

Additive Scales

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1.	Introduction.....	1
2.	Numerical Addition.....	2
3.	Mereology	3
4.	The Principles of Mereology.....	4
5.	An Aside on Identity	6
6.	Atoms	7
7.	Disjointness	7
8.	Mereological Sums	8
9.	Mereology and Set Theory.....	9
10.	Expanding Measurement to A Wider Context.....	9
11.	Weight.....	10
12.	Comparing Mereological Sums	12
13.	Expanding the Weight Scale	14
14.	Ratios	15
15.	The Archimedean Balance Scale	17
16.	Deflection Scales.....	17
17.	Units	18
18.	Standardized Units	19

1. Introduction

In investigating measurement scales, we have so far discussed nominal scales, ordinal scales, and interval scales. That leaves ratio scales, which are also called "additive scales". By way of review:

- (1) A **nominal scale** has categories that bear no special relations to each other. Simple examples include classifying individuals according to their (1) political affiliation, (2) religious affiliation, (3) ethnic group.
- (2) An **ordinal scale** has categories that are *linearly* ordered, but does not quantify differences (a.k.a., intervals) between the categories. Grades (A,B,C, etc.) are an example of an ordinal scale.¹
- (3) An **interval scale** has categories between which it makes sense to measure differences (intervals). More importantly, perhaps, it makes sense to *add intervals*, although it does not make sense to add the categories themselves. An example of an interval scale is a musical scale (e.g., do, re, mi, fa, so, la, ti, do). Intervals in music can be meaningfully added; for example, a fifth plus a fourth equals an octave.
- (4) A **ratio scale (additive scale)** has categories that can be meaningfully added. Key examples of additive scales include the measurement of weight (mass), length, area, and duration. All of these quantities can be meaningfully added.

¹ Although academic administrators treat them as more than an ordinal scale, by manufacturing grade-point averages.

The purpose of this chapter is to examine in detail what it means to add quantities. Toward this end we first review addition in the context of numbers, and then we discuss addition in the context of mereology. Finally, we discuss a particular additive scale – the measurement of weight – and show how one empirically and logically demonstrates that weight forms an additive scale.

2. Numerical Addition

The paradigmatic additive scale is set-size. In particular set-sizes can be added in accordance with the following *fundamental principle of addition*.²

if
 no A 's are B 's
then
 the number of A 's and B 's
 =
 the number of A 's
 plus
 the number of B 's

For example, since no apple is a banana, if we have 5 apples and 7 bananas, then we have 12 apples and bananas.

The above formula can also be written with special symbols, as follows.

$$\text{if } A \perp B \text{ then } \#(A \cup B) = \#(A) + \#(B)$$

Here, we use the following special symbols.

\perp	$=:$	disjoint
\cup	$=:$	set union
$\#$	$=:$	the size of ...
$+$	$=:$	addition

Two sets A and B are said to be *disjoint* – written $A \perp B$ – if they do not share any members in common. Officially:

$$A \perp B \quad =_{\text{df}} \quad \sim \exists x \{ x \in A \ \& \ x \in B \}$$

For example, any collection of apples is disjoint from any collection of bananas. On the other hand, the set of attorneys and the set of Bostonians are not disjoint, since there are Bostonian attorneys.

The *union* of two sets is obtained by adding them together. If we have a set of apples and a set of bananas, then together with have a set of apples and bananas. Officially, set-union is defined as follows.

$$A \cup B \quad =_{\text{df}} \quad \{ x : x \in A \vee x \in B \}$$

² See chapter “Arithmetic”.

In other words, an item is a member of $A \cup B$ if and only if it is a member of A and/or a member of B . For example, a person is a member of the set of actors-and-bakers if and only if that person is an actor and/or a baker.

3. Mereology

Set union is a special example of mereological union. Mereology is the study of parts and wholes.³ The *part-whole* concept is fundamental to our grasp of the world. We naturally think of concrete physical objects as having parts; for example, a bicycle has many parts, including two wheels, two pedals, a handlebar, a seat, etc.

Events also have parts; for example, a war has parts, including most notably battles, and a football game is divided into two halves, each of which is divided into two quarters. Similarly, a baseball game is divided into nine innings, each of which is divided into a "top" and "bottom", each of which is divided into three or more "at bats". A tennis match is divided into sets, which are divided into games which are divided into points.

Mediated material – i.e., content conveyed via media – also often have parts. For example, a book (the message, not the physical medium) is usually divided into chapters, which are divided into sections, which are divided into subsections, etc. Similarly, a play is usually divided into acts, which are divided into scenes. Similarly, a poem is usually divided into stanzas, which are divided into lines, which are divided into metrical "feet".

In the physical realm, science has pursued the part-whole relation to its extremes. As a result of centuries of astronomical investigation, we now believe that the Earth is part of the Solar System, which is part of the Milky Way Galaxy, which is part of the "local cluster" of galaxies, which is part of the "local super-cluster" of galactic clusters.⁴ Going the other direction, as a result of centuries of chemical and physical investigation, we now believe that chemicals have molecules as parts, which have atoms as parts, which have elementary particles as parts, many of which have quarks as parts.

There is evidently a *largest object* – what philosophers call "the world".⁵ Every object is a part of the world. As such, the world includes all its "local" parts – the "local world", or what astronomers call "the universe" – as well as any non-local parts. Here, the word 'local' means *physically accessible*. It is accordingly very difficult, or even impossible, to empirically confirm any hypotheses about whether the world has any non-local parts – for example, hypotheses concerning matter that "escaped" soon after the Big Bang but before the creation of the local space-time manifold, or hypotheses concerning matter that existed prior to the Big Bang, or might exist after the Big Crunch.⁶

At one end, we have the world. What do we have at the other end? According to the philosophical position known as *atomism*, which was first postulated by Democritus (~460-~370 BC),

³ The prefix 'mer' comes from the Greek word 'meros' which means 'part'. Other words with this root include 'polymer' [many-part] and 'meridian' [middle part of the day, as in *ante meridian* (a.m.) and *post meridian* (p.m.)].

⁴ Even super-cluster is a bit of a reach at the moment.

⁵ This is the "big" sense of the word 'world' not to be confused with the "small" sense according to which 'world' is synonymous with 'the planet earth'. So the 'biggest building in the world' means the biggest building on the planet earth. There is yet another use of 'world' which uses it to refer to a coherent segment of reality, such as the "world of baseball".

