

A4

Multi-Modal Systems

1.	Introduction.....	2
2.	Absolute Modal Logic Resurrected.....	2
	1. Absolute Necessity Rules.....	2
	2. Absolute Possibility Rules.....	2
3.	Examples of Derivations Involving Both Modalities.....	3
4.	Other Rules.....	4
	1. Strict-Arrow Rules.....	4
	2. Modal Negation Rules.....	4
	3. Short-Cut Modal Negation Rules.....	4
	4. Modal Repetition Rules.....	5
5.	Combining Different Relative Modalities.....	5
6.	Relatively-Absolute Modalities.....	7

1. Introduction

As we have seen, whereas there is essentially one absolute modal logic, characterized by System L, there are infinitely many relative modal logics. In the present chapter, we consider combining them into multi-modal systems.

2. Absolute Modal Logic Resurrected

The easiest combination to construct takes absolute modal logic and grafts it onto System K(+), after making appropriate notational adjustments. Let us now make that official. In particular, henceforth, we will use the modified box symbol ‘ \Box ’ to stand for the absolute necessity operator, and we will correspondingly use ‘ \Diamond ’ as the absolute possibility operator.

The rules have to be adjusted to take into account the nature of indices in Relative Modal Logic. As usual, i and j are indices (numerical sequences) and m and n are individual numerals.

1. Absolute Necessity Rules

$$\begin{array}{l} \Box O \\ \frac{\Box \mathcal{A} \quad /i}{\mathcal{A}} \quad /j \text{ (old)} \end{array}$$

$$\begin{array}{l} \Box D \\ \begin{array}{l} \text{SHOW: } \Box \mathcal{A} \quad /i \\ | \text{SHOW: } \mathcal{A} \quad /n \text{ (new)} \\ | \\ | \end{array} \end{array}$$

2. Absolute Possibility Rules

$$\begin{array}{l} \Diamond O \\ \frac{\Diamond \mathcal{A} \quad /i}{\mathcal{A}} \quad /n \text{ (new)} \end{array}$$

$$\begin{array}{l} \Diamond I \\ \frac{\mathcal{A} \quad /i \text{ (old)}}{\Diamond \mathcal{A}} \quad /j \end{array}$$

Notice the following changes.

- (1) Whereas the rules of the original System L only apply to singleton indices, the revised rules apply to all to our general indices, which are sequences of numerals.
- (2) The revised $\diamond O$ and $\Box D$ require a new singleton index – as indicated by the letter ‘n’.

So, for example, the following counts as a valid instance of our new $\Box O$ (supposing 0134 is old).

- | | | | |
|-----|----------|-------|-------------|
| (1) | $\Box P$ | /012 | |
| (2) | P | /0134 | 1, $\Box O$ |

And the following counts as a valid instance of the new $\diamond O$ (supposing 4 is new).

- | | | | |
|-----|--------------|------|-----------------|
| (1) | $\diamond P$ | /013 | |
| (2) | P | /4 | 1, $\diamond O$ |

3. Examples of Derivations Involving Both Modalities

Now that we have two sets of modal operators, and two sets of derivation rules, we can consider evaluating arguments involving combinations of \Box , \diamond , \Box , and \diamond . The following are examples of derivations, where we use System K for \Box and \diamond .

- | | | | |
|-----|-----------------------------------|-----|----------|
| (1) | SHOW: $\Box P \rightarrow \Box P$ | /0 | CD |
| (2) | $\Box P$ | /0 | As |
| (3) | SHOW: $\Box P$ | /0 | ND |
| (4) | SHOW: P | /01 | DD |
| (5) | P | /01 | $\Box O$ |
-
- | | | | |
|-----|--|-----|-----------------|
| (1) | SHOW: $\diamond P \rightarrow \Box \diamond P$ | /0 | CD |
| (2) | $\diamond P$ | /0 | As |
| (3) | SHOW: $\Box \diamond P$ | /0 | $\Box D$ |
| (4) | SHOW: $\diamond P$ | /1 | DD |
| (5) | P | /02 | 2, $\diamond O$ |
| (6) | $\diamond P$ | /1 | 5, $\diamond I$ |
-
- | | | | |
|-----|--|-----|-----------------|
| (1) | SHOW: $\diamond P \rightarrow \Box \diamond P$ | /0 | CD |
| (2) | $\diamond P$ | /0 | As |
| (3) | SHOW: $\Box \diamond P$ | /0 | $\Box D$ |
| (4) | SHOW: $\diamond P$ | /01 | DD |
| (5) | P | /2 | 2, $\diamond O$ |
| (6) | $\diamond P$ | /01 | 5, $\diamond I$ |
-
- | | | | |
|-----|--|-----|-----------------|
| (1) | SHOW: $\diamond \diamond P \rightarrow \diamond P$ | /0 | Pr |
| (2) | $\diamond \diamond P$ | /0 | As |
| (3) | SHOW: $\diamond P$ | /0 | DD |
| (4) | $\diamond P$ | /01 | 2, $\diamond O$ |
| (5) | P | /2 | 4, $\diamond O$ |
| (6) | $\diamond P$ | /0 | 5, $\diamond I$ |

4. Other Rules

There are other rules of absolute modal logic that we adopt.

