Scales in Music

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1. Introduction

As noted earlier, the use of the word 'scale' in measurement theory is closely related to its use in music; in particular, this use of 'scale' derives from the Latin word *sc³lae* [ladder]. In this section we examine musical scales from the viewpoint of measurement theory (and of course music theory) . As we will see, from the viewpoint of measurement theory, a musical scale is basically an interval scale; indeed, in music theory, the distances between notes are even called 'intervals'! The question then is – how are these intervals measured?

Music is based on pitch. For example, when a guitar string is plucked, it vibrates, thereby producing sound with a characteristic fundamental frequency, which we perceive as *pitch* or *tone*. Pitch recognition is part of the human sensory endowment; we innately recognize when frequency/pitch goes up, and when it goes down. The evolution of this ability is most likely associated with communication, which played a vital role in the development of modern humans.[1] Pitch production and recognition

[1] Needless to say, humans are not the only animals that sing, or even the only animals that produce "messages" involving pitch.

allows for more subtle and varied communication. This shows up in modern languages, in which pitch distinctions are crucial for conveying grammatical (and even lexical[2]) information. For example, the *question* "you *did* ?" is intonated quite differently from the *statement* "you did"; this tonal distinction would be impossible without our ability to produce and hear pitch.

Singing basically involves producing sequences of syllable-tones. Some sequences seem more agreeable than others; and some combinations of pitches seem more harmonious than others. This is partly cultural, and it is partly psycho-physical. In any case, over the course of history, by way of organizing pitch combinations, various human cultures have devised various *musical scales*.

It is quite difficult to give a simple definition of musical scale, but it is extremely simple to give an example. In the Western world, probably the best known (and most widely used) musical scale is the *major scale*

do re mi fa so la ti do

which virtually everyone can recite, and which is immortalized in a song from the movie "Sound of Music".[3]

What is crucial to realize in describing the major scale is that the scale-values (do, re, …) do not correspond to particular pitches; for example, do does not correspond to a frequency of 400 Hz,[4] or any other frequency. In other words, a musical scale does not correspond to a conventional ratio-scale, like weight or volume. On the other hand, a musical scale is more than an ordinal scale; there is more to a musical scale than the order of the pitches, low to high. Rather, a major scale is a special kind of interval scale – one defined by the pitch-relations (intervals) among the various scale-values.

In addition to the major scale, there have been thousands of other scales that have been theoretically proposed and studied, and there have been hundreds of scales that have been actually used by people to produce music. What further complicates the study of musical scales is that the major scale is not a singular object, since there are different ways of calibrating its different components. In what follows, first we examine the various ways in which the major scale can be calibrated (i.e., tuned), and second we examine a few alternative musical scales, both theoretical and practical.

2. The Fundamental Unit – The Octave

The first scientific construction of the major scale was accomplished by Pythagoras (569-475 BC). Pythagoras discovered that the *fundamental* harmonious pitch-combinations correspond to the following mathematical ratios.

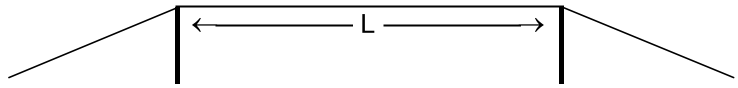
1:1 2:1 3:2

[2] In conveying lexical information, I have in mind tonal languages, the most notable of which are the various Chinese languages. In a tonal language, a given syllable has different meanings according to how it is intonated. These tonal distinctions make phonetic transcriptions very difficult, which partly explains why the official written language of China is not phonetic, but is instead logographic.

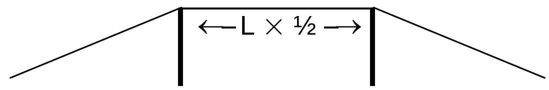
[3] This used to be a classic children's song – "do, a deer, a female deer; re, a drop of golden sun; mi, a name I call myself; fa, a long long way to run; so, a needle pulling thread; la, a note that follows so; ti, a drink with jam and bread; and that brings us back to do".

[4] The unit of frequency is cycles per second, which are called "Hertz" (abbreviated Hz), after the German physicist Heinrich Rudolf Hertz (1857-94), who experimentally demonstrated that energy can be transmitted in electromagnetic waves, which travel at the speed of light and which possess many other properties of light. These discoveries ultimately led to the development of all the wonderful "wireless" communication devices – telegraph, radio, television, etc.

In this section, we concentrate on the 1:1 and 2:1 ratio. By way of an idealized reconstruction of Pythagoras's experimental setup, we imagine constructing a single-string "guitar" by stretching a musical string between two "bridges", separated by a distance of L, as follows.

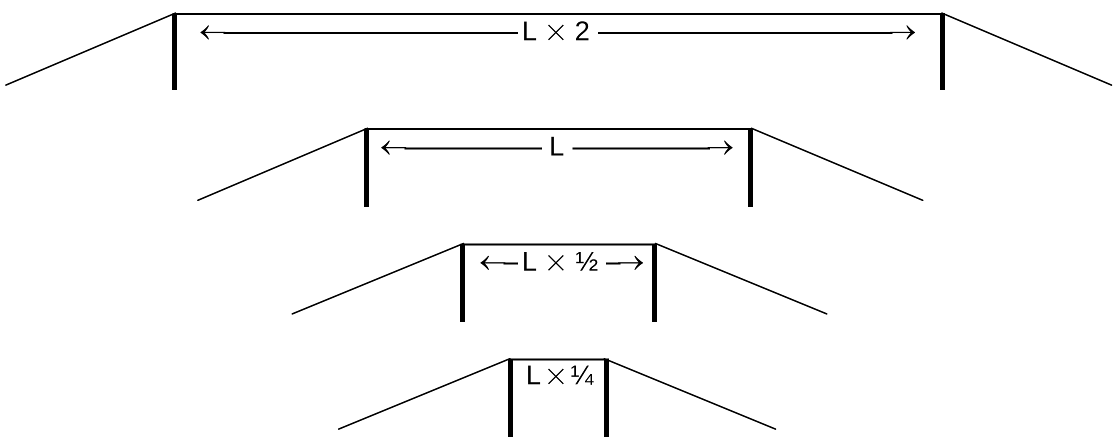


Now, when we pluck the string, it produces a characteristic pitch. Next, we construct another such instrument, but one in which the bridge distance is one-half L.



Now, here is the trick. If we make sure that the tensions are the same for both strings, the resulting pitches will be very harmonious. From the modern viewpoint, the shorter string will vibrate at a frequency that is exactly *twice* that of the longer string. This ratio is so fundamental to human pitch-recognition that two pitches that stand in this ratio are regarded as instances of the very same "note". For example, if I sing a 440 Hz tone, and you sing an 880 Hz tone, we are basically singing the same note – which is called "A" according to modern tuning standards.[5] This interchangeability of note pitches is what enables hundreds of people to collectively sing the same hymn or anthem, but in widely different registers (e.g., bass, baritone, tenor, alto, and soprano).

The procedure described above can be repeated, in either direction, so that we could for example produce the following series of single-string guitars.



Under appropriate conditions, these strings would then produce frequencies with the following ratios.

1:2 1:1 2:1 4:1

There is a natural ordering of these ratios, from smaller to larger, so we have a natural ordinal scale. This is in turn part of an infinite ordinal scale given as follows.

[5] The modern concert A is 440 Hz, which is the result of standardization. Over the centuries, many other pitches have been used for tuning a "concert A". For example, in the premier of Handel's *Messiah*, Handel used a tuning fork (which we still have!) that vibrates at 423 Hz. The difference from modern tuning is partly a matter of aesthetics, but is also partly a matter of technology. We can now make instrument components that accept higher tensions.

… $\frac{1}{4}$ < $\frac{1}{2}$ < $\frac{1}{1}$ < $\frac{2}{1}$ < $\frac{4}{1}$ < $\frac{8}{1}$ …

A particular scale of frequencies can be produced by assigning 1/1 to a particular frequency – say 220Hz – which yields the following derivative scale.

… 55 Hz … 110 Hz < 220 Hz < 440 Hz < 880 Hz < 1760 Hz …

Notice that this scale has neither a beginning nor an end, although of course we quickly exceed normal human hearing in either direction.[6]

So far, we have an ordinal scale with (in principle) neither a beginning nor an end. This can be converted into an interval scale by declaring that all intervals between adjacent ratios/frequencies are equal.

$\frac{1}{4}$ – $\frac{1}{2}$ = $\frac{1}{2}$ – $\frac{1}{1}$ = $\frac{1}{1}$ – $\frac{2}{1}$ = 1 octave

55 Hz – 110 Hz = 110 Hz – 220 Hz = 220 Hz – 440 Hz = 1 octave

The resulting interval-unit is musically fundamental, and is called an "octave"[7]; it is the tonal difference between a pitch and the next pitch with the same note-name. For example, 220 Hz corresponds to the note A, and 440 Hz is the next frequency that corresponds to A; so the interval between 220 Hz and 440 Hz is one octave. Similarly, the interval between 440 Hz and 880 Hz is one octave.