⁶ It is presently unknown whether the universe has enough matter that the current expansion of the universe will eventually be reversed by gravitation. But if it does, then the universe would eventually collapse into a super black hole, known as the Big Crunch. This scenario has a beautiful symmetry to it. It seems aesthetically preferable to endless expansion, in which the universe eventually "freezes to death", and thereafter *eternally* lies dormant. The universe, however, was not built to satisfy our aesthetic criteria.

the world divides into ultimate parts – called atoms – which have no further parts. The word ‘atom’ derives from the Greek word ‘atomos’ which combines ‘a’ [not] and ‘tomos’ [cutting].⁷ The original atomic hypothesis concerned matter, but we can consider similar hypotheses about events, space, and time.

Needless to say, the modern usage of the word ‘atom’ as in ‘hydrogen atom’ derives from this usage. Unfortunately, however, when chemists first discovered the chemical "atom" they were a bit hasty in declaring that they had discovered atoms. For, it wasn't that much later that these so-called "atoms" were discovered to have further parts – including electrons, protons, and neutrons! Nevertheless, the word ‘atom’ stuck, in spite of its inappropriateness, and indeed became engraved into our collective consciousness soon after the first "atom bombs" were detonated.⁸

This usage is not *completely* unfortunate, however, since we can use the term ‘atom’ *contextually*, as described in the first entry in *American Heritage Dictionary*:

a part or particle considered to be an irreducible constituent of a specified system.

In other words, we define the relevant system first; then relative to that system we define ‘atom’. For example, "chemical atoms" are the smallest chemicals; chemical atoms have parts, but they are not chemicals.⁹ Logic uses the word ‘atom’ in precisely the same manner; in particular, the word ‘atomic sentence’ refers to sentences that do not have other sentences as parts. Atomic sentences have further parts, including subjects and predicates, but these are not sentences.¹⁰ Other disciplines posit their own special atomic objects. For example, in linguistics, the smallest phonetic unit is called a "phoneme", and the smallest grammatical/morphological unit is called a "morpheme". On the other hand, in biology the smallest genetic unit is called a "gene", and the smallest life-unit is called a "cell". In making up words for the various fundamental units, chemistry struck first, and appropriated the word ‘atom’ for their own purposes. Everyone else has to come up with other words; for example, physicists sometimes talk of "fundamental particles".

4. The Principles of Mereology

Can we reduce the part-whole relation to a few fundamental principles that encompass its core meaning? We propose the following formal theory. First, we propose a special symbol, defined as follows.

$$a < b \quad \begin{array}{l} \text{=:} \quad a \text{ is a part of } b \\ \text{or:} \quad a \text{ is part of } b \\ \text{or:} \quad b \text{ contains } a \end{array}$$

We next propose the following fundamental postulates (axioms).

⁷ A similar word involving cutting is the word ‘anatomy’ which combines ‘ana’ [up] and ‘tomos’ [cutting]. So ‘anatomy’ literally means ‘cutting up’.

⁸ We are now more sophisticated in our speech, and refer to these devices as "nuclear" bombs, which is more appropriate since the processes involved are clearly not atomic (i.e., chemical) but are rather nuclear (i.e., involving protons and neutrons).

⁹ This is not completely accurate, although it is close. The problem is that ions count as chemicals, and the hydrogen ion H⁺ is basically a nucleus, consisting of just the proton.

¹⁰ The actual story is a bit more complicated, due to the presence of operators that form noun phrases from sentences, but this simple story works fine as far as introductory logic goes.

(p1)	$a < b \rightarrow b \nless a$	[asymmetry]
(p2)	$a < b \ \& \ b < c \rightarrow a < c$	[transitivity]
(p3)	$\forall x \{ x < a \leftrightarrow x < b \} \rightarrow a = b$	[uniqueness]
(p4)	$a < b \rightarrow \exists x \{ x < b \ \& \ x \nless a \ \& \ x \neq a \}$	[anti-linearity]

The first axiom claims that the part-whole relation is asymmetric; in other words:

if a is a part of b , then b is *not* a part of a

or: if b contains a , then a does not contain b

For example, since the Solar System contains the Earth, the Earth does not contain the Solar System.

The second axiom claims that the part-whole relation is transitive; in other words:

if a is a part of b , and b is part of c , then a is a part of c .

For example, the Earth is a part of the Solar System, which is part of the Milky Way Galaxy, so the Earth is a part of the Milky Way Galaxy.

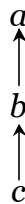
The third axiom claims that distinct objects cannot have exactly the same parts.

if a and b have precisely the same parts, then a and b are the same thing

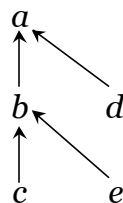
This is logically equivalent to the following.

if a and b are distinct,
then a has a part that b doesn't have
or b has a part that a doesn't have.

The fourth axiom is a bit more complicated, but ensures that the part-whole relation is not linear. It prohibits systems like the following.



In this system, there are just three individuals, where a contains b , which contains c . Although a is bigger than b , it doesn't contain anything over and above b . Likewise, although b is bigger than c , it doesn't contain anything over and above c . What Axiom (p4) requires is that in order to build a bigger object, one must add extra material. For example, the following would be an acceptable reconstruction of the above diagram.



In this system one builds a by adding b and d , and one builds b by adding c and e .¹¹

5. An Aside on Identity

The uniqueness postulate (p3) has some interesting ramifications for the problem of identity, and accordingly *might* be controversial. I offer two examples that might be problematic.

1. Example 1

Suppose we take an automobile A and disassemble it into various parts – muffler, engine, wheels, windshield, etc – and suppose we re-assemble these parts into a sculpture S . It seems that we have destroyed the car A , and used its parts to make a new object, the sculpture S . Common sense tells us that there is an object, the sculpture S , which didn't exist before, but exists now; it also tells us that there is an object, the automobile A , which existed before, but no longer exists. Since these objects differ in critical ways, they are distinct – i.e., $A \neq S$. On the other hand, by hypothesis, S and A have precisely the same parts, so according to (p3) they are identical; i.e., $A = S$.

This appears to be a counter-example to Axiom (p3). The most plausible response is to clarify our mereological principles so that they are time-dependent. In particular, we understand (p3) as follows.