1. Strict-Arrow Rules

$$\begin{array}{l} \text{Def } \prec \\ \mathcal{A} \prec \mathcal{B} \\ \hline \hline \Box(\mathcal{A} \rightarrow \mathcal{B}) \end{array}$$

$$\begin{array}{l} \text{Def } = \\ \mathcal{A} = \mathcal{B} \\ \hline \hline \Box(\mathcal{A} \leftrightarrow \mathcal{B}) \end{array}$$

2. Modal Negation Rules

$$\begin{array}{l} \text{MN} \\ \sim \Box \mathcal{A} \\ \hline \hline \Diamond \sim \mathcal{A} \\ \\ \sim \Diamond \mathcal{A} \\ \hline \hline \Box \sim \mathcal{A} \end{array}$$

3. Short-Cut Modal Negation Rules

$$\begin{array}{l} \sim \Box \mathcal{O} \\ \sim \Box \mathcal{A} \quad /i \\ \hline \sim \mathcal{A} \quad /m \text{ (new)} \end{array}$$

$$\begin{array}{l} \sim \Diamond \mathcal{O} \\ \sim \Diamond \mathcal{A} \quad /i \\ \hline \sim \mathcal{A} \quad /j \text{ (old)} \end{array}$$

4. Modal Repetition Rules

$\Box R$	$\frac{\Box \mathcal{A}}{\Box \mathcal{A}}$	$/i$ $/j$ (old)
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$\Diamond R$	$\frac{\Diamond \mathcal{A}}{\Diamond \mathcal{A}}$	$/i$ $/j$ (old)
--------------	---	--------------------

$\prec R$	$\frac{\mathcal{A} \prec \mathcal{B}}{\mathcal{A} \prec \mathcal{B}}$	$/i$ $/j$ (old)
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$=R$	$\frac{\mathcal{A} = \mathcal{B}}{\mathcal{A} = \mathcal{B}}$	$/i$ $/j$ (old)
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The last four rules are derived rules. They reflect the fact that absolute modal formulas are, well, absolute. A formula \mathcal{F} is absolute if and only if it is true everywhere or true nowhere.

5. Combining Different Relative Modalities

The absolute and relative modalities coexist in our system of modal logic. What happens if we try to combine two different relative modalities – say, a K4-modality and a KD-modality. For example, one might use a K4-modality to represent belief, as in epistemic logic, and a KD-modality to represent obligation, as in deontic logic. One might wish to combine these two modalities with two alethic modalities – a KT-modality, and an absolute modality.

In that case, at a minimum, one must use alternative symbols. The following seem natural in this situation.

\Box	absolute necessity	it is absolutely necessary that...
\square	relative alethic necessity	it is necessary that
\mathbb{B}	belief	...believes that...
\bigcirc	obligation	it ought to be the case that...

The symbolization is no problem. For example, the following multiply-modal sentence

Jay believes Kay believes Jay ought to be fair

is symbolized:

$$j\mathbb{B}k\mathbb{B}\mathbb{O}Fj$$

Symbolization is one thing; logic is another. Suppose we are asked to show this [based on some set of premises]. Consider the following initial derivation attempt.

(1)	SHOW: $j\mathbb{B}k\mathbb{B}\mathbb{O}Fj$	/0	$[j\mathbb{B}]D$
(2)	SHOW: $k\mathbb{B}\mathbb{O}Fj$	/01	$[k\mathbb{B}]D$
(3)	SHOW: $\mathbb{O}Fj$	/012	$\mathbb{O}D$
(4)	SHOW: Fj	/0123	

Note that $[j\mathbb{B}]$, $[k\mathbb{B}]$, and \mathbb{O} are box-modalities. Still, this is seriously flawed, because it treats all three modalities as the same modality. It is no different from:

(1)	SHOW: $\square\square\square Fj$	/0	$\square D3$
(2)	SHOW: Fj	/0123	

What we need is a scheme to keep separate all the different accessibility relations that are implicit in the above sentence – listed as follows.

0	engenders	01	in respect to	Jay's beliefs
01	engenders	012	in respect to	Kay's beliefs
012	engenders	0123	in respect to	obligation

So we need some way to encode this information in the derivation. No system will be entirely free of ugliness. We propose the following encoding.