Notice carefully that frequency-differences *per se* are musically irrelevant; only frequency-ratios matter. Another way to describe this is that a musical scale is a *logarithmic scale*, which is a special kind of interval scale. For example, if we take base-2 logarithms of the various ratios above, we get the following scale, which looks more familiar.

… –2 < –1 < 0 < +1 < +2 …

In particular, the distances between values are obtained by simple subtraction; for example, the distance between adjacent values is 1 – which in this case corresponds to 1 octave.

[6] Normal human hearing is usually regarded as ranging from 20 Hz to 20,000 Hz, plus or minus.

[7] In the standard eight-note scale, the "distance" from the first note to the eighth note is 8 (octave), even though the number of *scale-steps* is 7! In music, there is no zero; for example, a one-step interval is called a "second", a two-step interval is called a "third", etc.

1. An Aside on Logarithmic Scales

At this point it might be helpful to consider another well-known logarithmic scale that is occasionally useful. We can measure size (say, height) relative to a fixed unit (say meters or miles); this produces an *arithmetic scale* with an associated scale and a natural zero. Such scales are useful in comparing objects of roughly the same size; for example, we might compare two buildings by saying that one is 10 meters taller than the other. But arithmetic scaling is useless for comparing objects of widely different sizes. For example, if we compare the volume of the planet Jupiter with the volume of a human, or the volume of an amoeba, we arrive at the very same comparison – Jupiter is a lot bigger! In particular, if we subtract a typical human's volume from Jupiter's volume, and we subtract a typical amoeba's volume from Jupiter's volume, we pretty much get the same quantity. Alternatively stated, if we place these three objects on an arithmetic scale, Jupiter is placed to the left somewhere, and the human and the amoeba are both placed infinitesimally close to the zero point. Even seemingly similarly sized objects produce nearly useless comparisons on an arithmetic scale. For example, if we subtract Jupiter's volume from the Sun's volume, and we subtract the Earth's volume from the Sun's volume, we also get pretty much the same volume difference!

A much more useful comparison of disparately sized objects is to compare them using a geometric (or logarithmic) scale, which is to say to compare their sizes using ratios. For example, we can say (roughly) that the sun is 1000 times bigger than Jupiter, which is 1000 times bigger than the Earth. Or considering the microscopic world, a human is 10^{18} times bigger than a cell, which is 10^{12} times bigger than a molecule, which is 10^{15} times bigger than an atomic nucleus.

All logarithmic scales are based on the same underlying interval scale, given as follows.

$$\ldots \boxed{-2} < \boxed{-1} < \boxed{0} < \boxed{+1} < \boxed{+2} \ldots$$

What distinguishes logarithmic scales from each other is the underlying "base", which is usually a positive whole number, which provides the fundamental ratio. For example, if the base is 10, then the corresponding ratios are given as follows.

$$\ldots \boxed{\frac{1}{100}} < \boxed{\frac{1}{10}} < \boxed{\frac{1}{1}} < \boxed{\frac{10}{1}} < \boxed{\frac{100}{1}} \ldots$$

On the other hand, if the base is 1000, then the corresponding ratios are given as follows.

$$\ldots \boxed{\frac{1}{1{,}000{,}000}} < \boxed{\frac{1}{1000}} < \boxed{\frac{1}{1}} < \boxed{\frac{1000}{1}} < \boxed{\frac{1{,}000{,}000}{1}} \ldots$$

2. Cents and Sensibility

The system for measuring musical intervals is based on a logarithmic scale based on 2.

$$\cdots \quad \frac{1}{4} < \frac{1}{2} < \frac{1}{1} < \frac{2}{1} < \frac{4}{1} \quad \cdots$$

As already mentioned, the fundamental interval is called an "octave". In the contemporary measurement system for musical intervals, this unit is divided into smaller units, just as miles are divided into feet, which are divided into inches. In particular, an octave is officially divided into 12 semi-tones, each of which is divided into 100 cents. Note carefully that, since the scale under consideration is logarithmic and not arithmetic, the meaning of the words 'one-twelfth' and 'one-hundredth' have to be very carefully considered. In particular, in measuring musical intervals, we must vigilantly keep the following "equations" in mind.

intervals	=	ratios
adding intervals	=	multiplying ratios

For example, we have the following

3 octaves	=	1 octave	+	1 octave	+	1 octave
	=	2/1	×	2/1	×	2/1
	=	8/1				

Similarly, to say that a semi-tone is one-twelfth of an octave, we mean the following (where s is a semi-tone, and r is its associated ratio.

$$s + s + s + s + s + s + s + s + s + s + s + s = \text{octave}$$

$$r \times r \times r \times r \times r \times r \times r \times r \times r \times r \times r \times r = 2/1$$

Or more succinctly stated:

r ^ 12 = 2

which entails that:

r = 2 ^ (1/12)

Here, the symbol '^' is the familiar programming symbol for exponentiation (raising to a power); in particular

x ^ y = x raised to the y power

For example, 3 ^ 2 = 3^2 = 3-squared; on the other hand, 3 ^ ½= $3^{½}$= the square-root of 3.

Thus, the ratio r corresponding to a semi-tone interval is an irrational number, which is approximately equal to 1.05946309. By similar reasoning, we calculate that one cent corresponds to the ratio 1.0005777895.[8]

The following is a summary of the interval units.

Unit Name	Relative Value	Pitch Ratio	
octave	fundamental	2/1	
semi-tone	1/12 of an octave	2 ^ (1/12)	1.05946309
cent	1/100 of a semi-tone	2 ^ (1/1200)	1.0005777895

Notice that we have the following identities.

1 octave	=	12 semi-tones
1 semi-tone	=	100 cents
1 octave	=	1200 cents

In thinking about musical interval units, please keep in mind that a cent, a semi-tone, and an octave are not frequencies, but rather ratios. Once you have a frequency, say 440 Hz, you can add one octave to it, or subtract one octave from it, which is to say that you can multiply it by 2, or divide it by 2. Similarly, you can add one cent to a given frequency, or subtract one cent from it, which is to say that you can multiply the frequency by 1.0005777895, or divide it by 1.0005777895.

3. The Pythagorean Construction of the Major Scale

So far, we have discussed the octave, and we have discussed the contemporary division of the octave into semi-tones and cents. Semi-tones and cents are relatively recent inventions. Long before them, Pythagoras proposed a division of the octave into intervals based on the ratios 2:1 and 3:2, which he had discovered correspond to natural harmonies.

In particular, according to Pythagoras, we first interpolate the ratio 3:2 between 1:1 and 2:1, thereby producing the following ordinal scale.

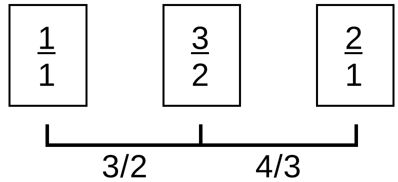
$$\boxed{\frac{1}{1}} < \boxed{\frac{3}{2}} < \boxed{\frac{2}{1}}$$

We next measure the interval between 3/2 and 2/1, which is calculated as follows.

the interval between	3/2	and	2/1
=	2/1	÷	3/2
=	2/1	×	2/3
=	4/3		

[8] We understand every number n to give rise to the ratio n:1 [n/1], even when n is irrational! This is an enlargement of official usage.

This can be depicted thus:



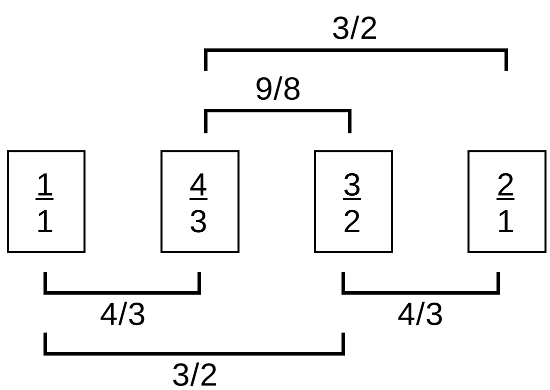
We now have a derived ratio – 4/3 – which we next interpolate between 1/1 and 3/2, thereby producing the following ordinal scale.

$$\frac{1}{1} < \frac{4}{3} < \frac{3}{2} < \frac{2}{1}$$

We next measure the interval between 4/3 and 3/2, which is calculated as follows.