(p3*) at any given time, t if an object a exists at t , and an object b exists at t , then if they have precisely the same parts at t , then they are identical at t .

Therefore, since A and S don't exist at the same time [one is destroyed to make the other], principle (p3*) does not apply to them.

2. Example 2

Our next example does not have a temporal component, but does have a heavy-duty metaphysical component. It concerns the relation between persons and their bodies. We have a number of intuitions about this. In many contexts, statements about a person are easily understood as statements about the person's body; for example, statements about one's location, height, weight, etc. are basically statements about one's body. For example, to weigh 170 pounds is for one's body to weigh 170 pounds. On the other hand, statements about one's hopes wishes and desires do not seem plausibly to be statements about one's body. For example, if I say that I yearn for world peace, it would be very strange to understand this as saying that *my body* yearns for world peace. Similarly, if I say that I decided to have pizza for dinner, it would be very strange to understand this as saying that *my body* decided to have pizza for dinner. Thus, common sense tells us that some attributes apply to persons, but not their bodies. Together with logic, this leads to the following principle.

(i1) no person is identical to his/her body

Common sense also tells us that one's body's parts are also one's parts. For example, there are various body parts that I refer to as "my hands", "my stomach", "my blood".¹² This leads to the following principle.

¹¹ We talk about mereological addition in a later section (8).

¹² Many items *inside* our body do not qualify as body parts, and are accordingly not "ours". These include the contents of our stomach, and otherwise ingested material, as well as invading organisms.

- (i2) every part of a person's body is a part of that person

We are now in position to make the following argument.

- | | | |
|-----|--|----------------|
| (1) | no person is identical to his/her body | (i1) |
| (2) | every part of a person's body is a part of that person | (i2) |
| (3) | if two things have the same parts, they are identical | (p3) |
| (4) | every person has some part that his/her body does not have | 1-3, logic |
| (5) | every person has a non-bodily part | 4, terminology |

I leave the reader to ponder his or her own response to this reasoning.

6. Atoms

Once we have a part-whole relation, we can define 'atom' as follows.

- (d) a is an atom $\equiv_{df} \sim \exists x \{ x < a \}$

In other words, an atom is an object that has no parts.¹³

Now, the notion of atomism includes at least the following principle.

- (a) $\forall x \{ x \text{ is an atom or } \exists y \{ y \text{ is an atom \& } y < x \} \}$

In other words, every object is an atom or contains at least one atom.¹⁴ This is a weak form of atomism. The stronger atomic thesis goes as follows.

- (A) $\forall x \{ x \text{ is an atom} \rightarrow \{ x < a \leftrightarrow x < b \} \} \rightarrow a = b$

Note that this is a logical strengthening of the uniqueness principle (p3). It says that if items a and b have the very same *atomic* constituents, then a and b are the very same item. Equivalently, if a and b are distinct (i.e., $a \neq b$), then one of them must contain an atom the other does not contain.

Another formulation of atomism treats it as a strengthening of anti-linearity, as follows.

- (A') $a < b \rightarrow \exists x \{ x \text{ is an atom \& } x < b \& x \nless a \& x \neq a \}$

In other words, if a is a part of b , then b contains at least one atom distinct from a that a doesn't contain.

7. Disjointness

The word 'two' is occasionally ambiguous between 'distinct' and 'disjoint' (or "completely distinct"). For example, consider the following well-known saying.

you can't be in two places at the same time

¹³ Here, we understand that we are quantifying over the relevant domain of objects. For example, material atoms are different from temporal atoms, spatial atoms, chemical atoms, phonetic atoms, grammatical atoms, etc.

¹⁴ Given anti-linearity (p4), every non-atom must contain at least *two* atoms.

If 'two' means 'distinct' (\neq), then this saying is obviously false. For one can be in distinct places at the same time, since surely one can be in Boston *and* Massachusetts at the same time, and surely Boston and Massachusetts are distinct (i.e., $\text{Boston} \neq \text{Massachusetts}$).

However, this is probably not what one "really" means in offering the above platitude. Rather, one probably means that

you can't be in two *disjoint* places at the same time

The main point is that disjoint pairs are automatically distinct, but not conversely. For example, Boston and Massachusetts are distinct but not disjoint, since in particular Massachusetts contains Boston. This leads to our official definition of 'disjoint' in the context of mereology.

$$(n) \quad a \perp b \quad =: \quad a \text{ and } b \text{ are disjoint}$$

$$(d) \quad a \perp b \quad =_{\text{df}} \quad \sim \exists x \{ x \prec a \ \& \ x \prec b \}$$

In other words, a and b are disjoint precisely if they have no part in common. It immediately follows (exercise!) that if a and b are disjoint, then neither contains the other.

$$(t) \quad a \perp b \rightarrow a \nprec b \ \& \ b \nprec a$$

8. Mereological Sums

Disjoint items can be put together, either physically or conceptually, to form a new item. This process is called "(disjoint) mereological summation". In particular, we can take any two *disjoint* items a and b , and we can *abstractly* consider the mereological sum $a \oplus b$.¹⁵ Some summations make more sense than others. For example, if we take my left hand as a , and my right foot as b , it seems that there is no natural (organic) object $a \oplus b$. Along the same lines, if we take Al Gore as a , and the super red giant star Betelgeuse as b , then the object $a \oplus b$ seems too absurd to even contemplate.¹⁶ On the other hand, if we take the Earth as a , and the Moon as b , then the object $a \oplus b$ seems like a natural object. It even behaves in a highly predictable manner – the Moon orbits the Earth, and the unit orbits the Sun. It similarly seems reasonable to sum all my cells and all the interstitial "goo" into a biological unit known as my body.

The following principles apply to mereological sums.

- (s1) $a \prec a \oplus b$
 $b \prec a \oplus b$
- (s2) $a \prec c \ \& \ b \prec c \rightarrow a \oplus b \prec c$
- (s3) $a \oplus b = b \oplus a$
- (s4) $a \perp b \ \& \ a \perp c \rightarrow a \perp b \oplus c$
- (s5) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
- (s6) $a \oplus b = a \oplus c \rightarrow b = c$

¹⁵ In principle, we can also consider summing overlapping objects, but we do not consider this option. The circle-plus operator only applies to disjoint pairs of individuals.

¹⁶ Not thereby implying that the individual objects are too absurd to contemplate.

