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Basic Concepts

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1. Introduction

Contemporary modal logic is founded on two distinct, but closely related, problems – one concerning intensionality (with an ‘s’), the other concerning intentionality (with a ‘t’). Neither of these is adequately addressed in ordinary first-order logic.

In the present chapter, we examine these motivations.

2. The Problem of Intensionality (with an ‘s’)

As the reader may recall, ordinary sentential logic is concerned exclusively with truth-functional connectives (see Chapter 2). This is the source of a number of problems in analyzing natural languages, including English. The most serious problem concerns the conditional (if-then) connective. The story goes something like this. On the one hand, we need a conditional connective, because conditional reasoning pervades mathematical discourse [for example, “let A be a right triangle”, or “suppose r is a prime number”]. On the other hand, we want our connectives to be truth-functional, because those are the ones that we fully understand.

What works for mathematical discourse does not necessarily work for all human discourse, where the two desiderata are seriously in conflict. Logic instructors (honest ones, at least) inform their students that, in natural language (e.g., English), ‘if...then’ is almost always used in a manner that is definitely not truth-functional. The inquisitive students are then directed to more advanced logic courses, where they might learn more about the more unruly (but natural!) if-then connectives.

In my own introductory logic book (*Symbolic Logic: a First Course*), there is a section called “Yet Another Problem with the Truth-Functional If-Then”. Here, I mention a problem with my translation scheme for *necessary* and *sufficient* conditions. I review that discussion here.

I propose that we translate sentences of the form¹

\mathcal{A} is sufficient for \mathcal{B} ,

as:

if \mathcal{A} , then \mathcal{B} .

For example, we translate

studying is sufficient for getting an A

as

if you study, then you get an A

if S, then A

$S \rightarrow A$

This implies, of course, that the negated sentence

studying is not sufficient for getting an A

is correspondingly translated as

¹ Note that these two sentence forms are not completely on a par. Whereas {if \mathcal{A} , then \mathcal{B} } takes ordinary indicative sentences as input, { \mathcal{A} is sufficient for \mathcal{B} } does not; rather, it takes nominalized sentences as input. More about this later.

it is not true that if you study, then you get an A.

not(if S, then A)

$\sim(S \rightarrow A)$

So far, so good, it would seem. The problem arises, not in the paraphrase, but in the analysis of the if-then connective. In particular, if we insist on a truth-functional analysis of if-then, then we are stuck with the following logical equivalence.

$\sim(S \rightarrow A)$

\equiv

$S \ \& \ \sim A$

But the latter formula presumably says something like:

you will study, but you will not get an A

This is surely not what we mean when we say that studying is not sufficient for getting an A.

3. What to Do

At this point, we can work backwards perhaps. When I say

S is not sufficient for A

I am definitely not saying

it will happen that: S but not A.

Rather, it seems more plausible that I am saying something like

it could happen that: S but not A

For example,

it could happen that: you study but do not get an A

Now the word ‘could’ is an example of what linguists call a *modal qualifier*. Other examples include: ‘will’, ‘shall’, ‘must’, ‘may’, ‘would’, ‘can’.²

Now, we can take the expression

it could happen that

and further sanitize it by paraphrasing it as:

it is possible that

² Note: both ‘will’ and ‘shall’ are used as modal auxiliaries, but they are not both “truly” modal in the same circumstances; the “truly” modal form of ‘it will’ is ‘it shall’ [also recall the well-known example ‘thou shalt not kill’]. Note, however, that these roles are reversed for first person pronouns(!) – an oddity of English syntax. The modal qualities of ‘will’ versus ‘shall’ are disappearing from English, although the subjunctive forms (‘would’ versus ‘should’) are still clearly distinct.

or:

possibly

4. Possibility and Context

At this point it is important to note that, in ordinary language at least, possibility is almost always *context-dependent*. When we use modal words like ‘could’, ‘might’, ‘possibly’, etc., there is usually an implicit context with respect to which we are speaking.