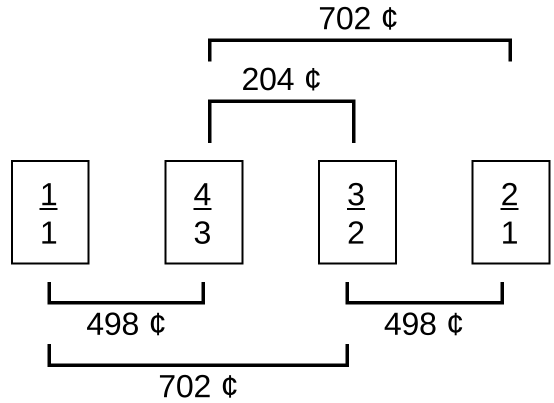
the interval between	3/2	and	4/3
=	3/2	÷	4/3
=	3/2	×	3/4
=	9/8		

The following diagram summarizes what we have so far.



So far, we have measured all these intervals as ratios; we can also measure them in cents, which yields the following.[9]

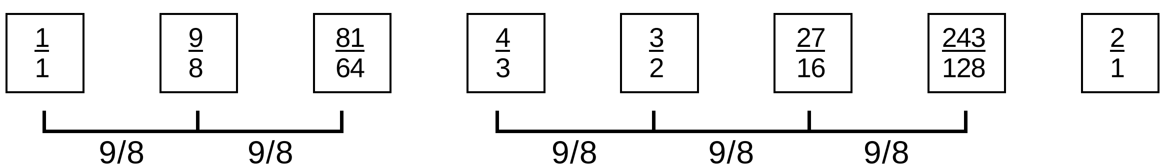
[9] These are approximations of irrational numbers; better approximations are: 203.910, 498.0445, 701.955



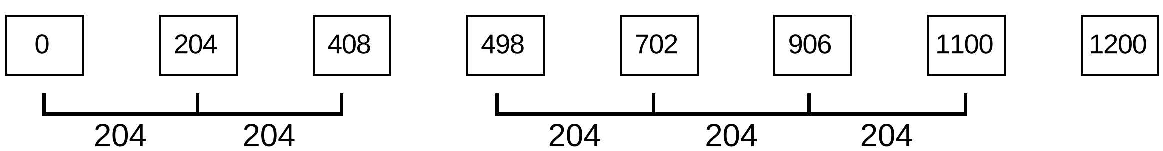
These intervals have special names – "perfect fifth", "perfect fourth", and "perfect second", which we add to our list of interval units as follows.[10]

Unit Name	Pitch Ratio[11]	Cent Value
octave	2:1	1200
perfect fifth	3:2	702
perfect fourth	4:3	498
perfect second	9:8	204
semi-tone	1.05946309	100
cent	1.0005777895	1

Our next move in the Pythagorean construction is to take the ratio 9/8 (204 cents) as a fundamental step-increment, and interpolate four more ratios, as follows.



Or using cents as our unit of measurement, we can write the following.



Two increments still need to be computed. This is easy using cents – the missing increments are each equal to 90 cents. The corresponding ratio, however, is a bit untidy – 256/243. Although 90 is a bit less than half of 204, we will nevertheless call this increment a "half-step", whereas we call the standard increment a "whole-step".

[10] The ordinal number words – 'second', 'third', etc. – refer to the position on the eight-note scale; for example, do is first, re is second, etc.

[11] We use the term 'ratio' in an enlarged way. A single number *n* automatically counts as the ratio *n*:1, or *n*-times.

When we are done, we have the Pythagorean construction of the major scale, which is presented as follows in its familiar form, where '<<' marks a whole-step, and '<' marks a half-step.

do	<<	re	<<	mi	<	fa	<<	so	<<	la	<<	ti	<	do
	204		204		90		204		204		204		90	

Or, using the white keys on a Piano keyboard, this is a C-major scale.

C	<<	D	<<	E	<	F	<<	G	<<	A	<<	B	<	C′

4. Ptolemaic Tuning

The Pythagorean calibration (tuning) technique constructs the major scale out of two fundamental ratios – 2:1 (octave) and 3:2 (perfect fifth); in particular, all the intervals are obtained from these two using interval addition and interval subtraction. The Pythagorean scale was widely used in the Middle Ages, but has generally fallen out of favor. There are a number of reasons for this, both practical and aesthetic. Let us consider the aesthetic problems first. Here, the most conspicuous problem is that certain harmonic intervals are less than ideal.

A tuning system that attempts to solve this problem is variously called "just intonation" and "Ptolemaic tuning", named after the great Greek astronomer Claudius Ptolemy (AD 85-165) who first proposed it. According to the Ptolemaic scheme, the Pythagorean ratios 2:1 and 3:2 are replaced by the following fundamental ratios

2:1 5:4 6:5

From these, one can derive the Pythagorean ratios. In particular,

$$\frac{3}{2} = \frac{5}{4} \times \frac{6}{5}$$

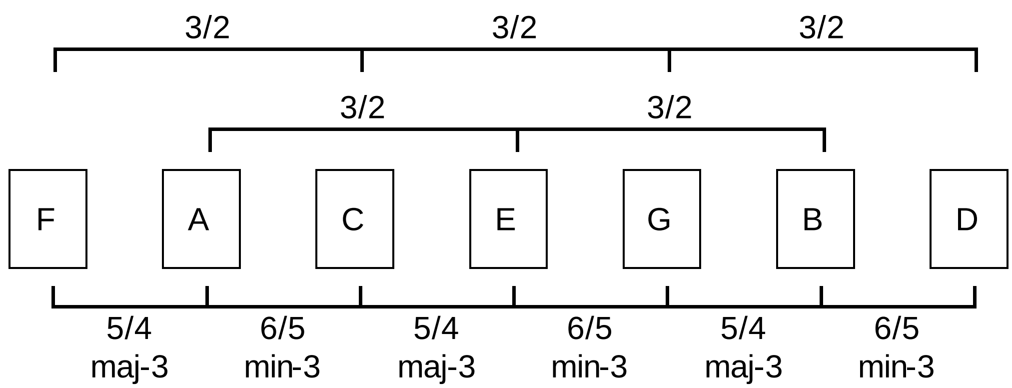
This equation corresponds to the harmonically ideal sub-division of a perfect-fifth into a perfect major-third and a perfect minor-third. In other words:

a perfect fifth = a perfect major-third + a perfect minor-third

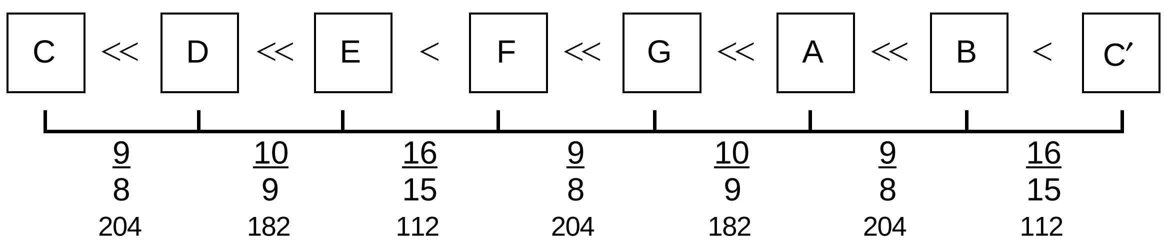
Let us add these new units to our list.

Unit Name	Pitch Ratio	Cent Value
octave	2:1	1200
perfect fifth	3:2	702 cents
perfect fourth	4:3	498 cents
perfect major-third	5:4	386.3 cents
perfect minor-third	6:5	315.6 cents
perfect second	9:8	204 cents

Next, the notes of an *expanded* C-major scale can all be inter-connected by these intervals, as displayed in the following chart.



Once the intervals are normalized,[12] we obtain the following C-major scale.



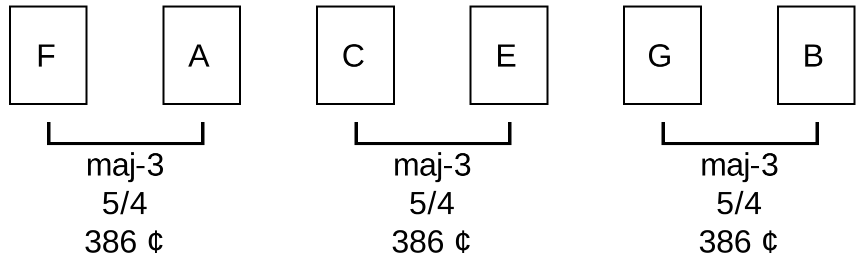
Most people find the resulting harmonies quite pleasing; instruments tuned in this manner really "sing". There are nevertheless a few problems. The first problem is that there are three different step-sizes – 204 cents, 182 cents, and 112 cents, rather than just two. This makes melodies sound a bit irregular. The other problem is that Ptolemaic tuning has a limited range. We discuss this more fully in a later section (6.3).