Note carefully that these are abbreviations of much more complicated expressions that clarify the dependence of $x \oplus y$ on the disjointness of x and y . For example, (s3) is more properly written as follows.

$$(s3^*) \quad \text{if } a \perp b, \text{ then } a \oplus b = b \oplus a$$

9. Mereology and Set Theory

Set theory provides a vivid example of a mereological system. In particular, we consider non-empty sets, and we take the proper-inclusion relation as the part-whole relation. Recall that set-inclusion is defined as follows.

$$A \subseteq B \quad =_{\text{df}} \quad \forall x \{ x \in A \rightarrow x \in B \}$$

In other words, A is included in B if and only if every member of A is also a member of B . This is not the part-whole relation, since a set is automatically included in itself. Rather, the part-whole relation corresponds to the *proper*-inclusion relation, defined as follows.

$$A \subset B \quad =_{\text{df}} \quad A \subseteq B \ \& \ A \neq B$$

It is fairly easy to demonstrate that the proper-inclusion relation satisfies all the postulates of mereology, including the atomism postulate, provided we exclude the empty set from consideration.

- (p1) $A \subset B \rightarrow B \not\subset A$
- (p2) $A \subset B \ \& \ B \subset C \rightarrow A \subset C$
- (p3) $\forall X \{ X \subset A \leftrightarrow X \subset B \} \rightarrow A = B$
- (p4) $A \subset B \rightarrow \exists X \{ X \subset B \ \& \ X \not\subset A \ \& \ X \neq A \}$
- (d) $A \text{ is an atom} \quad =_{\text{df}} \quad \sim \exists X \{ X \subset A \}$
- (a) $\forall X \{ X \text{ is an atom or } \exists Y \{ Y \text{ is an atom} \ \& \ Y \subset X \} \}$
- (A) $\forall X \{ X \text{ is an atom} \rightarrow \{ X \subset A \leftrightarrow X \subset B \} \} \rightarrow A = B$
- (A') $A \subset B \rightarrow \exists X \{ X \text{ is an atom} \ \& \ X \subset B \ \& \ X \not\subset A \ \& \ X \neq A \}$

Note that we explicitly exclude the empty set \emptyset from consideration in these formulas. For example, the stricter formulation of (d) is as follows.

$$(d) \quad A \text{ is an atom} \quad =_{\text{df}} \quad A \neq \emptyset \ \& \ \sim \exists X \{ X \neq \emptyset \ \& \ X \subset A \}$$

10. Expanding Measurement to A Wider Context

Set-size provides the paradigm for measurement, but not all measurements conform to its rigorous standards. Absolute scales conform exactly, but I know of no absolute scale that does not reduce to counting individuals. Below absolute scales in the hierarchy are *additive scales*. What makes a scale additive? Well, addition! But what is addition? Well, mereological summation – whose exact definition depends upon the context. For example, in set-theory, addition corresponds to disjoint-union. In particular,

$$A \oplus B \text{ exists precisely if } A \perp B, \text{ in which case } A \oplus B = A \cup B$$

For example, any collection A of apples is disjoint from any collection B of bananas, so the disjoint union – apples-and-bananas – is the sum $A \oplus B$.

We can now state the additivity requirement for set sizes as follows.

$$\#(A \oplus B) = \#(A) + \#(B)$$

This in turn allows us to informally define additive scale as follows.

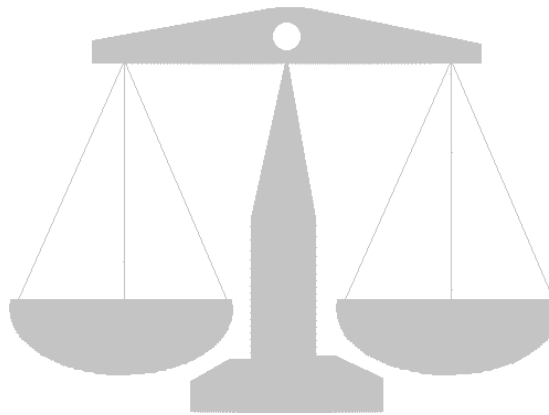
A measurement scale is additive if and only if it "admits" a summation operation \oplus .

The problem is the vagueness of the term 'admits'. So, in the next few sections, we examine a particular example of addition, in order to illustrate what we mean.

11. Weight

Probably the most straightforward physical example of an additive measurement scale is provided by the measurement of weight. The measurement of weight probably goes back very far in the history of civilization, and played an increasingly important role in commercial transactions. Most commodities had their own special methods of measurement – whether by volume or some other evident unit. But it seems that it would be very difficult to buy and sell metal and gemstones (for the production of tools and decorations) without a reliable way of comparing and measuring weights. And even today, many products are sold by weight, not volume.

The fundamental device for measuring weight is the balance scale (or beam balance), which is graphically depicted as follows.



There are two ways of using a balance scale. According to the fundamental method, one keeps the balance point (the fulcrum) at a fixed location, calibrated so that equal weights exactly balance each other. According to the more sophisticated method, first discovered by Archimedes (a287-212 BC) in his study of levers, one adjusts the balance point, and measures the corresponding distances once the two weights are exactly balanced. A variant of this method is used in scales in doctors' offices.

Let us concentrate on the fundamental method. How do we use a balance scale to construct an additive measurement scale for weight? What is the logic? First, we must realize that a balance scale does not measure weight *per se*, but only compares weights of various objects placed in the pans.¹⁷ Here is how it works (for those who have never actually seen an "analog" measuring device work). In order to compare two objects *a* and *b*, we place them in separate pans. Then, according to the way the balance tips, we conclude one of the following.

¹⁷ The word 'scale' as used here comes from the old Norse word *skál* which means 'bowl'.

- | | | | |
|-----|--------------|-------------------------------|--|
| (1) | $a > b$ | a is heavier than b | if the scale tips toward a |
| (2) | $b > a$ | b is heavier than a | if the scale tips toward b |
| (3) | $a \equiv b$ | a and b are equally-heavy | otherwise (i.e., if a and b balance exactly) |

At this point, we need to consider the issue of reliability. In particular, it is critical that the result we obtain – $a > b$, $b > a$, $a \equiv b$ – does not depend upon which pan we place each item in. So for example, if the placement a – b [a left, b right] judges a and b to be equally-heavy, then the opposite placement b – a should likewise judge them to be equally-heavy. This is an issue of mechanical calibration; a measurement instrument that is out of calibration will provide misleading results. Many further issues surround this one, but let us for the moment presuppose that our device is reliable in at least this sense.

