A2 Rules of Derivation

1. Sentential Logic 2
    1. Inference Rules 2
    2. Strategic Rules 3
2. System L 4
    1. Basics 4
    2. Modal Rules 4
    3. Modal-Negation Rules 4
    4. Short-Cut Modal-Negation Rules 4
    5. Strict Conditional Rules 4
3. System K and its Extensions 5
    1. Basics 5
    2. Modal Rules 5
    3. Modal-Negation Rules 5
    4. Short-Cut Modal-Negation Rules 5
    5. Strict-Arrow Rules 6
    6. Rules For Systems That Extend K [K+ Systems] 6
4. Free First-Order Modal Logic with Actuality 7
    1. Basics 7
    2. Constants 7
    3. Basic Quantifier Rules 7
    4. Actualist Quantifiers 8
    5. Simple Identity Rules 8
    6. Identity Repetition 8
    7. Leibniz's Law – Modal Version 8
    8. Description Rules 9
    9. Scoped-Terms 9
    10. Rigid Actuality 9
    11. Flexible Actuality 9
5. Two-Dimensional Modal Logic (K-version) 10
1. Second-Order Modal Logic (Monadic Plurals) 11
    1. Quantifier Rules 11
    2. Identity Rules 11
    3. Further Second-Order Rules 12
2. Short-Cut Rules 13
    1. Short-Cut Rules for Modal Sentential Logic 13
    2. Short-Cut Rules for First-Order Modal Logic 13

1. Sentential Logic

Henceforth, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are closed formulas.

1. Inference Rules

&I	**&O**		**~&I/O**
$\mathcal{A}$ $\mathcal{B}$ ——— $\mathcal{A}\&\mathcal{B}$	$\mathcal{A}\&\mathcal{B}$ ——— $\mathcal{A}$	$\mathcal{A}\&\mathcal{B}$ ——— $\mathcal{B}$	$\sim(\mathcal{A}\&\mathcal{B})$ ===== $\mathcal{A} \rightarrow \sim\mathcal{B}$

vI		**vO**		**~vI/O**
$\mathcal{A}$ ——— $\mathcal{A}\vee\mathcal{B}$	$\mathcal{B}$ ——— $\mathcal{A}\vee\mathcal{B}$	$\mathcal{A}\vee\mathcal{B}$ $\sim\mathcal{A}$ ——— $\mathcal{B}$	$\mathcal{A}\vee\mathcal{B}$ $\sim\mathcal{B}$ ——— $\mathcal{A}$	$\sim(\mathcal{A}\vee\mathcal{B})$ ===== $\sim\mathcal{A}\ \&\ \sim\mathcal{B}$

→I		**→O**		**~→I/O**
$\sim\mathcal{A}$ ——— $\mathcal{A}\rightarrow\mathcal{B}$	$\mathcal{B}$ ——— $\mathcal{A}\rightarrow\mathcal{B}$	$\mathcal{A}\rightarrow\mathcal{B}$ $\mathcal{A}$ ——— $\mathcal{B}$	$\mathcal{A}\rightarrow\mathcal{B}$ $\sim\mathcal{B}$ ——— $\sim\mathcal{A}$	$\sim(\mathcal{A}\rightarrow\mathcal{B})$ ===== $\mathcal{A}\ \&\ \sim\mathcal{B}$

↔I	**↔O**		**~↔I/O**
$\mathcal{A}\rightarrow\mathcal{B}$ $\mathcal{B}\rightarrow\mathcal{A}$ ——— $\mathcal{A}\leftrightarrow\mathcal{B}$	$\mathcal{A}\leftrightarrow\mathcal{B}$ ——— $\mathcal{A}\rightarrow\mathcal{B}$	$\mathcal{A}\leftrightarrow\mathcal{B}$ ——— $\mathcal{B}\rightarrow\mathcal{A}$	$\sim(\mathcal{A}\leftrightarrow\mathcal{B})$ ===== $\sim\mathcal{A}\leftrightarrow\mathcal{B}$

※I	**※O**	**DN**
$\mathcal{A}$ $\sim\mathcal{A}$ ——— ※	※ ——— $\mathcal{A}$	$\sim\sim\mathcal{A}$ ===== $\mathcal{A}$

Note: The ~O and ~I rules are combined, using a long equals sign '==='. Henceforth, any rule that is displayed with '===' is a bi-directional rule, which can be used both as an in-rule and as an out-rule.