For example, suppose I tell you that

getting a hundred on every exam is *not sufficient for* getting an A,

which is presumably equivalent to saying that

you *could* get a hundred on every exam *without* getting an A.

What do I mean by ‘could’ here. Let us consider a few *possibilities*; I *could* mean any of the following.

- ◆ the course has further requirements that you must satisfy;
- ◆ I could change the scale, so that a perfect score is 200, rather than 100;
- ◆ you are not officially in the course, so you will get no grade;
- ◆ I am lying, or mistaken, about the grading scale;
- ◆ I, or someone in the administration, could make a clerical error in recording your grade;
- ◆ the sun could explode, vaporizing the Earth and all its inhabitants, just before the grades are recorded, so no one gets anything except vaporized;
- ◆ through some truly amazing "event", the whole world could simply cease to exist;
- ◆ "they" could pull the plug on the vat in which our brains collectively reside, thus bringing our phenomenal world to a screeching halt;

These are all possibilities (some considerably more remote than others). The point is that, when we use the word ‘could’, we *could* mean just about anything.

5. A Symbol for ‘Possibly’

Keeping stringently in mind that the words ‘could’ and ‘possible’ are highly context-dependent, we nevertheless propose a single syntactic analysis of it, and a single *overall* semantic analysis of it.

To begin with, we follow tradition, and use a diamond symbol ‘ \diamond ’ to stand for the word ‘possibly’ [equivalently, the phrase ‘it is possible that’]. Formally speaking, ‘ \diamond ’ is a one-place connective, so we have the following syntactic rule.

if \mathbb{F} is a formula, then so is: $\diamond\mathbb{F}$

Accordingly, we translate

it is possible that: you study, but you do not get an A

as follows.

$$\diamond(S \ \& \ \sim A)$$

6. Other Modalities Definable in terms of ‘Possibly’

Following usual logical custom, let us say that a *modality* is a string of instances of ‘ \sim ’ and ‘ \diamond ’. Accordingly, the following expressions are all modalities, in this sense.

\sim , $\sim\sim$, $\sim\sim\sim$, etc.
 \diamond , $\diamond\diamond$, $\diamond\diamond\diamond$, etc.
 $\sim\diamond$
 $\diamond\sim$
 $\sim\diamond\sim$

Given the associated syntactic rules, each one of these combinations can be regarded as a one-place connective in its own right, although some will of course turn out to be logically equivalent. On the other hand, not every such combination has a simple English paraphrase.

For example, let ‘P’ abbreviate the atomic sentence ‘I pass’. Now consider the following formulas, which employ four of our modalities.

- (a) $\diamond P$
- (b) $\sim\diamond P$
- (c) $\diamond\sim P$
- (d) $\sim\diamond\sim P$

How do we translate these formulas into English? Well, let’s start by translating the logical symbols according to the following "standard" scheme.³

\sim : it is not true that
 \diamond : it is possible that

Then we have:

- (a) it is possible that P
- (b) it is not true that it is possible that P
- (c) it is possible that it is not true that P
- (d) it is not true that it is possible that it is not true that P

Now, it seems plausible that

it is not true that it is possible that P

is equivalent to

it is not possible that P

or better still:

it is impossible that P

³ As we see later, this “standard” translation scheme suppresses a great deal of interesting grammar.

Also, of course,

it is not true that I pass

is equivalent to:

I don't pass.

This permits us to rewrite our translations as follows.

- (a) it is possible that I pass
- (b) it is impossible that I pass
- (c) it is possible that I don't pass
- (d) it is impossible that I don't pass

Or, if we prefer to nominalize the component sentence, we can rewrite these as follows.

- (a) passing is possible
- (b) passing is impossible
- (c) not-passing is possible
- (d) not-passing is impossible

7. A Somewhat Trickier Example

Let us consider another example. First, let us symbolize

you *may* leave your term papers in my mailbox⁴

as

- (a) $\Diamond P$

Then how do we translate the other modal formulas?

- (b) $\sim \Diamond P$
- (c) $\Diamond \sim P$
- (d) $\sim \Diamond \sim P$

Curiously, the colloquial translation of (b) is:

you may not leave your term papers in my mailbox.