The issue of reliability is actually subordinate to a much bigger issue – whether the *logic* of the terms ‘heavier’ and ‘equally-heavy’ match the empirical facts. As we saw earlier, the logical principles of comparison words like ‘heavier’ and ‘equally-heavy’ reduce to the following four principles.¹⁸

- (h1) a and b are equally-heavy iff a is not heavier than b and b is not heavier than a
- (h2) if a is heavier than b , then b is not heavier than a
- (h3) if a is heavier than b , and b is heavier than c , then a is heavier than c
- (h4) if a and b are equally-heavy, and b and c are equally-heavy, then a and c are equally-heavy

Notice that, since ‘heavier’ and ‘equally-heavy’ have been operationally defined by reference to our balance scale, there is no *a priori* guarantee that these four principles obtain. We could certainly imagine other operational definitions that would clearly not yield (h1)-(h4).¹⁹ Another way to describe the logical situation is to say that we have *proposed* a system for measuring weight, and we have four minimal logical criteria – (h1)-(h4) – by which we judge the proposed system.

So let us judge our balance-scale system against these four criteria. We first note that both (h1) and (h2) follow logically from the manner in which we have operationally defined ‘heavier’ and ‘equally-heavy’. Concerning (h1), on any given occasion, we judge a and b to be equally-heavy precisely if the scale does not tip either direction; on each such occasion, we simultaneously judge a is not be heavier than b and b is not to be heavier than a . Concerning (h2), given the manner in which we compare a and b , by placing them in opposite pans, if the scale tips toward a , then *a fortiori* it does not tip toward b , and vice versa. This is built into our understanding of the function of the balance scale.

There is a bit of a logical ruse in the above argument. Principles (h1)-(h4) are disguised universally-quantified formulas, which permit us to substitute the same item for two different variables. So in particular, the following are logical consequences.

- (h1') a and a are equally-heavy iff a is not heavier than a [and a is not heavier than a]
- (h2') if a is heavier than a , then a is not heavier than a

In connection with our operational definition of ‘equally heavy’, (h1) and (h2) logically yield the following.

a balances a

¹⁸ Note also that (h3) and (h4) can be replaced by the following:

(h3*) if a is *not* heavier than b , and b is *not* heavier than c , then a is *not* heavier than c

¹⁹ For example, in Ancient Rome, we could let the Emperor estimate weights. In particular, whenever we wish to compare two objects, we present them to the Emperor and let him deem which one, if either, is heavier. Even if the Emperor was not insane (as many apparently were!), the resulting scale would most likely fail to meet the logical criteria (h1)-(h4).

Now, here is the operational problem. A balance scale can only compare the weights of two *different* (disjoint) objects; it cannot compare the weight of an object with itself. This is because, quite simply, we cannot place a single object on both pans simultaneously to check whether it balances itself! So how can we empirically confirm that a single object balances itself?

There are a couple of approaches to this problem. According to one approach we simply declare by fiat that an object balances itself. In effect, we imagine balancing the object with an exact duplicate. The second approach is to alter the question from “how do we confirm that a single object balances itself?” to “how do we refute that a single object balances itself?”. It seems that we cannot refute this claim because we cannot place an object in both pans and observe that it doesn’t balance. If we take the impossibility of refutation as confirmation, then we can confirm the hypothesis that every object balances itself.

So far, we have verified two principles – (h1) and (h2) – simply by examining the logical character of our operational definitions. This changes when we examine the remaining two principles, which do not follow merely from the definition of the terms, and accordingly must be tested empirically. The question is whether the following are born out by the facts.

if we compare a and b , and judge that $a > b$ [$a \equiv b$]
 and we compare b and c , and judge that $b > c$ [$b \equiv c$],
 then when we compare a and c , we will judge that $a > c$ [$a \equiv c$].

For the sake of argument, let’s suppose that we have hundreds of test-subjects in our laboratory that we have inter-compared laboriously (it’s a laboratory, yes?) Let us further suppose that our observations confirm (h3) and (h4), which is to say that we test every possible triple a - b - c , and we discover no refutation.²⁰

At this point, we have constructed an ordinal classification of our test-subjects, which we can picture as follows.

$$\dots \boxed{W_1} < \boxed{W_2} < \boxed{W_3} < \boxed{W_4} < \boxed{W_5} < \boxed{W_6} \dots$$

Each of these categories corresponds to a *weight-class*, which is simply a collection of items that are all equally-heavy. Thinking of the categories as bins, two items are placed in the same bin precisely when they balance each other on the scale. Furthermore, the bins are arranged left-to-right in ascending order according to weight. For example, W_2 is to the right of W_1 , so every item in W_2 outweighs every item in W_1 .

12. Comparing Mereological Sums

Although we now have a perfectly respectable ordinal classification of our test-subjects, we still don’t have anything resembling “numbers”, which is what we ultimately would like to have. Towards this goal, we must construct an additive scale. As mentioned earlier, the key to an additive scale is having a method of mereological summation. In our example, mereological summation is completely straightforward – we can form the appropriate sum of two objects simply by placing them together in the same pan. Notice that, in this case, the compound object $a \oplus b$ need not be a completely natural organic unit.

²⁰ This includes the trivial confirmation that if $a \equiv b$, then $b \equiv a$, so $a \equiv a$.

The next step is to perform a number of rather tedious measurements of the following sort. Specifically, we go through all our weights and make judgments of the following two forms.

$a > b \oplus c$ $a < b \oplus c$ $a \equiv b \oplus c$		$a \oplus b > c \oplus d$ $a \oplus b < c \oplus d$ $a \oplus b \equiv c \oplus d$
---	--	--

In the first comparison, we place item a on one pan, and we place items b and c on the other pan, and we observe which direction, if either, the balance tips. In the second comparison, we place items a and b on one pan, and we place items c and d on the other pan, and we observe which direction, if either, the balance tips.

At this point, we must further check empirically whether $>$ and \equiv continue to be transitive. If they are not, we are dead in the water; our attempt to construct an additive scale is already doomed. However, let's suppose the gods of weight and balance continue to smile upon us, and we find *no* experimental refutations of transitivity.