2. Strategic Rules

Direct Derivation (DD)

~~SHOW~~: $\mathcal{A}$	DD
$\mathcal{A}$	

(Generic) Indirect Derivation (ID)

~~SHOW~~: $\mathcal{A}$	ID
$\sim\mathcal{A}$	As
~~SHOW~~: ⨳	

Conditional Derivation (CD)

~~SHOW~~: $\mathcal{A}\to\mathcal{B}$	CD
$\mathcal{A}$	As
~~SHOW~~: $\mathcal{B}$	

Tilde Derivation (~D)

~~SHOW~~: $\sim\mathcal{A}$	~D
$\mathcal{A}$	As
~~SHOW~~: ⨳	

Ampersand Derivation (&D)

~~SHOW~~: $\mathcal{A}\&\mathcal{B}$	&D
~~SHOW~~: $\mathcal{A}$	
~~SHOW~~: $\mathcal{B}$	

Biconditional Derivation (↔D)

~~SHOW~~: $\mathcal{A}\leftrightarrow\mathcal{B}$	↔D
~~SHOW~~: $\mathcal{A}\to\mathcal{B}$	
~~SHOW~~: $\mathcal{B}\to\mathcal{A}$	

Wedge (Indirect) Derivation (∨D)

~~SHOW~~: $\mathcal{D}_1 \vee \mathcal{D}_2 \vee \ldots \vee \mathcal{D}_k$	∨D
$\sim\mathcal{D}_1$	As
$\sim\mathcal{D}_2$	As
…	
$\sim\mathcal{D}k$	
~~SHOW~~: ⨳	

Separation of Cases (SC)

$\mathcal{D}_1 \vee \mathcal{D}_2 \vee \ldots \vee \mathcal{D}_k$	
~~SHOW~~: $\mathcal{C}$	SC
c_1: $\mathcal{D}_1$	As
~~SHOW~~: $\mathcal{C}$	
c_2: $\mathcal{D}_2$	As
~~SHOW~~: $\mathcal{C}$	
.	
.	
.	
c_k: $\mathcal{D}_k$	As
~~SHOW~~: $\mathcal{C}$	

2. System L

1. Basics

Henceforth, i, j, k are index points [which in System L are represented by (decimal Arabic) numerals]. Henceforth, in any given rule, if indices are not explicitly indicated, then it is understood that the premises and conclusion of that rule pertain to the same index. This applies (retroactively) to the SL-rules, except for contradiction (※) lines. Contradictions are "absolute", and hence do not require indices, but for the sake of symmetry, we mark them all by "wildcard" index symbol '✳'.

An index counts as ***old*** precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as ***new***.

2. Modal Rules

□O	□D (ND)
$\Box\mathcal{A}$ /i ——— $\mathcal{A}$ /j (old)	~~SHOW~~: $\Box\mathcal{A}$ /i | ~~SHOW~~: $\mathcal{A}$ /k (new) | |

◇O	◇I
$\Diamond\mathcal{A}$ /i ——— $\mathcal{A}$ /k (new)	$\mathcal{A}$ /i (old) ——— $\Diamond\mathcal{A}$ /j

3. Modal-Negation Rules

MN	
$\sim\Box\mathcal{A}$ ═══════ $\Diamond\sim\mathcal{A}$	$\sim\Diamond\mathcal{A}$ ═══════ $\Box\sim\mathcal{A}$

4. Short-Cut Modal-Negation Rules

~□O = MN+◇O	~◇O = MN+□O
$\sim\Box\mathcal{A}$ /i ——— $\sim\mathcal{A}$ /k (new)	$\sim\Diamond\mathcal{A}$ /i ——— $\sim\mathcal{A}$ /j

5. Strict Conditional Rules

Def ⥽	Def =
$\mathcal{A}$⥽$\mathcal{B}$ ═══════ $\Box(\mathcal{A}\rightarrow\mathcal{B})$	$\mathcal{A}=\mathcal{B}$ ═══════ $\Box(\mathcal{A}\leftrightarrow\mathcal{B})$

3. System K and its Extensions

1. Basics

In what follows, i, j, k are indices [which in System K and its extensions are sequences of numerals]. Also, m and n are numerals, and i+m is the result of appending m to i.

