  $\Diamond(P \rightarrow \Diamond Q) // \Box P \rightarrow \Diamond Q$
73.  $\Box(P \leftrightarrow \Box Q) / \Box P \leftrightarrow \Box Q$
74.  $\Box(P \leftrightarrow \Diamond Q) / \Box P \leftrightarrow \Diamond Q$
75.  $\Diamond P \leftrightarrow \Box Q / \Diamond(P \leftrightarrow \Box Q)$
76.  $\Diamond P \leftrightarrow \Diamond Q / \Diamond(P \leftrightarrow \Diamond Q)$
77.  $P \prec Q // \sim Q \prec \sim P$
78.  $P \prec Q // P \prec (P \prec Q)$
79.  $P \prec Q // P \prec (P \& Q)$
80.  $P \prec Q // (P \vee Q) \prec Q$
81.  $P \prec (Q \prec R) / \Box Q \prec (P \prec R)$
82.  $P \prec (Q \& R) // (P \prec Q) \& (P \prec R)$
83.  $(P \vee Q) \prec R // (P \prec R) \& (Q \prec R)$
84.  $(P \prec Q) \vee (P \prec R) / P \prec (Q \vee R)$
85.  $P \prec Q; Q \prec R / P \prec R$
86.  $/ (P \prec Q) \prec [(Q \prec R) \prec (P \prec R)]$
87.  $/ (Q \prec R) \prec [(P \prec Q) \prec (P \prec R)]$
88.  $P \prec (Q \prec R) / (P \prec Q) \prec (P \prec R)$
89.  $(P \prec Q) \prec (P \prec R) / \Box P \prec (\Box Q \prec R)$
90.  $/ \Box P \prec [Q \prec (P \& Q)]$
91.  $P = Q / (P \prec R) = (Q \prec R)$
92.  $P = Q / (R \prec P) = (R \prec Q)$
93.  $Q \prec R / P \prec (Q \prec R)$
94.  $P \prec (Q \prec R) / (P \& Q) \prec R$
95.  $/ (\Box P \prec \Box Q) \vee (\Box Q \prec \Box P)$
96.  $P = (Q \prec R) / \Box P \vee \Box \sim P$
97.  $((P \rightarrow Q) \rightarrow R) = (P \rightarrow (Q \rightarrow R)) / \Box(P \vee R)$
98.  $/ (P \prec Q) \vee (Q \prec \Diamond P)$
99.  $\Box P \rightarrow \Diamond Q / \Diamond(P \rightarrow Q)$
100.  $(P \& Q) \prec \Box R / P \prec (\Box Q \prec R)$
101.  $\Box P \rightarrow \Box Q / \Diamond(P \rightarrow Q)$

## 2. Derivations in WT(L)

**Directions:** For each of the argument forms in the previous section, construct a derivation of the conclusion from the premises (if any) in WT(L).

## 3. Counter-Models in System L

**Directions:** Demonstrate that each of the following arguments is invalid in System L, by constructing a counter-model in System L.