Our next step is to examine judgments of the form:

$$a \equiv b \oplus c$$

These judgments are especially important, because (if all goes well) they will enable us to define addition for weight-classes. In an additive scale, it makes sense to add categories. In the present operational setting, each category is a weight-class, which consists of concrete objects that all balance each other on the scale. So, how do we add weight-classes?

This is our proposal – to add two weight-classes W_1 and W_2 operationally, we take an item a from W_1 , and we take an item b from W_2 , and we compare the mereological sum $a \oplus b$ with all other items. Now, suppose that we observe the following.

$$a \oplus b \equiv c$$

where c comes from category W_3 . We then define weight-class addition so that:

$$W_1 \oplus W_2 = W_3$$

There is a potentially very serious problem with this definition. Specifically, the following scenario is still *logically* possible.

we pick a different item from W_1 , say a' ;
 we pick a different item from W_2 , say b' ;
 we judge that $a' \oplus b'$ balances item d ,
 BUT d does not balance c , so d is in a different weight-class from c

In this case, we would have conflicting sums.

$$\begin{aligned} W_1 \oplus W_2 &= W_3 \\ W_1 \oplus W_2 &= W_4 \\ \text{but} \\ W_3 &\neq W_4 \end{aligned}$$

This is contradictory by identity logic, and is therefore unacceptable.

We accordingly must empirically confirm that the imagined scenario does not in fact arise. This amounts to testing the following key principle, which we add to our list of principles.

$$(h5) \quad a_1 \equiv a_2 \ \& \ b_1 \equiv b_2 \ \rightarrow \ a_1 \oplus b_1 \equiv a_2 \oplus b_2$$

The actual experiment is easy to imagine. We take two items – a_1 and a_2 – from one bin, and two items – b_1 and b_2 – from another bin, and pair them appropriately in the pans – $a_1 \oplus b_1$ and $a_2 \oplus b_2$ – and observe whether the scale balances. Each time it balances, we have a confirmation of (h5), but each time it doesn't we have a (conclusive) refutation of (h5).

As we have done previously, we presume that the weight-gods are smiling on us, and (h5) stands up to rigorous testing.

13. Expanding the Weight Scale

The ever alert reader may have noticed in the previous section that it is surely conceivable that we discover *no equivalences* of the form:

$$a \oplus b \equiv c$$

In that case, how do we define addition of weight-classes?

So far, we have presumed that the variables in (h1)-(h5) range over unitary test-subjects. For example, in the expression ' $a \oplus b$ ', we have presumed that a and b are unitary objects, and not mereological sums. The next step in our logical construction is to remove this restriction and understand (h1)-(h5) as pertaining to arbitrary (disjoint) mereological sums *in addition to* unitary test-subjects (which can be thought of as "unit" sums). Alternatively stated, our domain of measurement now includes all the original test-subjects (in effect, the atoms of this context, since we are not sub-dividing them) as well as all their disjoint sums.²¹

We next construct a database recording all comparisons of the following forms.

$a_1 \oplus \dots \oplus a_m$	$>$	$b_1 \oplus \dots \oplus b_n$
$a_1 \oplus \dots \oplus a_m$	$<$	$b_1 \oplus \dots \oplus b_n$
$a_1 \oplus \dots \oplus a_m$	\equiv	$b_1 \oplus \dots \oplus b_n$

As before, this provides an ordinal scale of weight-classes. The key change is that we are considering a much wider class of test-subjects which includes all the original test-subjects and additionally all the mereological compounds. We can no longer picture this by way of bins, since the various compounds will share many atoms in common, which cannot accordingly be placed in two different bins. We can still make a simple database, a tiny fragment of which might look like the following.

²¹ There is an immediate practical problem – we might not be physically able to place all the test-subjects on the pans of our balance scale, because there are too many! However, let's pretend we can. Either the test-subjects are very small or the balance scale is very big.

	simple		compound	
W_1	a	b		
W_2	c		$a \oplus b$	
W_3			$a \oplus c$	$b \oplus c$
W_4			$a \oplus b \oplus c$	
...

In this table are listed four weight-classes, one of which only has a single member. The columns are subdivided into atomic/simple and molecular/compound items. For example, weight-class W_1 consists of two simple items a and b and no compound items, and weight-class W_2 consists of one simple item c and one compound item $a \oplus b$. Notice also that some compounds are equivalent to atoms, and some are not.

14. Ratios

The next task is to introduce ratios. Suppose that we observe the following.

$$\begin{aligned} a &\in W_1 \\ b &\in W_2 \\ c &\in W_2 \\ a &\equiv b \oplus c \end{aligned}$$

Thus, b and c are equally-heavy and together balance a . From what we have learned so far, we can deduce the following.

$$\text{for any } x, y, z: \text{ if } x \in W_1, \text{ and } y, z \in W_2, \text{ then } x \equiv y \oplus z.$$

This can be described as follows.

any **one** item from W_1 is equal-in-weight to any **two** items from W_2

We can even go so far as to say that:

any item from W_1 is **twice** as heavy as any item from W_2

The word ‘twice’ is a ratio word; other ratio words include:

two times	three times	...
one-half	one-third	...
three-halves	four-thirds	...
two-thirds	three-fourths	...

Ratio words appear in comparison statements such as:

there are **twice** as many apples as bananas
there are **half** as many bananas as apples

Ratio words denote special set-relations called "ratios", which correspond to the positive rational numbers.²²

We can construct a whole host of ratios from our laboratory items by comparing arbitrary mereological sums, including the following.

$a_1 \equiv b_1 \oplus b_2$	$a_1 \equiv b_1 \oplus b_2 \oplus b_3$	$a_1 \equiv b_1 \oplus b_2 \oplus b_3 \oplus b_4$...
$a_1 \oplus a_2 \equiv b_1 \oplus b_2 \oplus b_3$	$a_1 \oplus a_2 \equiv b_1 \oplus b_2 \oplus b_3 \oplus b_4$
$a_1 \oplus a_2 \oplus a_3 \equiv b_1 \oplus b_2 \oplus b_3$	$a_1 \oplus a_2 \oplus a_3 \equiv b_1 \oplus b_2 \oplus b_3 \oplus b_4$

Here, items named by the same letter are presumed to be equivalent [e.g., $a_1 \equiv a_2$, $b_2 \equiv b_3$].