Here is a standard example in which the negative 'not' is placed within the verb phrase 'may leave' in order to negate the whole sentence. As logicians are apt to describe the situation, 'not' has *wide scope*.

But what if we want to make 'not' have narrow scope? This is easy in symbolic languages, because symbolic languages do scope very well. We just write:

- (c) $\Diamond \sim P$

which is literally translated as follows.

⁴ Here, 'may' pertains to *permission*, which is a special type of possibility. Other interpretations of 'may' are considered in a later section.

it is permissible that it is not true that you leave your term papers in my mailbox,
or *somewhat* more colloquially,

it is permissible that you don't leave your term papers in my mailbox.

A slightly more natural translation invokes the infinitive construction for embedded sentences, thus:

$\diamond P$ it is permissible FOR you to leave your term papers in my mailbox;
 $\diamond \sim P$ it is permissible FOR you not to leave your term papers in my mailbox

Now, a native speaker can probably do a lot better than this. Unlike logically constructed languages, natural languages convey scope through a number of oblique and subtle techniques. To make scope distinctions in English, you simply have to know the appropriate scope-like phrases. The following seems to be the appropriate colloquial reading of the above formula.

$\diamond \sim P$ you need not leave your term papers in my mailbox

8. Duality – Necessity

We have one remaining modality to consider from the above list.

(d) $\sim \diamond \sim P$

Sentences with multiple-negations are notoriously difficult to parse semantically. For this reason, if for no other, English comes equipped with paraphrase techniques to help reduce negation-clutter. An excellent example is the term 'only'; consider the following example,

no one who is not a member is allowed inside the club,

which has two negatives – 'no' and 'not'. I can just as easily say the following.

only members are allowed inside the club.

Similarly, I can paraphrase

if you don't study, then you don't get an A

as

you will get an A only if you study.

Or, if I only want to simplify one of the negations, I can say

you will not get an A unless you study.

Still other examples come from simple quantification. If I say

there is no one who is not happy,

I can just as easily say

everyone is happy.

The standard translations of these two sentences, into predicate logic, are given as follows.

$$\sim\exists x\sim Hx$$

$$\forall xHx$$

More generally, the quantifier combo $\lceil\sim\exists x\sim\rceil$ is equivalent to $\lceil\forall x\rceil$. This is sometimes described by saying that \forall and \exists are "duals" of each other.⁵

By analogy we are led to ask what the dual of \diamond is. Is there a natural simplification of the following negation-cluttered formula?

$$\sim\diamond\sim P$$

it is not permissible for you not to leave your term-papers in my mailbox

Well, it seems obvious enough that a more tidy rendering of this statement goes as follows.

you must leave your term papers in my mailbox

or:

you are required to leave your term papers in my mailbox

or:

it is necessary that you leave your term papers in my mailbox

This suggests the following list of dual-pairs of modal expressions.

may	must
permitted	required
possible	necessary

At this point, we note that the notion of necessity gets its own symbol, the "box" \square , which is accompanied by the following official syntactic rule.

if F is a formula, then so is $\square F$.

Also, just as the "standard" reading of \sim is 'it is not true that', and the standard reading of \diamond is 'it is possible that', the standard reading of \square is 'it is necessary that'.

9. Back to Sufficiency

We analyzed not-sufficient using \diamond . We now see that \diamond has an alter-ego \square . Using this information, let us go back to the problem of analyzing 'S is sufficient for A'. First, we have

$$S \text{ is not sufficient for } A$$

$$\equiv:$$

$$\diamond(S \ \& \ \sim A)$$

So, negating both halves we obtain:

⁵ Also note that conjunction (&) and disjunction (\vee) are dual to each other. This is illustrated in their truth-tables; take the truth-table for one, reverse all the T's and F's, and one obtains the truth-table for the other one. One is the "upside-down version" of the other. This duality is sometimes symbolically captured by using an upside-down wedge (i.e., \wedge) to represent conjunction. Other authors take the duality-inspired notation one step further. In particular, they use a large wedge for existential-quantification, and a large upside-down wedge for universal quantification.