- |     |   |     |  |
|-----|---|-----|--|
| 1.  | $/ \Diamond P \prec P$  | 23. | $P \prec Q / \Diamond P \prec \Box Q$  |
| 2.  | $/ P \prec \Box P$  | 24. | $\Diamond P = \Diamond Q / P = \Diamond Q$                                       |
| 3.  | $/ \Diamond P \prec \Box P$                                   | 25. | $\Box P = \Diamond Q / P = \Diamond Q$   |
| 4.  | $/ \Box \Diamond P \prec P$                                   | 26. | $\Diamond P = \Box Q / P = \Box Q$   |
| 5.  | $/ P \prec \Diamond \Box P$                                   | 27. | $P \prec \Diamond Q / \Diamond(P \rightarrow Q)$                                 |
| 6.  | $/ \Box \Diamond P \prec \Diamond \Box P$                     | 28. | $\Diamond P \& \Diamond \sim Q / \sim(P \prec Q)$                                |
| 7.  | $/ \sim P \prec \sim \Diamond P$                              | 29. | $\Diamond(P \vee Q) / \Diamond \Box P$   |
| 8.  | $/ \Diamond \sim P \prec \sim \Diamond P$                     | 30. | $\Diamond P \leftrightarrow \Box Q / \Diamond(P \leftrightarrow \Box Q)$         |
| 9.  | $/ \sim \Box P \prec \Box \sim P$                             | 31. | $\Diamond P \leftrightarrow \Diamond Q / \Diamond(P \leftrightarrow \Diamond Q)$ |
| 10. | $/ \Box P \vee \Box \sim P$                                   | 32. | $/ P \prec (Q \prec P)$  |
| 11. | $/ \Diamond P \& \Diamond \sim P$                             | 33. | $/ \sim P \prec (P \prec Q)$   |
| 12. | $\sim(P \& \sim Q) / P \prec Q$                               | 34. | $/ P \prec ((P \prec Q) \prec Q)$  |
| 13. | $\sim(P \prec Q) / P \& \sim Q$                               | 35. | $/ ((P \prec Q) \prec P) \prec P$  |
| 14. | $P \rightarrow Q / P \prec Q$                                 | 36. | $/ (P \prec Q) \vee (Q \prec P)$   |
| 15. | $P \leftrightarrow Q / P = Q$                                 | 37. | $\sim P \vee Q / P \prec Q$  |
| 16. | $\Box P \rightarrow \Box Q / \Box(P \rightarrow Q)$           | 38. | $\sim(P \prec Q) / P \& \sim Q$  |
| 17. | $\Box P \leftrightarrow \Box Q / \Box(P \leftrightarrow Q)$   | 39. | $\sim(P = Q) / \sim P \leftrightarrow Q$   |
| 18. | $\Box(P \vee Q) / \Box P \vee \Box Q$                         | 40. | $P \prec (Q \vee R) / (P \prec Q) \vee (P \prec R)$                              |
| 19. | $(\Diamond P \rightarrow \Diamond Q) / \Box(P \rightarrow Q)$ | 41. | $(P \prec Q) \prec (P \prec R) / P \prec (Q \prec R)$                            |
| 20. | $\Diamond P \& \Diamond Q / \Diamond(P \& Q)$                 | 42. | $P \prec (\Box Q \prec R) / (P \prec Q) \prec (P \prec R)$                       |
| 21. | $\Diamond(P \& Q) / \Diamond P \& \Box Q$                     | 43. | $(P \& Q) \prec R / P \prec (Q \prec R)$   |
| 22. | $\Diamond P = \Diamond Q / P = Q$                             |     |  |

## 4. Answers to Selected Exercises

### 1. Derivations in System L

#3:

- |     |                                    |    |             |
|-----|------------------------------------|----|-------------|
| (1) | SHOW: $\Box P \prec P$             | /0 | Def $\prec$ |
| (2) | SHOW: $\Box(\Box P \rightarrow P)$ | /0 | ND          |
| (3) | SHOW: $\Box P \rightarrow P$       | /1 | CD          |
| (4) | $\Box P$                           | /1 | As          |
| (5) | SHOW: $P$                          | /1 | DD          |
| (6) | $P$                                | /1 | 4, $\Box O$ |

## #5:

|     |   |    |                 |
|-----|---|----|-----------------|
| (1) | SHOW: $P \prec \Box \Diamond P$             | /0 | Def $\prec$     |
| (2) | SHOW: $\Box(P \rightarrow \Box \Diamond P)$ | /0 | ND              |
| (3) | SHOW: $P \rightarrow \Box \Diamond P$       | /1 | CD              |
| (4) | P   | /1 | As.             |
| (5) | SHOW: $\Box \Diamond P$                     | /1 | ND              |
| (6) | SHOW: $\Diamond P$                          | /2 | 4, $\Diamond I$ |

## #6:

|     |   |    |                 |
|-----|---|----|-----------------|
| (1) | SHOW: $\Diamond \Box P \prec P$             | /0 | Def $\prec$     |
| (2) | SHOW: $\Box(\Diamond \Box P \rightarrow P)$ | /0 | ND              |
| (3) | SHOW: $\Diamond \Box P \rightarrow P$       | /1 | CD              |
| (4) | $\Diamond \Box P$                           | /1 | As              |
| (5) | SHOW: P                                     | /1 | DD              |
| (6) | $\Box P$                                    | /2 | 4, $\Diamond O$ |
| (7) | P   | /1 | 6, $\Box O$     |

## #24:

|     |   |    |             |
|-----|---|----|-------------|
| (1) | SHOW: $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$ | /0 | CD          |
| (2) | $\Box(P \rightarrow Q)$   | /0 | As          |
| (3) | SHOW: $\Box P \rightarrow \Box Q$                                     | /0 | CD          |
| (4) | $\Box P$  | /0 | As          |
| (5) | SHOW: $\Box Q$  | /0 | ND          |
| (6) | SHOW: Q   | /1 | DD          |
| (7) | $P \rightarrow Q$   | /1 | 2, $\Box O$ |
| (8) | P   | /1 | 4, $\Box O$ |
| (9) | Q   | /1 | 7,8,SL      |