This investigation will produce a database that can be tabulated as follows.

	a_1	a_2	a_3	a_4	...
a_1	R_{11}	R_{12}	R_{13}	R_{14}	...
a_2	R_{21}	R_{22}	R_{23}	R_{24}	...
a_3	R_{31}	R_{32}	R_{33}	R_{34}	...
a_4	R_{41}	R_{42}	R_{43}	R_{44}	...
...

Here the test-subjects are enumerated – a_1, a_2, \dots – and the weight-ratios are double-enumerated – R_{11}, R_{12}, \dots . For example, R_{34} is the ratio of a_3 to a_4 .

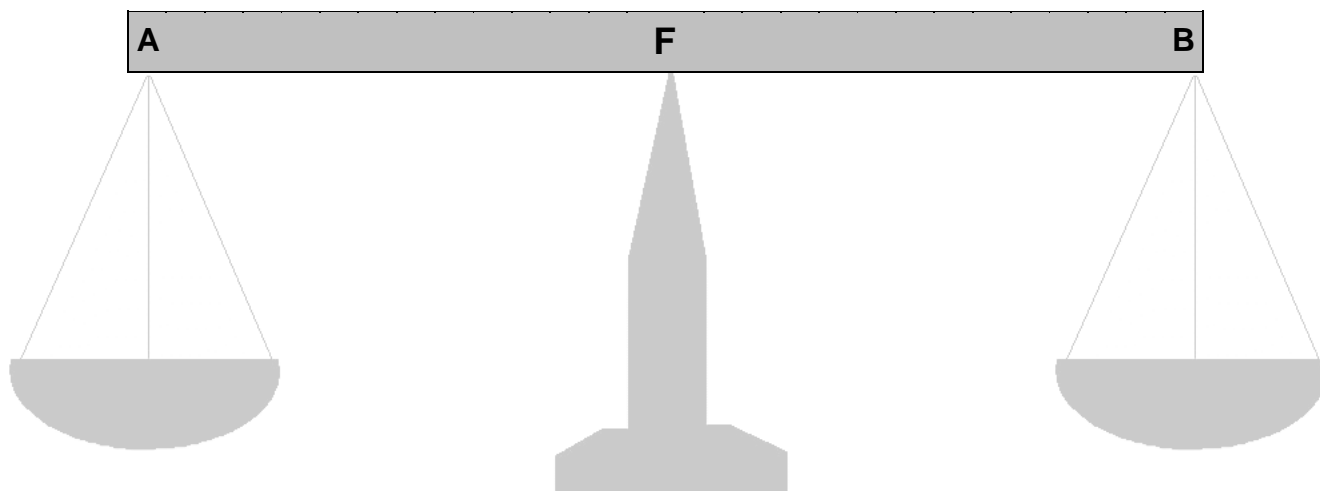
An investigation of the database will suggest that it contains a lot of redundant information. For example, the operational definition of weight-ratios implies that the ratio of a to b will automatically be the inverse of the ratio of b to a . For example, if a is twice as heavy as b , then b is half as heavy as a . More generally, if a is m/n as heavy as b , then b is n/m as heavy as a . This has a further consequence that all the diagonal elements are 1/1 – every object is 1/1 times as heavy as itself. That means we can compress the table of ratios as follows.

	a_1	a_2	a_3	a_4	...
a_1		R_{12}	R_{13}	R_{14}	...
a_2			R_{23}	R_{24}	...
a_3				R_{34}	...
a_4					...
...

²² See chapter "Other Numbers".

15. The Archimedean Balance Scale

Earlier we mentioned that a balance scale has a fundamental method, and a more sophisticated method, for comparing weights. According to the fundamental method, the balance point is fixed, so when we compare two items (simple or compound), all we can judge is whether the balance tips toward one of the items. According to the more sophisticated method, originally discovered by Archimedes, and in principle employed in official medical scales in doctors' offices, one adjusts the balance point. The adjusted picture is as follows.



In this apparatus, the balance point (fulcrum) **F** can be adjusted in location until the scale achieves balance, at which point one compares the distance between **A** and **F**, with the distance between **F** and **B**. The weight-ratio is then operationally defined as follows.

$$\text{weight-ratio}(a/b) = \text{distance}(\mathbf{F}, \mathbf{B}) / \text{distance}(\mathbf{A}, \mathbf{F})$$

The advantage of this type of scale is that it allows us to compute weight-ratios very quickly; the disadvantage is that its precision is limited by our ability to make precise movements of **F**, and to make correspondingly precise measurements of distance.

In any case, it is still a purely empirical matter whether an Archimedean balance scale agrees in its weight judgments with the conventional balance scale. Let us presume that, in principle, the two devices can be adjusted so that they are in complete agreement.

16. Deflection Scales²³

The common bathroom scale is built on different physical principles – that a spring is deflected (squeezed or stretched) by a specifiable amount as a function of the force applied to it. To measure the weight of something, one measures the deflection of the spring. Although most bathroom scales are neither very precise nor very accurate (because they don't need to be!), the *general* method of deflection, according to which a change in applied force results in a specifiable alteration ("deflection")

²³ As far as I know, the term 'deflection scale' is my own invention. I have been unable to find a general term for these kinds of scales.

in a physical system, applies to all manner of modern weight scales, some of which are astonishingly precise and accurate.²⁴

Let us concentrate on scientific instruments. The chief difference between a balance scale and a deflection scale is that the latter must be calibrated *on site* before it can be properly used. There is simply no way that a manufacturer can know exactly what the force of gravity is at the site in question. Accordingly, standard weights must be sent along with the device. One weighs the standard masses, and adjusts the scale until the read-out gives the correct values.²⁵ But we are getting ahead of ourselves, since we still haven't officially discussed either "values" or "standard weights". For this we need some further logical machinery.

17. Units

The next step in the construction of a genuine ratio-scale is to test the following hypothesis against the data.

if a is **m -times** as big as b ,
and b is **n -times** as big as c ,
then a is **$m \times n$ -times** as big as c

Here, we introduce ratio-multiplication in the standard manner.²⁶ The following is a simple and natural example of ratio-multiplication.

if a is **twice** as big as b ,
and b is **twice** as big as c ,
then a is **four-times** as big as c

Supposing this hypothesis stands up to scrutiny, we can compress our database by eliminating all but one row from the original table. For example, we can summarize our data as follows.

	a_1	a_2	a_3	a_4	...
a_1	R_{11}	R_{12}	R_{13}	R_{14}	...