5. Mean-Tone Temperament (Tuning by Major Thirds)

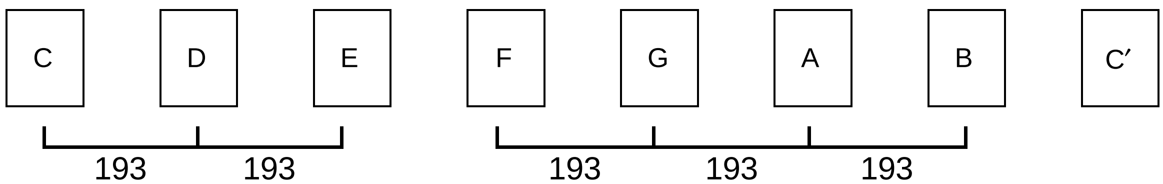
So far, we have examined two different tunings of the major scale. Whereas the Pythagorean tuning is based on the ratios 2:1 and 3:2, Ptolemaic tuning is based on the ratios 2:1, 5:4, and 6:5. Since Ptolemaic tuning can produce the ratio 3:2 [3/2 = 5/4 × 6/5], Ptolemaic tuning is more inclusive.

[12] For example, the distance from C to D on the expanded scale is two fifths – 3/2 × 3/2 = 9/4; subtracting one octave from 9/4 yields 9/8.

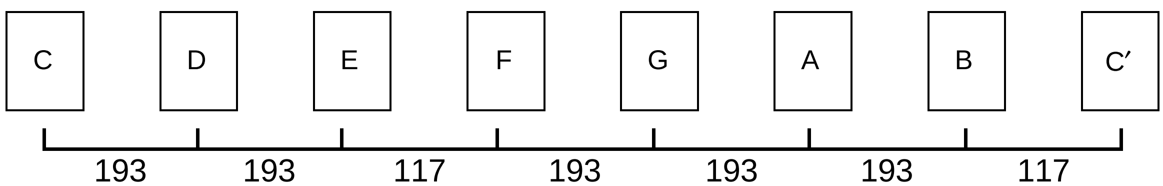
Another tuning technique – mean-tone temperament – is based on the ratios 2:1 (octave) and 5:4 (perfect major-third). In order to construct the C-major scale using mean-tone technique, we first set all the major thirds as follows.



We then divide each major-third into two equal intervals of approximately 193 cents each (square root of 5/4), which gives us the following.



All that remains is to assign the two half-step intervals – E-F and B-C′. These are posited to equal each other. Since the distance from C to C′ is assumed to be 2:1 (1200 cents), this means that the missing half-steps are each equal to 117 cents,[13] which gives us the following major scale.



6. Problems with Perfect Tuning

We have now discussed three classical tuning systems that are based on one or more "perfect" intervals, summarized as follows.

Pythagorean	2:1	3:2	
Ptolemaic	2:1	5:4	6:5
Mean-Tone	2:1	5:4	

The problem with "perfect" tuning is that it is not perfect. This becomes evident as soon as we try to expand our tuning to the chromatic keys (black keys on the piano). The reason that a piano has these extra keys is that very little music is written for the C-major scale. Just counting *major* scales, there are 15 official scale degrees (usually called "keys"), listed as follows.[14]

[13] There are rounding errors. 5/4 = 386.313714 cents; so $(5/4)^{\frac{1}{2}}$= 193.156857; so the left-over half-step = 117.107858.
[14] It is reputed that J.S. Bach wrote in as many as 31 distinct scale degrees.

Key	Scale	Key Signature
C♭ major	C♭ ≪ D♭ ≪ E♭ < F♭ ≪ G♭ ≪ A♭ ≪ B♭ < C♭	7 flats
G♭ major	G♭ ≪ A♭ ≪ B♭ < C♭ ≪ D♭ ≪ E♭ ≪ F < G♭	6 flats
D♭ major	D♭ ≪ E♭ ≪ F < G♭ ≪ A♭ ≪ B♭ ≪ C < D♭	5 flats
A♭ major	A♭ ≪ B♭ ≪ C < D♭ ≪ E♭ ≪ F ≪ G < A♭	4 flats
E♭ major	E♭ ≪ F ≪ G < A♭ ≪ B♭ ≪ C ≪ D < E♭	3 flats
B♭ major	B♭ ≪ C ≪ D < E♭ ≪ F ≪ G ≪ A < B♭	2 flats
F major	F ≪ G ≪ A < B♭ ≪ C ≪ D ≪ E < F	1 flat
C major	C ≪ D ≪ E < F ≪ G ≪ A ≪ B < C	∅
G major	G ≪ A ≪ B < C ≪ D ≪ E ≪ F♯ < G	1 sharp
D major	D ≪ E ≪ F♯ < G ≪ A ≪ B ≪ C♯ < D	2 sharps
A major	A ≪ B ≪ C♯ < D ≪ E ≪ F♯ ≪ G♯ < A	3 sharps
E major	E ≪ F♯ ≪ G♯ < A ≪ B ≪ C♯ ≪ D♯ < E	4 sharps
B major	B ≪ C♯ ≪ D♯ < E ≪ F♯ ≪ G♯ ≪ A♯ < B	5 sharps
F♯ major	F♯ ≪ G♯ ≪ A♯ < B ≪ C♯ ≪ D♯ ≪ E♯ < F♯	6 sharps
C♯ major	C♯ ≪ D♯ ≪ E♯ < F♯ ≪ G♯ ≪ A♯ ≪ B♯ < C♯	7 sharps

We quickly note, however, that on a standard piano, the first three scales [C♭ major, G♭ major, D♭ major] are respectively equivalent to the last three scales [B major, F♯ major, C♯ major]. This is because a standard piano only has 12 keys per octave, and accordingly cannot distinguish more than 12 of these scales. Intimately related to this, a standard piano cannot distinguish the following note pairs.[15]

C♯ ≡ D♭	D♯ ≡ E♭	E ≡ F♭	E♯ ≡ F	F♯ ≡ G♭	G♯ ≡ A♭	A♯ ≡ B♭	B ≡ C♭	B♯ ≡ C

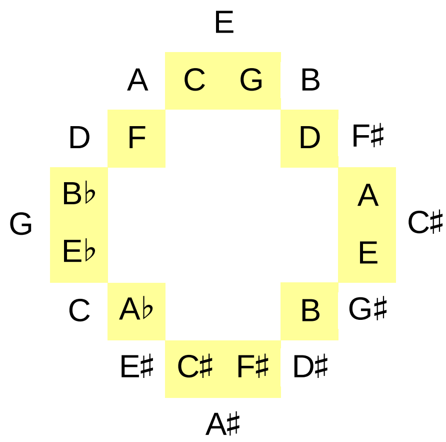
In music theory, these pairs are said to be "enharmonic"; they are distinguishable, as we see below, but not on a standard piano.

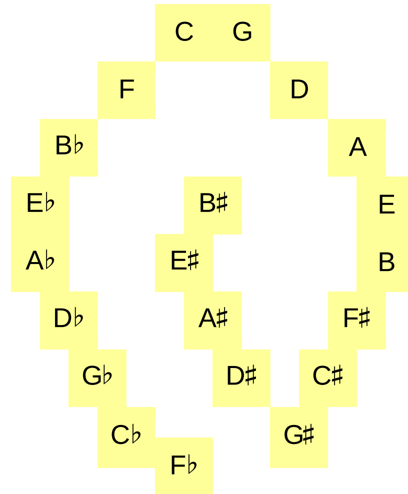
This begins to reveal the fundamental problem with tuning a piano, or any instrument that permits only 12 steps per octave. Specifically, if we tune by perfect ratios, then we cannot perfectly tune all 15 major scales at once; rather, we can perfectly tune only six *adjacent scales* at a time. Ptolemaic tuning is even more limited – each of the six adjacent scales has one dissonant triad.

[15] A "hyper" piano – which has twice as many black keys – can distinguish enharmonic pairs. Such a keyboard instrument is theoretically desirable, but impractical both from the viewpoint of construction, and from the viewpoint of playing.

1. An Aside on The Circle of Fifths and the Circle of Thirds

Adjacency of scales is measured against the quasi-ordinal scale known as the "circle of fifths", which we have already alluded to in the above list of major scales, and which is diagrammed as follows.





The left diagram assumes a particular tuning of the black piano keys; the right diagram is a more accurate portrayal, which depicts a "spiral of fifths". The left diagram actually depicts three circles:

(1)	shaded inside circle	C	G	D	A	E	B	F♯	…
(2)	unshaded outside circle	E	B	F♯	C♯	G♯	D♯	A♯	…
(3)	interleaved circle	C	E	G	B	D	F♯	A…	

All three can be read clockwise. The inner circle and the outer circle proceed by fifths; for example, a fifth above F is C, and a fifth above C is G, and a fifth above G is D. The interleaved circle – which is called the "circle of thirds" – proceeds by thirds, alternating major-thirds and minor-thirds. Moreover, a major scale can be constructed by starting on any shaded element and following the interleaved circle six steps; the third item will constitute the "tonic" key. The following are a few examples.

