Infinite Sets and Infinite Sizes

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1. Review

We have seen that, in English, number-words are used both as adjectives and as nouns, as illustrated in the following examples.

> I have <u>two</u> brothers
> <u>two</u> is the only even prime number

Our semantic account of number-words furthermore maintains a straightforward relation between numerical-nouns and numerical-adjectives, given as follows.

> a numerical-noun *names* what a numerical-adjective *expresses*

Adjectives express properties, and numerical-adjectives in particular express properties of sets, specifically their sizes. To put it succinctly:

numbers are sizes

The *size* of a set is how many members it has. The following list summarizes the various cases.

$\underset{\sim}{0}[A]$	i.e., A is memberless
$\underset{\sim}{1}[A]$	i.e., A is single-membered
$\underset{\sim}{2}[A]$	i.e., A is double-membered
etc.	

These concepts in turn have rigorous logical formulations as follows.

$$\underset{\sim}{0}[A] =_{df} \sim\exists x \{ x \in A \}$$
$$\underset{\sim}{1}[A] =_{df} \exists x \{ A = \{x\} \}$$
$$\underset{\sim}{2}[A] =_{df} \exists x \exists y \{ x \neq y \ \& \ A = \{x,y\} \}$$
$$\underset{\sim}{3}[A] =_{df} \exists x \exists y \exists z \{ x \neq y \ \& \ x \neq z \ \& \ y \neq z \ \& \ A = \{x,y,z\} \}$$
etc.

The *size* of set A – denoted $\#(A)$ – is then defined as follows.

$$\#(A) = 0 \leftrightarrow \underset{\sim}{0}[A]$$
$$\#(A) = 1 \leftrightarrow \underset{\sim}{1}[A]$$
$$\#(A) = 2 \leftrightarrow \underset{\sim}{2}[A]$$
etc.

2. Infinite Sets

How many natural numbers are there? How big is the set $\mathbb{N}$ of natural numbers, which is defined as follows.

$$\mathbb{N} =_{df} \{ x : x \text{ is a natural number} \}$$
$$=_{df} \{0, 1, 2, \ldots\}$$

Upon reflection, the following should be apparent.

$\#(\mathbb{N}) \neq 0$	$\mathbb{N}$ is not empty
$\#(\mathbb{N}) \neq 1$	$\mathbb{N}$ does not have exactly one member
$\#(\mathbb{N}) \neq 2$	$\mathbb{N}$ does not have exactly two members
etc.	etc.

This is associated with the intuition that, no matter how far you count, you can always count farther. The natural numbers are endless! Another way to describe this is to say that:

there are ***infinitely-many*** natural numbers

This poses a serious conceptual problem. Natural numbers are sizes, but no natural number is the size of the set of all natural numbers. Given this result, there seem logically to be two choices.

(1) the natural numbers do not form a set, contrary to what we presumed above, and accordingly the natural numbers do not have a size (no set, no size);

(2) there is *at least one* size that is not a natural number.

A minority of mathematicians and philosophers deny set-hood to the natural numbers, so for them the problem of how many natural numbers there are does not arise. For these mathematicians and philosophers, there are no *actual* infinities, but only *potential* infinities.[1]

[1] By similar reasoning, although there *may* be a boundless future, which is a *potential* infinity, there *cannot* be a boundless past, which would be an *actual* infinity. On the other hand, it seems that whether the past is infinite or not is an *empirical* matter, not a *logical* matter. To be sure, our minds are boggled when we consider the possibility of an infinite past, which might even contain infinitely-many actual humans. But boggling the human mind is not a reliable sign of impossibility, as history has repeatedly attested. I accordingly reject this sort of reasoning.

Most mathematicians and philosophers, however, are perfectly happy to grant set-hood to the natural numbers, and even more vast collections, and accordingly must come to terms with the question.

what is the size of the set of natural numbers?
$\#(\mathbb{N}) = ?$

The glib answer is "infinity!". Many modern school children can even recite this, and may even be able to reproduce a symbol for it – ∞. More interesting perhaps, the glib answer seems to have satisfied humanity for thousands of years, and even the greatest minds were unable to get past it.