Or, we can equally well summarize our data by *any* of the following single-row tables.

	a_1	a_2	a_3	a_4	...
a_2	R_{21}	R_{22}	R_{23}	R_{24}	...

²⁴ One such scale, invented by Setra Systems (Boxborough, Mass), measures capacitance across ceramic beams, which changes as a function of the strain on them. See <http://www.setra.com/wei/tec/tec.htm>.

²⁵ This is because we are usually interested in an object's mass, rather than it's weight, which is proportional to it's mass (unless the object is weightless!) For example, a dieter would not qualify as losing "weight" if he/she merely goes to the Moon, where the force of gravity is considerably less than on the Earth. But even across the Earth, the force of gravity varies enough that scientifically exact measurements of mass often must take this into account.

²⁶ For example, two-thirds multiplied by three-fifths is two-fifths.

	a_1	a_2	a_3	a_4	...
a_3	R_{31}	R_{32}	R_{33}	R_{34}	...

	a_1	a_2	a_3	a_4	...
a_4	R_{41}	R_{42}	R_{43}	R_{44}	...

...

Logically speaking, which row we choose is completely arbitrary. Practically speaking, selecting a particular row amounts to selecting a "standard" object – either a_1 , or a_2 , or a_3 , or ... – and stating all weight information by reference to this standard.

For example, suppose we select a standard object, and name it "Libby",²⁷ which we derive from 'Libra' which is Latin for balance scale. We can then summarize all weight information by reference to Libby. For example:

a weighs *twice* as much as Libby
 b weighs *half* as much as Libby
 c weighs *six times* as much as Libby
 d weighs *two-thirds* as much as Libby

We can abbreviate this information as follows.

$\text{weight}(a) = 2 \mathcal{L}$
 $\text{weight}(b) = 1/2 \mathcal{L}$
 $\text{weight}(c) = 6 \mathcal{L}$
 $\text{weight}(d) = 2/3 \mathcal{L}$

The critical point is that if we know how two objects compare to Libby, we can deduce how they compare to each other. For example, since a is twice \mathcal{L} , and b is half \mathcal{L} , we can deduce that a is four-times b , and accordingly b is one-fourth a .

18. Standardized Units

At this point we have introduced a measurement unit into our weighing – the "Libby". At the moment, its role is primarily to simplify bookkeeping; in particular, a single row of data is far preferable to a correspondingly large matrix of data. Also, granting scientific induction, we can surmise that to measure the weight of a new test-subject, we do not have to compare it to all the existing test-subjects; rather, we only need to compare it to our standard test subject Libby, which we now keep handy for this purpose.

The next step in the development of weight measurement arises from the need to communicate our measurement results to other people (or even to ourselves at a later time!). For intra-laboratory comparisons, an intra-laboratory standard – Libby – is all we need. But for extra-laboratory

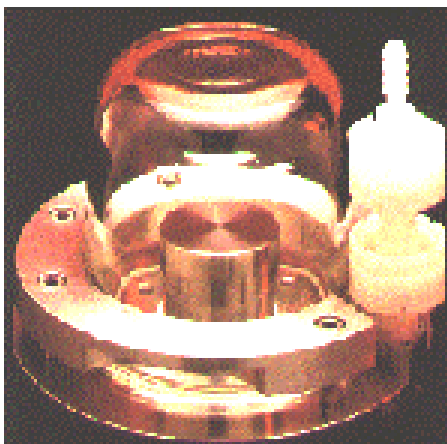
²⁷ The actual name is largely irrelevant. We can also call our standard object "Jack" or "Jill" or "Graham" or "Lucilla" or whatever!

comparisons, we need a *transmittable standard*, over space and over time. If a later investigator reads that my sample weighed 20 Libbys, that investigator must be able to ascertain what a Libby is! If I somehow transmit a copy of Libby to the investigator, then this problem is solved.²⁸

If we want to communicate our measurements to a wide range of people, then a more publicly available standard must be used. Various standards have been adopted over the centuries. It was not until the rise of science that international standards became important for the communication of scientific results. The need for standards was later seen to be equally critical to industrial development and international commerce. The so-called "metric" system was first officially proposed in France soon after the French Revolution (1791), based on these concerns, and it was disseminated to the rest of Europe during the 19th Century (partly aided by Napoleon's conquest of Europe). The International Bureau of Weights and Measures still resides in France.

Standards have changed over the years. The meter was originally defined to be 1/10,000,000 of a quadrant of a great circle of the Earth,²⁹ an absurdly difficult distance to measure. Once this distance was actually measured, in 1798, a standard prototype meter platinum meter stick was constructed and installed at the Archives of the Republic, in Paris. Later (1889), a new prototype object was constructed, and the meter was defined by explicit reference to this concrete object. More recently (1983), the standard has been changed so that a meter is defined to be the distance light travels (through a vacuum) in 1/299,792,458 second. The second, in turn, is currently defined by reference to atomic clocks.³⁰

Most measurement standards are defined in terms of reproducible *types* of physical phenomena, such as atomic resonance and the transmission of light through a vacuum. The notable exception is weight, which continues to be measured against a unique concrete object (prototype), currently housed at the International Bureau of Weights and Measures, in Sèvres, near Paris.³¹ At present, I am unable to locate a picture of the "Paris kilogram", so well-hidden is it, but secondary standards also exist around the world, which are copies of the Paris kilogram.³² The following is a picture of the standard kilogram in the U.K.,³³ which (like all the others) is a Platinum-Iridium cylinder housed in a specially sealed container.



²⁸ Mostly! A number of issues arise. For example, does the standard change weight over time? Also, does it weight different amounts in different places.

²⁹ In particular, the great circle passing through the North Pole and Paris.

³⁰ A second is officially defined as "the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom."

³¹ Bureau International des Poids et Mesures (BIPM).

³² The production of a standard is largely arbitrary, but the production of a copy is an incredibly difficult process, since in principle there is absolutely no room for error. This level of exactness can of course never be achieved.

³³ This picture comes from <http://www.dti.gov.uk/NMD/nmspu.htm>.