F	A	C	E	G	B	D	C-major
C	E	G	B	D	F♯	A	G-major
G	B	D	F♯	A	C♯	E	D-major

2. Back to the Problem of Tuning by Fifths

The problem is that when we tune by perfect fifths, we face the problem of adjudicating the various enharmonic pairs, which are repeated from above.

C♯ ≡ D♭	D♯ ≡ E♭	E ≡ F♭	E♯ ≡ F	F♯ ≡ G♭	G♯ ≡ A♭	A♯ ≡ B♭	B ≡ C♭	B♯ ≡ C

Now, as everyone "knows", C♯ = D♭! Alternatively stated, if you are asked to find these keys on a piano within a given octave, you go the same key – the black key between C and D! Nevertheless, this identity does not hold if we use Pythagorean tuning! Rather, by Pythagorean tuning, we must tune as follows.

D♭	=	five fifths "below" C
	=	$C \times 2/3 \times 2/3 \times 2/3 \times 2/3 \times 2/3$ $(\times 8)$
	=	$C \times 1.05350$
	=	C + 90.23 cents

C♯	=	seven fifths "above" C
	=	$C \times 3/2 \times 3/2 \times 3/2 \times 3/2 \times 3/2 \times 3/2 \times 3/2$ $(\times 1/16)$
	=	$C \times 1.06787$
	=	C + 113.6 cents

In other words, a Pythagorean-tuned C♯ is quite a bit sharper than a Pythagorean-tuned D♭! In the Middle Ages, this discrepancy was called the "Pythagorean comma"; its modern cent value is approximately 23.

Now, go back and look at the above major scales. Every major scale has a characteristic "key signature", which corresponds to which notes are sharped or flatted. Moreover, if a given scale calls for a particular key, say C♯, then in order for that scale to be in *perfect* tune, the black key between C and D must be tuned as an C♯, not as a D♭. Conversely, if a scale calls for a D♭, then in order for that scale to be in *perfect* tune, the key between C and D must be tuned as a D♭, not as an C♯. This holds *mutatis mutandis* for every enharmonic pair in the above chart.

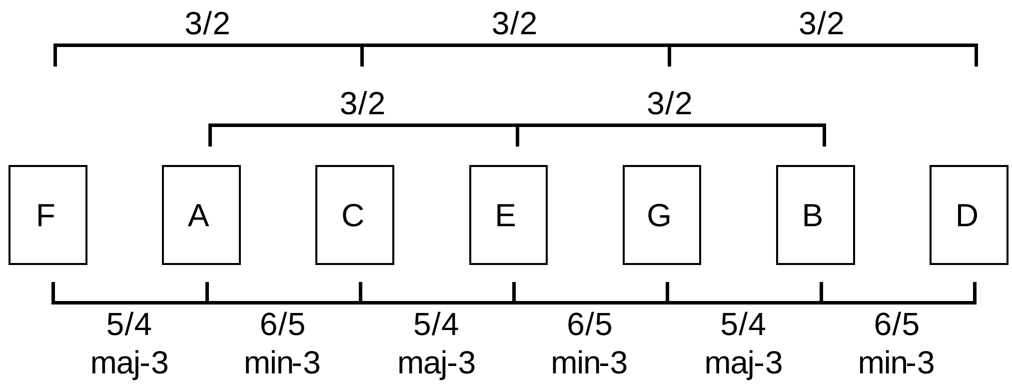
When the dust has settled, this is what we have. We can perfectly tune six adjacent scales, but the remaining scales will be variously imperfectly tuned. Some of these will be usable; others will be unusable. For example, if we perfectly tune the six scales E♭ through D, then the interval G♯ – E♭ will be a *very bad* fifth. In the Middle Ages, this interval was called a "wolf" (because on an organ it "growls").

3. Ptolemaic Tuning Makes Matters Even Worse!

So far we have discussed tuning by perfect fifths, and we have seen that we can perfectly tune at most 6 adjacent scales. In the present section, we briefly examine what happens when we attempt to tune by perfect major-thirds (5:4) and perfect minor-thirds (6:5), as prescribed by Ptolemaic tuning. Recall that the Ptolemaic C-major scale is given as follows.

C	<<	D	<<	E	<	F	<<	G	<<	A	<<	B	<	C′
	$\frac{9}{8}$		$\frac{10}{9}$		$\frac{16}{15}$		$\frac{9}{8}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{16}{15}$	
	204		182		112		204		182		204		112	

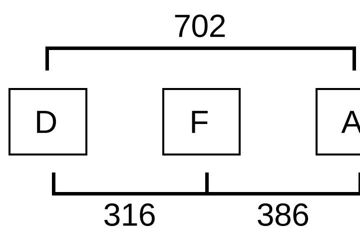
This tuning provides five perfectly-tuned triads (3-note chords), given by the following chart.



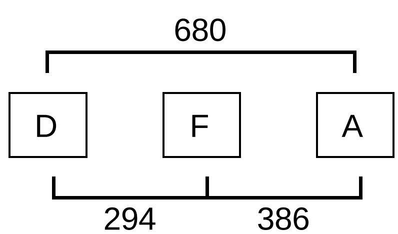
These chords are respectively:

F-major	F–A–C
A-minor	A–C–E
C-major	C–E–G
E-minor	E–G–B
G-major	G–B–D

What is missing is the triad D–F–A [D-minor]. For this chord to be perfect, we need the following intervals.

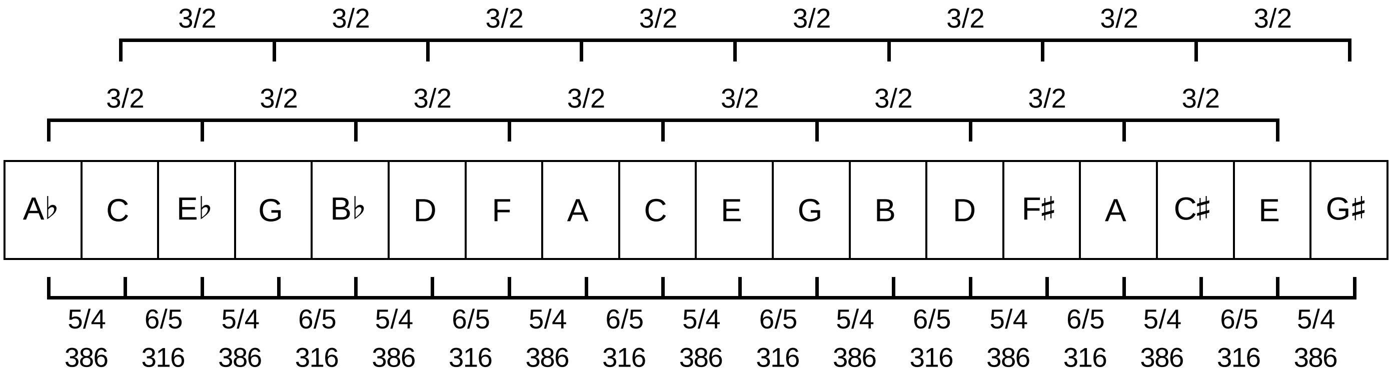


On the other hand, Ptolemaic tuning provides the following intervals.



Given that the fifth interval is flat by 22 cents, this surely qualifies as a "wolf".

The problem worsens as we move up or down the "circle of thirds". Let us expand our tuning from C-major as follows.

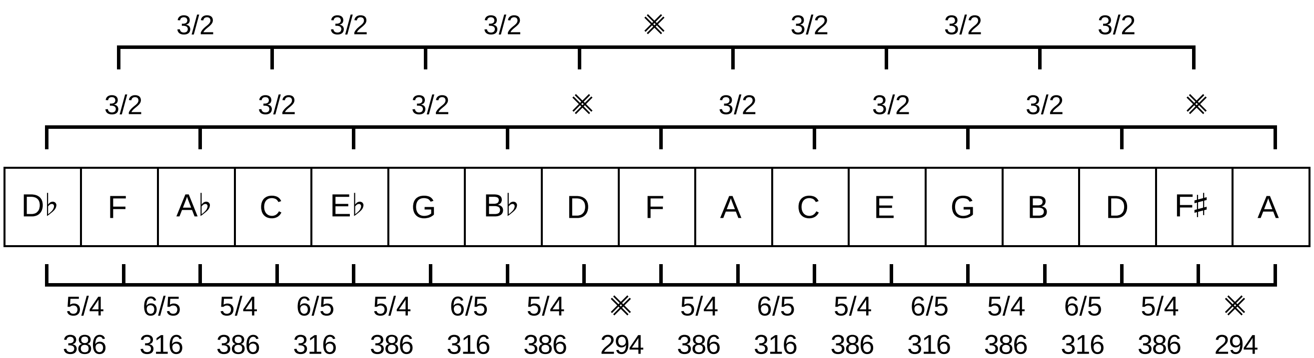


Notice that the end points – A♭ and G♯ – are enharmonic. We already know from our work on Pythagorean tuning (by fifths) that we must tune the key between G and A as G♯ or A♭, which differ by 24 cents. But matters are much more serious; the above sequence of thirds harbors a much more fundamental discrepancy. In particular, notice that C, G, D, A, and E all repeat in the above chart. Now, if we tune using the above chart, as required by Ptolemaic tuning, then the interval between the first C [G, D, A, E] and the second C [G, D, A, E] is:

$$6/5 \times 5/4 \times 6/5 \times 5/4 \times 6/5 \times 5/4 \times 6/5 \quad = \quad 4.04999 \quad = \quad 2422 \text{ cents}$$

On the other hand, octave-tuning requires this distance to be an octave-multiple, the closest of which is 2400 cents, which produces a discrepancy of 22 cents! In other words, the above chart involves tuning C and C″ to two different pitches. This is considered completely unacceptable by nearly everyone who has studied tuning. The one sacrosanct tuning rule is that adjacent notes of the same note-class (e.g., C) are exactly one octave apart.