## #27:

(done using ordinary 110 moves)

|      |                        |    |              |
|------|------------------------|----|--------------|
| (1)  | $\Box P \vee \Box Q$   | /0 | Pr           |
| (2)  | SHOW: $\Box(P \vee Q)$ | /0 | ND           |
| (3)  | SHOW: $P \vee Q$       | /1 | ID           |
| (4)  | $\sim(P \vee Q)$       | /1 | As           |
| (5)  | SHOW: $\ast$           | /* | DD           |
| (6)  | $\sim P$               | /1 | 4,SL         |
| (7)  | $\sim Q$               | /1 | 4,SL         |
| (8)  | SHOW: $\sim \Box P$    | /0 | ID           |
| (9)  | $\Box P$               | /0 | As           |
| (10) | SHOW: $\ast$           | /* | DD           |
| (11) | P                      | /1 | 9, $\Box O$  |
| (12) | $\ast$                 | /0 | 6,11,SL      |
| (13) | $\Box Q$               | /0 | 1,8,SL       |
| (14) | Q                      | /1 | 13, $\Box O$ |
| (15) | $\ast$                 | /* | 7,14,SL      |

**#27:**

(done using separation of cases)

|      |                        |    |             |
|------|------------------------|----|-------------|
| (1)  | $\Box P \vee \Box Q$   | /0 | Pr          |
| (2)  | SHOW: $\Box(P \vee Q)$ | /0 | 1,SC        |
| (3)  | c1: $\Box P$           | /0 | As          |
| (4)  | SHOW: $\Box(P \vee Q)$ | /0 | ND          |
| (5)  | SHOW: $P \vee Q$       | /1 | DD          |
| (6)  | P                      | /1 | 3, $\Box O$ |
| (7)  | $P \vee Q$             | /1 | 6,SL        |
| (8)  | c2: $\Box Q$           | /0 | As          |
| (9)  | SHOW: $\Box(P \vee Q)$ | /0 | ND          |
| (10) | SHOW: $P \vee Q$       | /1 | DD          |
| (11) | Q                      | /1 | 8, $\Box O$ |
| (12) | $P \vee Q$             | /1 | 11,SL       |

**#27:**

(also done using separation of cases; compare with previous example)

|      |                        |    |             |
|------|------------------------|----|-------------|
| (1)  | $\Box P \vee \Box Q$   | /0 | Pr          |
| (2)  | SHOW: $\Box(P \vee Q)$ | /0 | ND          |
| (3)  | SHOW: $P \vee Q$       | /1 | 1,SC        |
| (4)  | c1: $\Box P$           | /0 | As          |
| (5)  | SHOW: $P \vee Q$       | /1 | DD          |
| (6)  | P                      | /1 | 4, $\Box O$ |
| (7)  | $P \vee Q$             | /1 | 6,SL        |
| (8)  | c2: $\Box Q$           | /0 | As          |
| (9)  | SHOW: $P \vee Q$       | /1 | DD          |
| (10) | Q                      | /1 | 8, $\Box O$ |
| (11) | $P \vee Q$             | /1 | 10,SL       |

**#29:**

|     |   |    |                 |
|-----|---|----|-----------------|
| (1) | SHOW: $\Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q)$ | /0 | CD              |
| (2) | $\Box(P \rightarrow Q)$   | /0 | Pr              |
| (3) | SHOW: $\Diamond P \rightarrow \Diamond Q$                                     | /0 | CD              |
| (4) | $\Diamond P$  | /0 | As              |
| (5) | SHOW: $\Diamond Q$  | /0 | DD              |
| (6) | P   | /1 | 4, $\Diamond O$ |
| (7) | $P \rightarrow Q$   | /1 | 2, $\Box O$     |
| (8) | Q   | /1 | 6,7,SL          |
| (9) | $\Diamond Q$  | /0 | 8, $\Diamond I$ |