- (a) indices are finite sequences of alpha-numeric characters
- (b) i engenders $i+\alpha+m$ in respect to α
- (c) the null alpha-index refers to (relative) alethic modality

So for example, we can use the following scheme for the example above.

0	engenders	0j1	in respect to	Jay's beliefs
0j1	engenders	0j1k2	in respect to	Kay's beliefs
0j1k2	engenders	0j1k2o3	in respect to	obligation

And, as before, 0 engenders 01, which engenders 012, etc., in respect to relative alethic necessity.

So, when we go back and clean up our earlier derivation, we obtain:

(1)	SHOW: $j\mathbb{B}k\mathbb{B}\mathbb{O}Fj$	/0	$[j\mathbb{B}]D$
(2)	SHOW: $k\mathbb{B}\mathbb{O}Fj$	/0j1	$[k\mathbb{B}]D$
(3)	SHOW: $\mathbb{O}Fj$	/0j1k2	$\mathbb{O}D$
(4)	SHOW: Fj	/0j1k2o3	

If we want to throw in relative alethic modality, here and there, we can produce the following example.

(1)	SHOW: $\Box j \mathbb{B} \Box k \mathbb{B} \bigcirc Fj$	/0	$\Box D$
(2)	SHOW: $j \mathbb{B} \Box k \mathbb{B} \bigcirc Fj$	/01	$[j \mathbb{B}] D$
(3)	SHOW: $\Box k \mathbb{B} \bigcirc Fj$	/01j2	$\Box D$
(4)	SHOW: $k \mathbb{B} \bigcirc Fj$	/01j23	$[k \mathbb{B}] D$
(5)	SHOW: $\bigcirc Fj$	/01j23k4	$\bigcirc D$
(6)	SHOW: Fj	/01j23k4o5	

6. Relatively-Absolute Modalities

Once we adopt the new alpha-numeric indexing system, we can reconsider alethic modal logic. In particular, we can devise a tidy system that combines absolute alethic modality with various non-alethic modalities. Specifically, we can go back and revise our rules for System L, to bring them into accord with the new alpha-numeric indexing system. This may be done as follows.

$\Box O$	$\Box \mathcal{A}$	/i
	\mathcal{A}	/i-1+m (old)

$\Box D$	SHOW: $\Box \mathcal{A}$	/i
	SHOW: \mathcal{A}	/i-1+n (new)

$\Diamond O$	$\Diamond \mathcal{A}$	/i
	\mathcal{A}	/i-1+n (new)

$\Diamond I$	\mathcal{A}	/i
	$\Diamond \mathcal{A}$	/i-1+m (old)

Notice the new notation (which will be seen again later in a completely different context). Specifically,

$i-1 \quad \text{=}_{df} \quad$ the sequence obtained by removing the last item in i
 [which is the null sequence, if i is a singleton.]

Notice that this reduces to our earlier rules for Absolute Modal Logic when all the indices are singleton sequences, as in System L. However, we don't obtain this simplification for general indices.

On the one hand, in the case of general numerical indices, these rules are equivalent to a combination of the ϵ -rules and the 5-rules (exercise). In other words, we have devised a more concise formulation of System KT5 (S5)!

On the other hand, in the case of general alpha-numeric indices, we have yet another L-like system. To see this, consider the following incomplete multi-modal derivation.

necessarily P			
/ Jay believes that necessarily P			
(1)	$\Box P$	/0	Pr
(2)	SHOW: $j\mathbb{B}\Box P$	/0	$[j\mathbb{B}]D$
(3)	SHOW: $\Box P$	/0j1	$\Box D(s5)$
(4)	SHOW: P	/0j2	???

As we can see, this argument form is invalid, because we cannot go from $\Box P/0$ to $P/0j2$. In other words, alethic modalities in Jay's "belief worlds" need not correspond to alethic modalities in the "real world", even if both modalities are (in some sense) absolute modalities.

If we try to reproduce this result using our original absolute alethic modality, we obtain the following completed derivation.

absolutely P			
/ Jay believes that absolutely P			
(1)	$\Box P$	/0	Pr
(2)	SHOW: $j\mathbb{B}\Box P$	/0	$[j\mathbb{B}]D$
(3)	SHOW: $\Box P$	/0j1	$\Box D$
(4)	SHOW: P	/2	DD
(5)	P	/2	$1, \Box O$

This may sound very odd. But if we take \Box to range over all worlds, including metaphysically possible worlds, and including goofy Jay-belief worlds, even including metaphysically impossible Jay-belief worlds, then this result is completely acceptable.

If we want the argument to fail, but still retain something resembling an absolute alethic modality, then we want to use the "relatively absolute" \Box -operator given above.