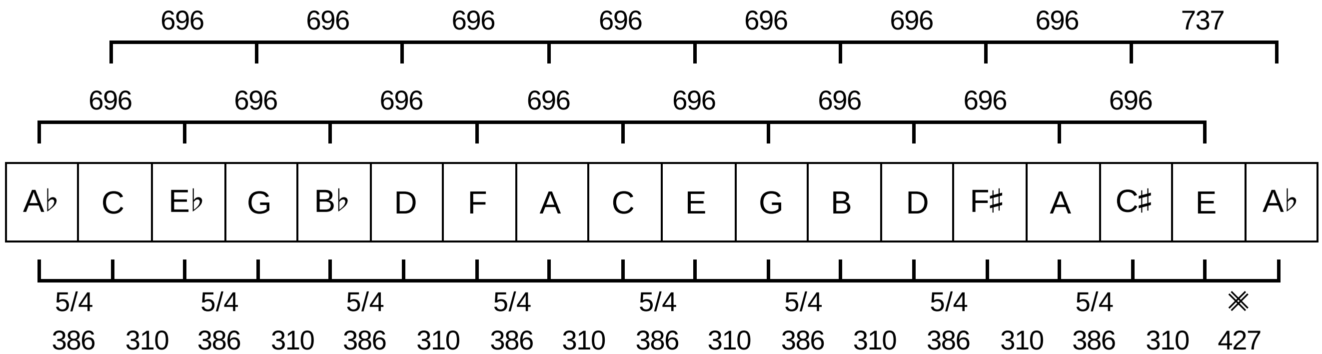
The question then is – how do we tune the keys on a piano using perfect-thirds? The following constitutes one tuning for the notes D♭ - A (on the circle of thirds)



Notice the intervals designated ※, which are flat by 22 cents (316 – 294), which makes the triad dissonant. Every one of the 6 major scales represented above – A♭, E♭, B♭, F, C, G – contains at least one dissonant triad. As we move farther away from the tuning center, we add more dissonant triads.

4. Mean-Tone Temperament's Wolves

Pythagorean tuning enables us to perfectly tune six adjacent scales (e.g., E♭ - D), but it has dissonant thirds. Ptolemaic tuning gives us perfect thirds, but each of the six adjacent scales has at least one dissonant triad. Mean-tone temperament, which was developed in the Baroque period (roughly 1650-1750), attempts to steer a middle course between Ptolemaic tuning and Pythagorean tuning. It produces perfect major-thirds, and produces decent approximations to perfect fifths and perfect minor-thirds. Moreover, six adjacent scales can be perfectly tuned. The remote scales remain out of tune, however, as seen in the following chart.



Except for the last one, all the triads in the above diagram are nearly perfect; the major-thirds are perfect; the fifths and minor-thirds are slightly flat (by 6 cents, which is completely acceptable to most musicians). This tuning permits playing the six major scales from E♭ major to D major. On the other hand, the remote scales contain dissonant triads. The one that is apparent above is the C♯ minor triad – C♯–E–G♯. The above scale approximates this triad with C♯–E–A♭. The interval C♯–E is fine, being 310 cents. But the interval E–A♭ is an astonishing 427 cents, which is 41 cents sharp! Any scale that needs this triad for harmony will be largely unusable (except for special-effects!).

7. What to do?

We have a dilemma. We have the following musical desiderata.

(1) playing all official scales (C♭, G♭, …, F♯, C♯) equally well [universality]
(2) with perfectly-tuned intervals [perfect-tuning]
(3) on instruments that can be reliably constructed, tuned, and played [practicality]

Unfortunately, these desiderata cannot be simultaneously satisfied. As in dealing with any dilemma or paradox, we must seriously reconsider the fundamental principles that lead to the difficulty. Which of these desiderata are we willing to give up, or to adjust?

1. Give up Universality

We have already discussed giving up universality. Dating back to the beginning of music, we have been able to construct and play instruments that can play perfect intonation. These instruments were all restricted in their tonal variety; they could only play one or two scale degrees. As technology became more advanced, instruments with greater tonal variety became available, including the organ and the piano. The problem is that, using perfect tuning, we are severely restricted in what keys we can play. Retuning an organ or piano is an arduous and time-consuming task. It is better to tune it once, and restrict what sorts of musical pieces one can play.

2. Give up Practicality – The Purist Approach

The purist solution is to play only instruments that can be tuned on the fly. We have several of these instruments already – the human voice, the trombone, and the four members of the violin family. If we restrict our ensembles to these instruments, played by musicians with very good pitch ability, then we can achieve the other two goals.

3. Give up Perfect Tuning – Equal Temperament

The almost universally accepted solution of the modern world, which nearly everyone pretty much takes for granted, abandons the ideal of perfect tuning, in order to achieve the other two goals. The proposed tuning system is known as "equal-temperament", or more strictly "12-tone equal temperament" or "12ET" for short.[16] According to 12ET, the octave is divided into 12 equal intervals, called "semi-tones". We have already seen semi-tones; a cent is defined to be one-hundredth of a semi-tone, which is defined as one-twelfth of an octave. As noted earlier, a semi-tone corresponds to an irrational number that is approximately 1.059463. This constitutes a half-step; a whole-step is obtained simply by adding two half-steps, which is approximately 1.12246.

The problem with 12ET is that all the fundamental harmonic ratios are missed. That's the bad news. The good news is that the discrepancies are not so big that it is musically atrocious. For example, the equal-tempered fifth consists of seven half-steps (700 cents), which is a completely acceptable approximation to a perfect fifth (702 cents). Similarly, the equal-tempered fourth consists of 5 half-steps (500 cents), which is a completely acceptable approximation to a perfect fourth (498 cents). Thirds are a different matter. The equal-tempered major-third is 400 cents, which is 14 cents sharp, and the equal-tempered minor-third is 300 cents, which is 15 cents flat. But the thirds are still better than Pythagorean tuning. Indeed, 12ET is probably best understood as a very slight adjustment of Pythagorean tuning in which the 24 cent discrepancy (the "wolf") is equally distributed across the scale, which requires us to flat each fifth by 2 cents. We then obtain a tuning that allows a piano to play 12 major scales.

Most people find equal temperament to be a workable compromise, and it is almost universally adopted in the manufacture of modern "fretted" instruments (guitars, flutes, horns, etc.), as well as electronic instruments (various synthesizers). Most musicians – including those with "perfect pitch" – are able to work within this system when it is necessary. For example, a violinist or singer who wishes to be accompanied by a piano is usually able to retune his/her playing to equal temperament.

There are hold-outs, however. For example, there are violinists who refuse to play with performers who play equal-tempered instruments.[17] Others are not so dogmatic, but are nevertheless frustrated by the strictures of equal temperament. For example, there are numerous guitar players who become frustrated when they cannot tune their guitars so that they really "sing". For these tortured souls, "enlightenment" comes only when they realize that it is *impossible* to perfectly tune a standard guitar, because it is manufactured to the standards of equal-temperament.

4. Alternative Instruments

As mentioned above, we already have three groups of instruments – the human voice, the violin family, and the trombone – that can play any scale in perfect intonation. This is because these instruments can be tuned "on the fly". This is not possible with "fretted" instruments, which include pianos, guitars, flutes, horns, clarinets, oboes, etc.. So why don't we manufacture fretless versions of these instruments, or versions that have more fixed intervals per octave (see Section 11). The reason is that this is somewhat impractical both in terms of the manufacturing and in terms of playing. For example, it is nearly impossible to get a decent chord on a fretless guitar.

[16] Also called "12 TET" and "12-Equal" in the literature. In a later section (11), we discuss other equal-interval scales, ones in which the number of intervals is not 12.
[17] These are usually amateurs; professionals are seldom given such latitude!