**#29 (done using ID):**

|      |   |    |                 |
|------|---|----|-----------------|
| (1)  | SHOW: $\Box(P \rightarrow Q) \rightarrow (\Diamond P \rightarrow \Diamond Q)$ | /0 | CD              |
| (2)  | $\Box(P \rightarrow Q)$   | /0 | Pr              |
| (3)  | SHOW: $\Diamond P \rightarrow \Diamond Q$                                     | /0 | CD              |
| (4)  | $\Diamond P$  | /0 | As              |
| (5)  | SHOW: $\Diamond Q$  | /0 | ID              |
| (6)  | $\sim \Diamond Q$   | /0 | As              |
| (7)  | SHOW: $\ast$  | /* | 11,12,SL        |
| (8)  | $\Box \sim Q$   | /0 | 6,MN            |
| (9)  | P   | /1 | 4, $\Diamond O$ |
| (10) | $P \rightarrow Q$   | /1 | 2, $\Box O$     |
| (11) | Q   | /1 | 9,10,SL         |
| (12) | $\sim Q$  | /1 | 8, $\Box O$     |

## #32:

|     |                              |    |                  |
|-----|------------------------------|----|------------------|
| (1) | $\Diamond P \ \& \ \Box Q$   | /0 | Pr               |
| (2) | SHOW: $\Diamond(P \ \& \ Q)$ | /0 | 5, $\Diamond I$  |
| (3) | P                            | /1 | 1a, $\Diamond O$ |
| (4) | Q                            | /1 | 1b, $\Box O$     |
| (5) | P & Q                        | /1 | 3,4, SL          |

## #58:

|      |                                  |    |                 |
|------|----------------------------------|----|-----------------|
| (1)  | $\Box \Diamond P$                | /0 | Pr              |
| (2)  | $\Diamond \Box(P \rightarrow Q)$ | /0 | Pr              |
| (3)  | SHOW: $\Diamond \Diamond Q$      | /0 | DD              |
| (4)  | $\Box(P \rightarrow Q)$          | /1 | 2, $\Diamond O$ |
| (5)  | $\Diamond P$                     | /1 | 1, $\Box O$     |
| (6)  | P                                | /2 | 5, $\Diamond O$ |
| (7)  | $P \rightarrow Q$                | /2 | 4, $\Box O$     |
| (8)  | Q                                | /2 | 6,7, SL         |
| (9)  | $\Diamond Q$                     | /0 | 8, $\Diamond I$ |
| (10) | $\Diamond \Diamond Q$            | /0 | 9, $\Diamond I$ |

## #75:

|      |  |    |                       |
|------|--|----|-----------------------|
| (1)  | $\Diamond P \leftrightarrow \Box Q$        | /0 | Pr                    |
| (2)  | SHOW: $\Diamond(P \leftrightarrow \Box Q)$ | /0 | ID                    |
| (3)  | $\sim \Diamond(P \leftrightarrow \Box Q)$  | /0 | As                    |
| (4)  | SHOW: $\ast$                               | /* | 5,17, SL              |
| (5)  | SHOW: $\sim \Box Q$                        | /0 | ID                    |
| (6)  | $\Box Q$                                   | /0 | As                    |
| (7)  | SHOW: $\ast$                               | /* | 12,13, SL             |
| (8)  | $\Diamond P$                               | /0 | 1,6, SL               |
| (9)  | P  | /1 | 8, $\Diamond O$       |
| (10) | $\sim(P \leftrightarrow \Box Q)$           | /1 | 3, $\sim \Diamond O$  |
| (11) | $\sim \Box Q$                              | /1 | 9,10, SL              |
| (12) | $\sim Q$                                   | /2 | 11, $\sim \Box O$     |
| (13) | Q  | /2 | 6, $\Box O$           |
| (14) | $\sim \Diamond P$                          | /0 | 1,5, SL               |
| (15) | $\sim P$                                   | /0 | 14, $\sim \Diamond O$ |
| (16) | $\sim(P \leftrightarrow \Box Q)$           | /0 | 3, $\sim \Diamond O$  |
| (17) | $\Box Q$                                   | /0 | 15,16, SL             |