An index counts as ***old*** precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as ***new***.
Unless stated otherwise, all indices must be old, which applies (retroactively) to SL rules. In particular, a zero-place rule may be applied only if the index is old.

2. Modal Rules

□O	□D (ND)
□$\mathcal{A}$ /i ——— $\mathcal{A}$ /i+m (old)	~~SHOW~~: □$\mathcal{A}$ /i │ SHOW: $\mathcal{A}$ /i+n (new) │ │

◇O	◇I
◇$\mathcal{A}$ /i ——— $\mathcal{A}$ /i+n (new)	$\mathcal{A}$ /i+m (old) ——— ◇$\mathcal{A}$ /i

3. Modal-Negation Rules

MN	
~□$\mathcal{A}$ ═════ ◇~$\mathcal{A}$	~◇$\mathcal{A}$ ═════ □~$\mathcal{A}$

4. Short-Cut Modal-Negation Rules

~□O = MN+◇O	~◇O = MN+□O
~□$\mathcal{A}$ /i ——— ~$\mathcal{A}$ /i+n (new)	~◇$\mathcal{A}$ /i ——— ~$\mathcal{A}$ /i+m (old)

5. Strict-Arrow Rules

Just like System L, but we basically ignore strict arrow and strict double-arrow in System K and its extensions.

6. Rules For Systems That Extend K [K+ Systems]

□O(k)		◇I(k)	
$\Box\mathcal{A}$	$/i$	$\mathcal{A}$	$/i{+}m$ (old)
$\mathcal{A}$	$/i{+}m$ (old)	$\Diamond\mathcal{A}$	$/i$

□O(d)		◇I(d)	
$\Box\mathcal{A}$	$/i$	$\mathcal{A}$	$/i{+}m$ (new)
$\mathcal{A}$	$/i{+}m$ (new)	$\Diamond\mathcal{A}$	$/i$

□O(b		◇I(b)	
$\Box\mathcal{A}$	$/i{+}m$ (old)	$\mathcal{A}$	$/i$
$\mathcal{A}$	$/i$	$\Diamond\mathcal{A}$	$/i{+}m$ (old)

□O(4)		◇I(4)	
$\Box\mathcal{A}$	$/i$	$\mathcal{A}$	$/i{+}m{+}n$ (old)
$\mathcal{A}$	$/i{+}m{+}n$ (old)	$\Diamond\mathcal{A}$	$/i$

□O(5)		◇I(5)	
$\Box\mathcal{A}$	$/i{+}m$ (old)	$\mathcal{A}$	$/i{+}m$ (old)
$\mathcal{A}$	$/i{+}n$ (old)	$\Diamond\mathcal{A}$	$/i{+}n$ (old)

NOTE: the rules ND and ◇O are the same for all K+ systems, and hence carry no suffixes.

4. Free First-Order Modal Logic with Actuality

1. Basics

In what follows, Φ is a formula, in which ν is the only variable (if any) that occurs free, and $\Phi[\varepsilon/\nu]$ is the formula that results when ε replaces every occurrence of ν that is free in Φ. In what follows, σ, τ, ρ are *closed singular-terms*. An expression is ***closed*** iff it contains no *free occurrence* of any variable. An occurrence of a variable ν is ***free*** in expression $\mathcal{E}$ iff that occurrence does not lie within the scope of an operator binding ν – i.e., $\forall\nu$, $\exists\nu$, $\iota\nu$, or (τ/ν).

2. Constants

A ***constant*** is an atomic singular-term that is introduced by UD or ∃O.
A constant counts as ***old*** precisely when it occurs in a line that is neither boxed nor cancelled; otherwise, it counts as ***new***.