A related solution is to build digital versions of all fretted instruments, which can then be retuned on the fly; tuning-on-the-fly is already feasible for digital pianos and organs (synthesizers). This is certainly a feasible choice for the orchestra of the future, but *current* musical aesthetics – both listening and playing – demands acoustic instruments for most applications; millions of people continue to listen to and play traditional acoustic instruments.[18]

5. Reconsidering Universality – Well-Tempered Scales

According to some people, the drawback of equal-temperament is not so much that it produces dissonant thirds, but that it produces homogeneously bland music. In particular, in the pursuit of universality, equal-temperament has produced 12 scale degrees that are exactly alike in terms of tonal color; E♭ major sounds pretty much just like C major.

This leads us to yet another approach to the tuning dilemma – to question what the requirement of universality amounts to. Does universality mean that all scale-degrees play *exactly alike*, as they do in 12-tone equal-temperament? Or, does universality mean that all scale-degrees play *equally well*? Thus arises the concept of *well-temperament*, according to which 12 scale degrees all play equally well, and nearly in perfect tune, although each one has its own tonal individuality (i.e., quirkiness!), which can be taken advantage of by the skillful composer. This is especially evident in Bach's *Well-Tempered Clavier*[19], in which he presents two books of preludes and fugues. In particular, in each book, Bach presents a prelude and fugue in each of 12 major scales and 12 minor scales. In each case, he takes advantage of the unique tonal personality of the scale degree.

It is not completely obvious what tuning Bach actually used, since it was never recorded, and there are in principle infinitely-many ways of accomplishing the vague criterion of well-temperament. However, a number of scholars and musicians have suggested that Bach's tuning was something *approximating* the following, which is known as Werckmeister temperament.

	A♭	E♭	B♭	F	C	G	D	A	E	B	F♯	C♯
1	0	0	0	0	0	0	0	0	0	0	0	0
2	204	204	204	198	192	192	198	204	198	198	204	204
3	408	402	396	390	390	396	396	402	402	402	408	408
4	498	498	498	498	498	504	504	504	498	498	504	498
5	702	702	702	702	696	696	696	702	702	696	702	702
6	906	906	900	894	888	894	900	900	900	900	906	906
7	1104	1098	1092	1092	1092	1092	1098	1104	1104	1104	1110	1110
8	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200	1200

This provides the intervals for the 12 major scale degrees A♭ through C♯. One can also inter-substitute enharmonic pairs (e.g., D♭ for C♯). To read the cumulative intervals, one goes down the column. Notice that some of the fifths are perfect (702 cents), but not all.

[18] In this context, an electric guitar counts as acoustic; a genuine acoustic process – a vibrating string – is involved in the production of sounds.

[19] A myth, perpetuated even in music schools based on some faulty musicology in the 19th Century, claims that Bach wrote the *Well-Tempered Clavier* for an equal-tempered instrument. Wrong! In Bach's time, the word 'equal tempered' was used, but it did not mean what it means today; rather, it meant that every scale degree could be played equally well. Thus, we have the more appropriate word 'well-tempered', which does not mean 'equal-tempered'.

8. Summary of Temperament Systems

The following chart summarizes the five major tuning systems, in which we give the various increments in cents for the C-major scale.

C << D << E < F << G << A << B < C′

Equal	200	200	100	200	200	200	100
Pythagorean	204	204	90	204	204	204	90
Ptolemaic	204	182	112	204	182	204	112
Mean-Tone	193	193	117	193	193	193	117
W-Well	192	198	118	192	194	194	118

The following chart, which is useful to synthesizer players, composers, and programmers, gives the "discrepancies" for each note compared to equal-temperament tuning.

	C	D	E	F	G	A	B
Equal	0	0	0	0	0	0	0
Pythagorean	0	+4	+8	–2	+2	+6	+10
Ptolemaic	0	+4	–14	–2	+2	–16	–12
Mean-Tone	0	–7	–14	+3	–4	–11	–18
W-Well	0	–8	–10	–2	–4	–6	–8

Finally, the following chart provides the discrepancies from perfect tuning.

	C	D	E	F	G	A	B
Equal	0	–4	+14	+2	–2	+16	+12
Pythagorean	0	0	+22	0	0	+22	+22
Ptolemaic	0	0	0	0	0	0	0
Mean-Tone	0	–11	0	+5	–6	+5	–6
W-Well	0	–12	+4	0	–6	+10	+4

9. Other Seven-Tone Scales

So far we have discussed one scale type (or *mode*) – the major scale. Although the major scale is the most widely used scale in Western music, there are many other scales/modes that have been proposed and used over the years. What characterizes a major scale is the following sequence of seven increments.

whole-step	whole-step	half-step	whole-step	whole-step	whole-step	half-step
<<	<<	<	<<	<<	<<	<

Needless to say, there are many other sequences consisting of exactly 5 whole-steps and 2 half-steps. Mathematically, there are 21 distinct possibilities (laboriously calculated!), so there are 21 mathematically possible 7-tone scales based on whole-steps and half-steps. Some of these are not musically very pleasing – for example, those scales involving successive half-steps, of which there are five. Let's get rid of these, which still leaves 16 scales. These can be judged according to whether they include the fundamental ratios 3/2 and 4/3. 9 of the 16 scales include these ratios, which are depicted as follows, beginning with the original C-major scale.

Ionian	C	<<	D	<<	E	<	F	<<	G	<<	A	<<	B	<	C′
Mixolydian	C	<<	D	<<	E	<	F	<<	G	<<	A	<	B♭	<<	C′
?	C	<<	D	<<	E	<	F	<<	G	<	A♭	<<	B♭	<<	C′
?	C	<<	D	<	E♭	<<	F	<<	G	<<	A	<<	B	<	C′
Dorian	C	<<	D	<	E♭	<<	F	<<	G	<<	A	<	B♭	<<	C′
Aeolian	C	<<	D	<	E♭	<<	F	<<	G	<	A♭	<<	B♭	<<	C′
?	C	<	D♭	<<	E♭	<<	F	<<	G	<<	A	<<	B	<	C′
?	C	<	D♭	<<	E♭	<<	F	<<	G	<<	A	<	B♭	<<	C′
Phrygian	C	<	D♭	<<	E♭	<<	F	<<	G	<	A♭	<<	B♭	<<	C′

Some of these have classical Greek names, given to the left, which correspond to Medieval modes. Note that the Aeolian scale corresponds to the modern natural minor scale. Two other Medieval modes do not appear on the above list.

Lydian	C	<<	D	<<	E	<<	F♯	<	G	<<	A	<<	B	<	C′
Locrian	C	<	D♭	<<	E♭	<<	F	<	G♭	<<	A♭	<<	B♭	<<	C′

The first one lacks the 4/3 ratio (F), but is still employed. The second one lacks the 3/2 ratio (G), which renders it musically questionable according to some authors.

10. Pentatonic Scales

All the scales we have thus far considered are predicated on dividing the octave into seven steps. Other scales have been considered that deviate from the seven-step rule. A favorite of mine is the major pentatonic scale, which is employed in both traditional Scots-Irish music (e.g., "Amazing Grace") and traditional Chinese music. The scale is obtained from the major scale (Ionian) by deleting the two single-step intervals, as follows.

Pentatonic$_1$	C	<<	D	<<	E	<<<	G	<<	A	<<<	C′

Here the step-pattern is given as follows.

whole-step	whole-step	3/2-step	whole-step	3/2-step
<<	<<	<<<	<<	<<<

So one naturally asks what other patterns of << and <<< are mathematically possible. Supposing we prohibit successive 3/2-step intervals, there are four other pentatonic scales based on these intervals, given as follows.

Pentatonic$_2$	C	<<	D	<<<	F	<<	G	<<<	B♭	<<	C′
Pentatonic$_3$	C	<<<	E♭	<<	F	<<	G	<<	A	<<<	C′
Pentatonic$_4$	C	<<<	E♭	<<	F	<<	G	<<<	B♭	<<	C′
Pentatonic$_5$	C	<<	D	<<<	F	<<	G	<<	A	<<<	C′

Pentatonic scales are characterized by the presence of 3/2-step adjacent intervals. Once we consider 3/2-step adjacent intervals, we can go back and reconsider 7-tone scales. We will not consider all the possibilities, but merely remark that the so-called "harmonic minor" scale involves a 3/2-step interval between the 6th and 7th notes.

Harmonic Minor	C	<<	D	<	E♭	<<	F	<<	G	<	A♭	<<<	B	<	C′

Such scales are occasionally employed in Middle Eastern music and Eastern European music.

11. Equal-Interval Scales

Equal-interval scales are based on dividing each octave into n equal intervals; for each whole number n, we have a scale, which we call "nET". The simplest such scales employ an inegral division of 1200 cents; these include 24, 12, 6, and 4 intervals, which are displayed as follows, using C as root.