S is sufficient for A

≡:

$\sim\Diamond(S \ \& \ \sim A)$

But, by the principle of modal duality, we know that $\sim\Diamond \equiv \Box\sim$, so

$\sim\Diamond(S \ \& \ \sim A)$

≡

$\Box\sim(S \ \& \ \sim A)$.

Now, this includes the formula

$\sim(S \ \& \ \sim A)$

which is equivalent by SL to:

$S \rightarrow A$.

So we have

S is sufficient for A

≡:

$\Box(S \rightarrow A)$

which we may read as:

it is necessary that, if you study, then you get an A

Note, of course, that necessity here is context-dependent in exactly the same manner as possibility.

10. C.I. Lewis and the Formulation of Modern Modal Logic

As this point, we are very near one of the modern motivations underlying modal logic. Specifically, in developing the modern version of modal logic, C. I. Lewis was originally concerned with developing an appropriate alternative to the truth-functional if-then connective. He devised a variety of systems for such a connective, which he symbolized by ‘ \prec ’ (called "hook" or "fish-hook"), but in each case, there was an associated box-operator such that

$$\mathcal{A} \prec C \equiv \Box(\mathcal{A} \rightarrow C)$$

So when we look at our original problem, analyzing ‘S is sufficient for A’, we see that the overall pattern was correct. The problem is that, in elementary logic, we have no means of writing down a non-truth-functional if-then connective. Modal logic provides that connective.

11. The Modal Counterpart of ‘if and only if’

Before continuing, just as there is a robust version of ‘if...then...’, there is also a robust version of the biconditional connective (if and only if), which is customarily symbolized by a bold-face equals sign ‘ \equiv ’. Unfortunately, this can be the source of much syntactic confusion, since ordinary equality ‘ $=$ ’ is a *predicate*, but ‘ \equiv ’ is a *connective*. See below.

In any case, ‘ \equiv ’ satisfies the following principles.

$$\mathcal{A} = \mathcal{B} \equiv \Box(\mathcal{B} \leftrightarrow \mathcal{A})$$

$$\mathcal{A} = \mathcal{B} \equiv (\mathcal{A} \prec \mathcal{B}) \& (\mathcal{B} \prec \mathcal{A})$$

12. Other Brands of Modalities

We have already seen that possibility is highly context-dependant. Suppose a student asks me whether he/she may turn in a term paper tomorrow (say, a day late). Suppose I say

yes, you may turn your paper in tomorrow

Ordinarily understood, I have granted the student permission relative to my grading of the paper. I have most likely not granted permission relative to God's prohibitions; who knows, it could be that late papers are bad (although presumably not as bad as murder, etc.)

I am most likely granting permission. But clearly the term 'may' does not automatically force a permission-type reading. Suppose you say

Jay may be out of town tomorrow,

and suppose I reply

yes, Jay may be out of town tomorrow.

Presumably, I am not granting Jay permission (although I might be!).

Still, there is at least a two-fold ambiguity. we might mean that

there is a subjective possibility that Jay will be out of town tomorrow;
in other words:

for all we know, Jay may be out of town tomorrow.

Or we might mean that

there is an objective possibility that Jay will be out of town tomorrow.
in other words:

no objective "fact" or "law" prohibits Jay from being out of town tomorrow.

The latter notion of possibility is furthermore ambiguous, since 'objective law' might refer to a logical law, or a metaphysical law, or perhaps a physical law.

In any case, there are roughly three basic brands of modal logic, corresponding respectively to the three ways we can use 'may'.

- | | | |
|-----|---------------------|--------------------------------|
| (1) | deontic logic | (permissibility) |
| (2) | epistemic logic | (for all we know) ⁶ |
| (3) | alethic modal logic | (objective possibility) |

⁶ Epistemic logic also includes the verbs 'knows' and 'believes', which we will later consider in the more general context of propositional attitude verbs.