3. Basic Quantifier Rules

Universal-Out (∀O)	Existential-In (∃I)
$\forall\nu\Phi$ ――― $\Phi[o/\nu]$	$\Phi[o/\nu]$ ――― $\exists\nu\Phi$
o is any ***old*** constant.	

Universal-Derivation (UD)	Existential-Out (∃O)
~~SHOW~~: $\forall\nu\Phi$ │ ~~SHOW~~: $\Phi[n/\nu]$ │ │	$\exists\nu\Phi$ ――― $\Phi[n/\nu]$
n is any ***new*** constant.	

Quantifier-Negation	
$\sim\forall\nu\Phi$ ═══ $\exists\nu\sim\Phi$	$\sim\exists\nu\Phi$ ═══ $\forall\nu\sim\Phi$

Tilde-Universal-Out (~∀O)	**Tilde-Existential-Out** (~∃O)
$\sim\forall\nu\Phi$ ――― $\sim\Phi[n/\nu]$	$\sim\exists\nu\Phi$ ――― $\sim\Phi[o/\nu]$
n is any ***new*** constant.	*o* is any ***old*** constant.

4. Actualist Quantifiers

Def ∀′	Def ∃′
$\forall'\nu\Phi$ ═════════ $\forall\nu(Æ[\nu] \rightarrow \Phi)$	$\exists'\nu\Phi$ ═════════ $\exists\nu(Æ[\nu]\ \&\ \Phi)$

∀′O = Def ∀′ + ∀→O	∃′O = Def ∃′ + ∃O
$\forall'\nu\Phi$ $Æ[o]$ ――――― $\Phi[o/\nu]$	$\exists'\nu\Phi$ ――――――――― $Æ[n]\ \&\ \Phi[n/\nu]$
o is any ***old*** constant.	*n* is any ***new*** constant.

5. Simple Identity Rules

Reflexivity (R=)	**Symmetry** (S=)	**Transitivity** (T=)
$\varnothing$ ――――― $\sigma = \sigma$	$\sigma = \tau$ ――――― $\tau = \sigma$	$\rho = \sigma$ $\sigma = \tau$ ――――― $\rho = \tau$
σ, τ, and ρ are any closed singular-terms.		

6. Identity Repetition

=Rep
$c_1 = c_2$ /i   ――――― $c_1 = c_2$ /j
Here, c_1 and c_2 are both constants.

7. Leibniz's Law – Modal Version

LL	
$\Phi[\sigma/\nu]$ $\sigma = \tau$ ――――― $\Phi[\tau/\nu]$	$\Phi[\sigma/\nu]$ $\tau = \sigma$ ――――― $\Phi[\tau/\nu]$
either both σ and τ are constants, or ν does not occur inside the scope of a modal operator.	

8. Description Rules

Iota-Out (ιO)	**Iota-In** (ιI)
$c = \iota\nu\Phi$ ———————— $\forall\nu(\Phi \leftrightarrow \nu{=}c)$	$\forall\nu(\Phi \leftrightarrow \nu{=}c)$ ———————— $c = \iota\nu\Phi$
c must be a ***constant***.	

9. Scoped-Terms

Def (τ/ν)
$(\tau/\nu)\Phi$ ========== $\exists\nu\{\nu{=}\tau \;\&\; \Phi\}$
Φ is any formula, ν is any variable, and τ is any closed singular-term.

10. Rigid Actuality

⊙O/⊙I	⊙O/⊙I
$c = \odot\tau$ /i ====== $c = \tau$ /$o(i)$	Φ /$o(i)$ === $\odot\Phi$ /i
c is any constant, and τ is any closed singular-term; $o(i)$ is the origin of i	

11. Flexible Actuality

○O/○I	○O/○I
$c = \bigcirc\tau$ /i ====== $c = \tau$ /$i{-}1$	Φ /$i{-}1$ === $\bigcirc\Phi$ /i
c is any constant, and τ is any closed singular-term; $i{-}1$ is the immediate predecessor of i.	
$i{-}1$ = the sequence obtained from i by removing its last element, *provided* i has at least two elements; $i{-}1$ is undefined if i does not have at least two elements.	