## #88:

|      |   |    |                |
|------|---|----|----------------|
| (1)  | $P \prec (Q \prec R)$                             | /0 | Pr             |
| (2)  | $\Box(P \rightarrow (Q \prec R))$                 | /0 | 1, Def $\prec$ |
| (3)  | SHOW: $(P \prec Q) \prec (P \prec R)$             | /0 | Def $\prec$    |
| (4)  | SHOW: $\Box((P \prec Q) \rightarrow (P \prec R))$ | /0 | ND             |
| (5)  | SHOW: $(P \prec Q) \rightarrow (P \prec R)$       | /0 | CD             |
| (6)  | $P \prec Q$                                       | /1 | As             |
| (7)  | $\Box(P \rightarrow Q)$                           | /1 | 6, Def $\prec$ |
| (8)  | SHOW: $P \prec R$                                 | /1 | Def $\prec$    |
| (9)  | SHOW: $\Box(P \rightarrow R)$                     | /1 | ND             |
| (10) | SHOW: $P \rightarrow R$                           | /2 | CD             |
| (11) | $P$   | /2 | As             |
| (12) | SHOW: $R$   | /2 | DD             |
| (13) | $P \rightarrow (Q \prec R)$                       | /2 | 2, $\Box O$    |
| (14) | $P \rightarrow Q$                                 | /2 | 7, $\Box O$    |
| (15) | $Q \prec R$                                       | /2 | 11,13,SL       |
| (16) | $Q$   | /2 | 11,14,SL       |
| (17) | $\Box(Q \rightarrow R)$                           | /2 | 15,Def $\prec$ |
| (18) | $Q \rightarrow R$                                 | /2 | 17, $\Box O$   |
| (19) | $R$   | /2 | 16,18,SL       |

## #98:

|      |   |    |                       |
|------|---|----|-----------------------|
| (1)  | SHOW: $(P \prec Q) \vee (Q \prec \Diamond P)$ | /0 | ID                    |
| (2)  | $\sim[(P \prec Q) \vee (Q \prec \Diamond P)]$ | /0 | As                    |
| (3)  | SHOW: $\ast$                                  | /* | DD                    |
| (4)  | $\sim(P \prec Q)$                             | /0 | 2,SL                  |
| (5)  | $\sim\Box(P \rightarrow Q)$                   | /0 | 4, Def $\prec (-)$    |
| (6)  | $\sim(Q \prec \Diamond P)$                    | /0 | 2,SL                  |
| (7)  | $\sim\Box(Q \rightarrow \Diamond P)$          | /0 | 6, Def $\prec (-)$    |
| (8)  | $\sim(P \rightarrow Q)$                       | /1 | 5, $\sim\Box O$       |
| (9)  | $\sim(Q \rightarrow \Diamond P)$              | /2 | 7, $\sim\Box O$       |
| (10) | $P \& \sim Q$                                 | /1 | 8,SL                  |
| (11) | $Q \& \sim\Diamond P$                         | /2 | 9,SL                  |
| (12) | $\sim P$                                      | /1 | 11b, $\sim\Diamond O$ |
| (13) | $\ast$  | /* | 10a,12,SL             |

## #99:

|      |                                   |    |                     |
|------|-----------------------------------|----|---------------------|
| (1)  | $\Box P \rightarrow \Diamond Q$   | /0 | Pr                  |
| (2)  | SHOW: $\Diamond(P \rightarrow Q)$ | /0 | ID                  |
| (3)  | $\sim\Diamond(P \rightarrow Q)$   | /0 | As                  |
| (4)  | SHOW: $\ast$                      | /* | DD                  |
| (5)  | SHOW: $\Box P$                    | /0 | ND                  |
| (6)  | SHOW: $P$                         | /1 | DD                  |
| (7)  | $\sim(P \rightarrow Q)$           | /1 | 3, $\sim\Diamond O$ |
| (8)  | $P$                               | /1 | 7,SL                |
| (9)  | $\Diamond Q$                      | /0 | 1,5,SL              |
| (10) | $Q$                               | /2 | 10, $\Diamond O$    |
| (11) | $\sim(P \rightarrow Q)$           | /2 | 3, $\sim\Diamond O$ |
| (12) | $\sim Q$                          | /2 | 12,SL               |
| (13) | $\ast$                            | /* | 11,13,SL            |

## #100:

|      |  |    |               |
|------|--|----|---------------|
| (1)  | $(P \ \& \ Q) \prec \Box R$                  | /0 | Pr            |
| (2)  | SHOW: $P \prec (\Box Q \prec R)$             | /0 | Def $\prec$   |
| (3)  | SHOW: $\Box[P \rightarrow (\Box Q \prec R)]$ | /0 | ND            |
| (4)  | SHOW: $P \rightarrow (\Box Q \prec R)$       | /1 | CD            |
| (5)  | P  | /1 | As            |
| (6)  | SHOW: $\Box Q \prec R$                       | /1 | Def $\prec$   |
| (7)  | SHOW: $\Box(\Box Q \rightarrow R)$           | /1 | ND            |
| (8)  | SHOW: $\Box Q \rightarrow R$                 | /2 | CD            |
| (9)  | $\Box Q$                                     | /2 | As            |
| (10) | SHOW: R                                      | /2 | DD            |
| (11) | $\Box[(P \ \& \ Q) \rightarrow \Box R]$      | /0 | 1,Def $\prec$ |
| (12) | $(P \ \& \ Q) \rightarrow \Box R$            | /1 | 11, $\Box O$  |
| (13) | Q  | /1 | 9, $\Box O$   |
| (14) | $\Box R$                                     | /1 | 5,12,13,SL    |
| (15) | R  | /2 | 14, $\Box O$  |