24ET	C	C+	C#	D–	D	D+	D#	E–	E	E+	F	F+	F#	G–	G	G+	G#	A–	A	A+	A#	B–	B	B+	C′

12ET	C	<	C#	<	D	<	D#	<	E	<	F	<	F#	<	G	<	G#	<	A	<	A#	<	B	<	C′

6ET	C	<<	D	<<	E	<<	F#	<<	G#	<<	A#	<<	C′

4ET	C	<<<	D#	<<<	F#	<<<	C′

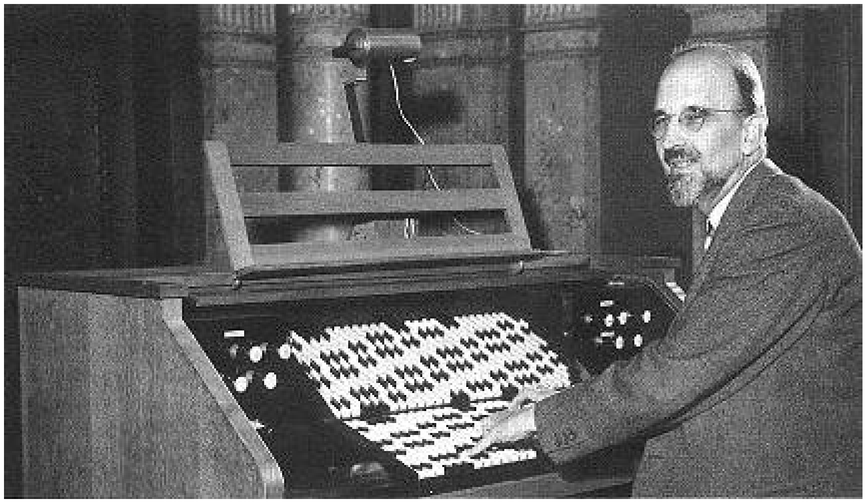
Here, we use a somewhat non-standard notation of plus '+', minus '–', and double-plus '#', which is approximately equal to sharp '♯'. Note that 12ET is intimately related to equal temperament, which we have already discussed.[20]

1. 31ET

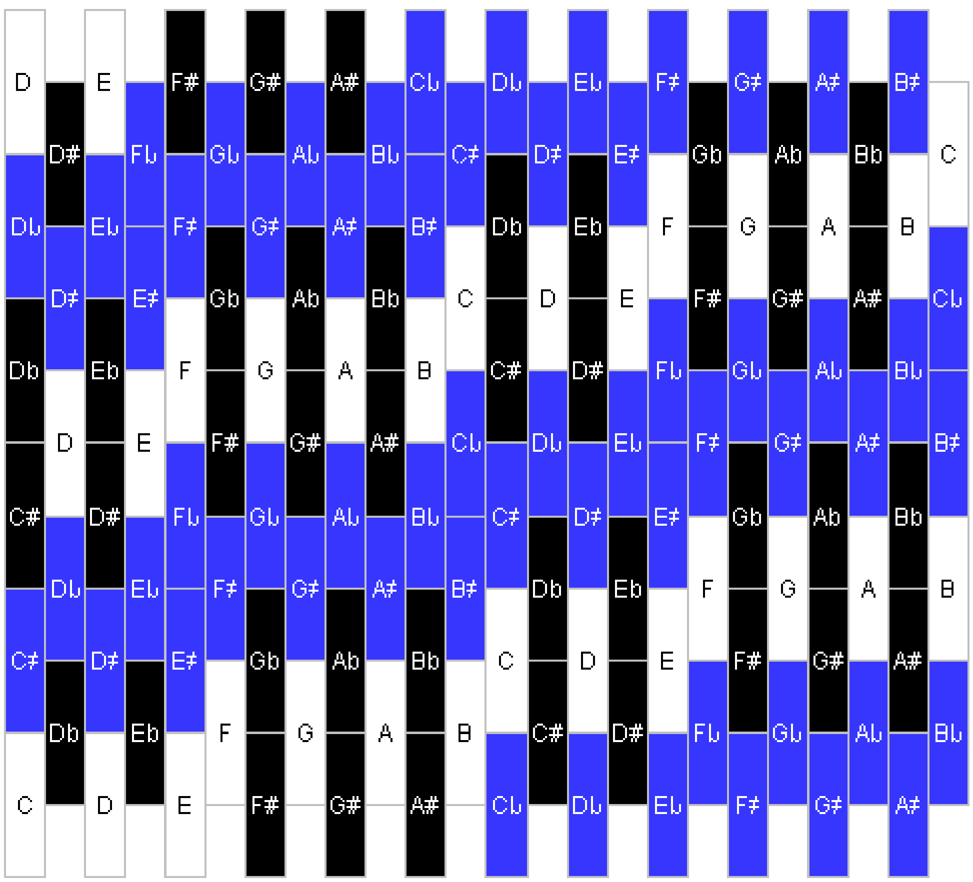
Other equal-interval scales have been seriously considered over the years, including divisions of the octave into 16, 19, 31, 43, and 53 equal intervals. After 12ET, the most widely examined equal-interval scale is 31ET, which has 31 equal intervals of 38.709677 cents each. It provides a close approximation to mean-tone temperament, and there have even been quite a few keyboard instruments built with 31 keys per octave, going back at least to the 16th Century. The following is a picture of an antique instrument – called a "archicembalo" – at L'Istituto Comunale di Musica Antica in Italy.

[20] Twelve-tone equal temperament should be carefully contrasted with 12-tone music, in which all 12 notes of the scale are given more or less equal representation.

The following is a picture of a more recent such instrument (1943), with its creator Adriaan Daniël Fokker (1887-1972).[21]

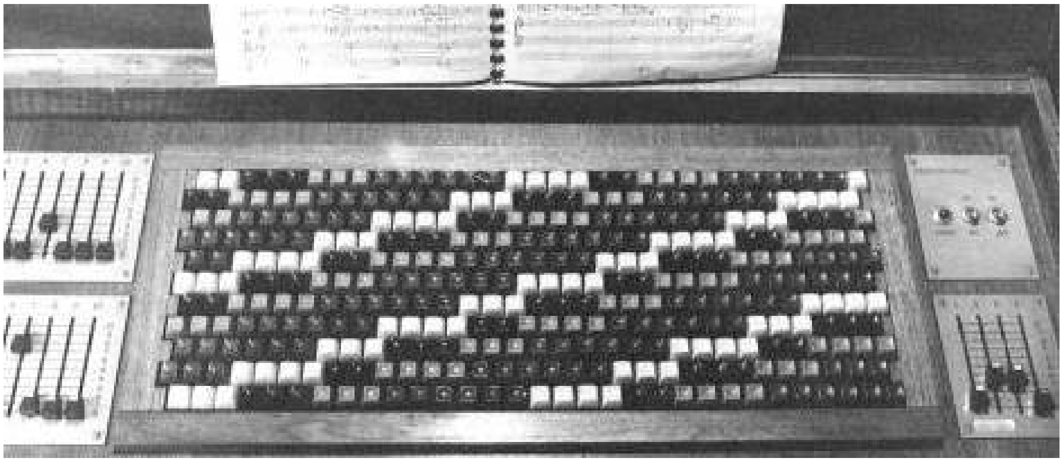


The following is a diagram of the keyboard layout. Note that there are semi-flats and semi-sharps. Also note that conventional scales have uniform fingering; for example, C-major has the same fingering pattern as D-major.



[21]The remaining pictures come from the archive of the Huygens-Fokker Foundation [http://www.xs4all.nl/~huygensf/english/]. The great Dutch physicist Christiaan Huygens (1629-1695), who is most famous for originating the wave theory of light, was also a pioneer in music theory and tuning, and wrote on the relation between 31 ET and mean-tone temperament.

Finally, the following is a picture of a 1970 electronic instrument, called an "archiphone". Notice the color coordinated keys – white, black, and gray; there is a method in the madness!



2. 53ET

My own personal favorite is 53ET, since I (re-)invented it from scratch, in my search for an ET that most closely approximates Ptolemaic tuning.[22] In particular, in 53ET, each fundamental step is 22.641509 cents, which provides the following conversions.

Unit Name	Pitch Ratio	Cent Value	Corresponding Number of 53E steps	Cent Value
octave	2:1	1200.000000	53	1200.000000
perfect fifth	3:2	701.955001	31	701.8867925
perfect fourth	4:3	498.044999	22	498.1132075
perfect major-third	5:4	386.313714	17	384.9056604
perfect minor-third	6:5	315.641287	14	316.9811321
perfect second	9:8	203.910002	9	203.7735849

The Ptolemaic major scale is then approximated by the following scale in which the numbers refer to fundamental 53ET steps (22.64 cents).

C << D << E < F << G << A << B < C′

9 8 5 9 8 9 5

[22] My "discovery" of 53ET was anticipated! There is even a 53-tone guitar called the "Dinarra", built by Eduardo Sabat-Garibaldi (Uruguay), which has 89 frets! Check the following address for more info and some music samples. http://dinarra.lookscool.com.