5. Two-Dimensional Modal Logic (K-version)

□O	□D
□Φ /*i* /*j* ――― Φ /*i*+*m* (old) /*j*	~~SHOW~~: □Φ /*i* /*j* ~~SHOW~~: Φ /*i*+*n* (new) /*j* |

◇O	◇I
◇Φ /*i* /*j* ――― Φ /*i*+*n* (new) /*j*	Φ /*i*+*m* (old) /*j* ――― ◇Φ /*i* /*j*

○O	○I
○$\mathcal{A}$ /*i* /*j* ――― $\mathcal{A}$ /*j* /∅	$\mathcal{A}$ /*j* /∅ ――― ○$\mathcal{A}$ /*i* /*j*

$O	$I
$$\mathcal{A}$ /*i* /*j* ――― $\mathcal{A}$ /*i* /*i*	$\mathcal{A}$ /*i* /*i* ――― $$\mathcal{A}$ /*i* /*j*

6. Second-Order Modal Logic (Monadic Plurals)

1. Quantifier Rules

Universal-Out (∀O)	Existential-In (∃I)
$\forall V\Phi$ ——— $\Phi[\mathbf{P}/V]$	$\Phi[\mathbf{P}/V]$ ——— $\exists V\Phi$
V is any monadic-predicate variable, and **P** is any closed monadic-predicate.	

Universal-Derivation (UD)	Existential-Out (∃O)
~~SHOW~~: $\forall V\Phi$ | ~~SHOW~~: $\Phi[N/V]$ | |	$\exists V\Phi$ ——— $\Phi[N/V]$
N is any ***new*** monadic-predicate constant.	

Quantifier Negation (QN)	
$\sim\forall V\Phi$ ═══ $\exists V\sim\Phi$	$\sim\exists V\Phi$ ═══ $\forall V\sim\Phi$

Tilde-Universal-Out (~∀O)	Tilde-Existential-Out (~∃O)
$\sim\forall V\Phi$ ——— $\sim\Phi[N/V]$	$\sim\exists V\Phi$ ——— $\sim\Phi[\mathbf{P}/V]$
N is any ***new*** monadic-predicate constant.	**P** is any closed monadic-predicate.

2. Identity Rules

Reflexivity (R=)	**Symmetry** (S=)	**Transitivity** (T=)
∅ ——— $\mathbf{P} = \mathbf{P}$	$\mathbf{P} = \mathbf{Q}$ ——— $\mathbf{Q} = \mathbf{P}$	$\mathbf{P} = \mathbf{Q}$ $\mathbf{Q} = \mathbf{R}$ ——— $\mathbf{P} = \mathbf{R}$
Here, **P**, **Q**, **R**, are *closed* monadic-predicates.		

Leibniz's Law (LL)	
$\Phi[\mathbf{P}/V]$ $\mathbf{P} = \mathbf{Q}$ ——— $\Phi[\mathbf{Q}/V]$	$\Phi[\mathbf{P}/V]$ $\mathbf{Q} = \mathbf{P}$ ——— $\Phi[\mathbf{Q}/V]$
V is **modally free** in Φ for both **P** and **Q**.	
α is ***modally free*** in Φ for β if and only if α does not occur inside the scope of a modal operator, or β is a constant.	

Identity Repetition (=Rep)	**Extensionality** (Ext)
$A = B$ /*i* ——— $A = B$ /*j* (old)	$\forall x(\mathbf{P}x \leftrightarrow \mathbf{Q}x)$ ——— $\mathbf{P} = \mathbf{Q}$
A and B are monadic-predicate ***constants***.	**P** and **Q** are closed monadic predicates.

In other words, just as with identity among individual constants, identity among monadic-predicate constants is absolute.