## #101:

|      |                                   |    |                      |
|------|-----------------------------------|----|----------------------|
| (1)  | $\Box P \rightarrow \Box Q$       | /0 | Pr                   |
| (2)  | SHOW: $\Diamond(P \rightarrow Q)$ | /0 | ID                   |
| (3)  | $\sim \Diamond(P \rightarrow Q)$  | /0 | As                   |
| (4)  | SHOW: $\ast$                      | /* | 9,10,SL              |
| (5)  | $\sim(P \rightarrow Q)$           | /0 | 3, $\sim \Diamond O$ |
| (6)  | $\sim Q$                          | /0 | 5,SL                 |
| (7)  | $\sim \Box Q$                     | /0 | 6, $\Box O(-)$       |
| (8)  | $\sim \Box P$                     | /0 | 1,7,SL               |
| (9)  | $\sim P$                          | /1 | 8, $\sim \Box O$     |
| (10) | $\sim(P \rightarrow Q)$           | /1 | 3, $\sim \Diamond O$ |

## 2. Derivations in WT(L)

under construction

### 3. Counter-Models in System L

Note: in each case, ‘0’ marks the actual world, whereas the remaining numerals mark the alternative worlds. ‘?’ indicates that the truth-value can be either T or F and it is still a counter-model.

Notice that strict conditionals must be first resolved, according to their definitions, into the corresponding necessary conditionals.

Note that the entry ‘?’ indicates that either truth-value is admissible.

#1:  $\neg \Diamond P \prec P$   
 $\text{=}_{df} \Box(\Diamond P \rightarrow P)$   
 $\Box(\Diamond P \rightarrow P)$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 0 | F | T | F | F | F |
| 1 | F | T | T | T | T |

#11:  $\neg \Diamond P \& \Diamond \sim P$   
 $\Diamond P \& \Diamond \sim P$

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 0 | T | T | F | F | F | T |
|---|---|---|---|---|---|---|

#16:  $\neg \Box P \rightarrow \Box Q / \Box(P \rightarrow Q)$   
 $\Box P \rightarrow \Box Q / \Box(P \rightarrow Q)$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 0 | F | T | T | F | F |
| 1 | F | F | T | F | ? |

|   |   |   |   |
|---|---|---|---|
| F | T | F | F |
| F | F | T | ? |

#20:  $\neg \Diamond P \& \Diamond Q / \Diamond(P \& Q)$   
 $\Diamond P \& \Diamond Q / \Diamond(P \& Q)$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 0 | T | T | T | T | F |
| 1 | T | F | T | T | T |

|   |   |   |   |
|---|---|---|---|
| F | T | F | F |
| F | F | F | T |

#32:  $\neg P \prec (Q \prec P)$   
 $\text{=}_{df} \Box(P \rightarrow \Box(Q \rightarrow P))$   
 $\Box(P \rightarrow \Box(Q \rightarrow P))$

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
| 0 | F | T | F | F | T | T | T |
| 1 | F | F | T | F | T | F | F |

#36:  $\neg(P \prec Q) \vee (Q \prec P)$   
 $\text{=}_{df} \Box(P \rightarrow Q) \vee \Box(Q \rightarrow P)$   
 $\Box(P \rightarrow Q) \vee \Box(Q \rightarrow P)$

|   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|
| 0 | F | T | F | F | F | F | F | T | T |
| 1 | F | F | T | T | F | F | T | F | F |

#43:  $(P \& Q) \prec R / P \prec (Q \prec R) \text{=}_{df} \Box[(P \& Q) \rightarrow R] / \Box[P \rightarrow \Box(Q \rightarrow R)]$   
 $\Box[(P \& Q) \rightarrow R] / \Box[P \rightarrow \Box(Q \rightarrow R)]$

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| 0 | T | T | T | T | T | T |
| 1 | T | F | F | T | T | F |

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| F | T | F | F | T | T | T |
| F | F | T | F | T | F | F |