3. Further Second-Order Rules

Scoped-Predicate Rule – Def (P/V)	**Lambda-Conversion** (λC)
$(\mathbf{P}/V)\Phi$ ═══════ $\exists V\{V{=}\mathbf{P}\ \&\ \Phi\}$	$[\lambda\nu\Phi][c]$ ═══════ $\Phi[c/\nu]$
P is any closed monadic predicate.	c must be an individual **constant**.

Predicate Existence (E![P])	**Predication Repetition** (P-Rep)
$\varnothing$ ——— $\exists X[X{=}\mathbf{P}]$	$A[c]$ /*i* ——— $A[c]$ /*j* (old)
P is any closed monadic predicate.	A is a monadic-predicate **constant**; c is an individual **constant**.

7. Short-Cut Rules

1. Short-Cut Rules for Modal Sentential Logic

1. The Rule SL

If a line can be derived from available lines using only SL rules, then it may be written down by the rule SL. This rule may be used in place of any combination of SL rules, including inference rules and show rules.

2. The Conjunction Rule

Any available conjunctive line (with any number of conjuncts) can be treated as the appropriate number of separate lines, numbered (e.g.) 7a, 7b, 7c. And conversely, any number of available lines can be treated as the corresponding conjunction.

3. Contraposition Rule

For every *genuine* one-place rule ℝ, there is an associated contrapositive rule ℝ(–), which is obtained by reversing and negating and the premise and conclusion.
NOTE CAREFULLY: ◇O, and ∃O are *not* genuine *inference rules*, but are rather ***assumption rules*.**

4. Rule-Multiplication

Any one-place rule can be multiplied, **provided** the particular rule also applies to the intermediate line. For example, ◇O+◇O = ◇O2; ~◇O+~◇O = ~◇O2.

5. The Immediate Show-Cancel Rule

If a show-line follows from available lines (**earlier** or **later**!) by a rule, then it can be cancelled by that rule. Annotation: cite the line number(s) and the rule.

2. Short-Cut Rules for First-Order Modal Logic

1. The Rule QL

If a line can be derived from previous available lines using only quantifier rules, but it cannot be derived using just SL rules, then it may be written down by the rule QL. This rule may be used in place of any QL inference rule, as well as any combination of QL rules, including inference rules and show rules.
NOTE CAREFULLY: The rule ∃O, and hence ~∀O, are **not** genuine inference rules, but **assumption rules**. No instance of ∃O is valid by QL! So when you cite ∃O or ~∀O, **do not** cite it as QL.

2. The Rule EQ(=)

If a line can be derived from previous available lines using only the simple identity-rules [i.e., R=, S=, T=] together with SL, then it may be written down by the rule EQ(=) [short for '= is an equivalence relation'].

3. Rule Combinations

As before, any one-place rule can be multiplied, so long as the rule applies to the intermediate formula(s). Notation – ∀O2, ∀O3, etc., UD2, UD3, etc.

Also ∀O, ∀O2, etc. can be combined with →O to produce ∀→O, ∀2→O, etc., and UD, UD2, etc., can be combined with CD to produce UCD, U2CD, etc.

Similarly, UD, UD2, etc., can be combined with ↔D to produce UBD, U2BD, etc.

4. A Few More Short-Cuts

LL(–)

$\Phi[\sigma/\nu]$ $\sim\Phi[\tau/\nu]$ ——— $\sigma \neq \tau$	$\sim\Phi[\sigma/\nu]$ $\Phi[\tau/\nu]$ ——— $\sigma \neq \tau$

ν must be modally-free in Φ for both σ and τ.

Only-Out (OO)

$\forall\nu(\Phi \leftrightarrow \nu{=}c)$
———
$\Phi[c/\nu]$

c is a constant.

Alphabetic Variance (AV)

$\Phi[\boldsymbol{u}]$
———
$\Phi[\boldsymbol{v}]$

Here, ***u***, ***v*** are variables, $\Phi[\boldsymbol{u}]$ is a formula in which ***v*** does *not* occur, $\Phi[\boldsymbol{v}]$ is a formula in which ***u*** does *not* occur, and $\Phi[\boldsymbol{v}]$ results when every occurrence of ***u*** in $\Phi[\boldsymbol{u}]$ is replaced by an occurrence of ***v***.