This all changed in the 19th Century when Georg Cantor[2] began to enquire a little more deeply into the issue of infinity, and in the process invented both set theory and the theory of infinite numbers. We have already discussed the basic insight – that two sets are equally large precisely when one can set up a one-to-one correspondence between them. What we haven't discussed, however, is how this insight can be applied to infinite sets.

First, a set is *finite* if and only if it can be counted by a natural number.

A is finite $=_{df}$ $\#(A) = n$ for some natural number n

The *definiens* (right side of the definition) can in turn be thought of as an infinite disjunction with the following disjuncts.

$\#(A) = 0$ or
$\#(A) = 1$ or
$\#(A) = 2$ or
etc.

On the other hand, a set is *infinite* if and only if it is not finite, which means that it cannot be counted by a natural number.

A is infinite $=_{df}$ A is not finite

In this case, the *definiens* can be thought of as an infinite conjunction with the following conjuncts.

$\#(A) \neq 0$ and
$\#(A) \neq 1$ and
$\#(A) \neq 2$ and
etc.

The set $\mathbb{N}$ of natural numbers is a prime example of an infinite set:

$\#(\mathbb{N}) \neq 0$ and
$\#(\mathbb{N}) \neq 1$ and
$\#(\mathbb{N}) \neq 2$ and
etc.

[2] Cantor was born in St Petersburg, Russia on March 3, 1845, and died in Halle, Germany on Jan 6, 1918. Cantor's father was German; his mother was Russian. By way of a little historical context, St. Petersburg was the capital of Russia. Czar Nicholas was Russian, Czarina Alexandra was German, and also a granddaughter of Queen Victoria of England. The Royal couple, along with their five children and several servants, were executed by firing squad by the Bolsheviks on July 17, 1918.

Notice, however, that ℕ is merely one of many examples of infinite sets; the following are a few more examples.

the set of even numbers	{0, 2, 4, 6, …}
the set of odd numbers	{1, 3, 5, …}
the set of prime numbers[3]	{2, 3, 5, 7, 11, 13, …}
the set of numbers ⩾ 10	{10, 11, 12, …}
the set of multiples of 10	{10, 20, 30, …}
the set of powers of 10	{1, 10, 100, 1000, …}

3. Comparing Infinite Sets

We have discussed size-comparison at great length, beginning with the Generic Theory of Size. We know how to compare the sizes of *finite* sets; how do we compare the sizes of *infinite* sets? Under what circumstances do we say that one infinite set is bigger than another infinite set? Does this even make sense? For example, which of the following sets is bigger?

the set of **even** numbers
the set of **all** numbers

Here is an argument that the latter is bigger than the former.

A1. half the numbers are even, and half the numbers are odd;
therefore, there are *twice as many* numbers as there are even numbers;
therefore, there are *more* numbers than even numbers.

Here is a more general argument.

A2. every even number is a number, but not every number is an even number;
therefore, the set of even numbers is a *proper subset*[4] of the set of all numbers;
therefore there are *more* numbers than even numbers.

Now, it is pretty clear that the above reasoning applies to finite sets and finite sizes. What is not so clear is whether it applies to infinite sets and infinite sizes. What Cantor discovered was that many of our intuitions about size do not apply to infinite sets. But what intuitions *do* apply?

Recall that in order to compare the sizes of two sets, we do not have to count them. Even before humans could count, they could compare sets directly; counting was an invention for *indirect* comparison of sets. If we can compare the sets *directly*, then counting is unnecessary. For example, I don't have to count the fingers on my right hand, and the fingers on my left hand, to know that my two hands have equally-many fingers. I don't even have to know *how* to count! All I have to do is pair up the fingers of the two hands.

Now, here is Cantor's insight.

[3] A number is *composite* if and only if it can be factored; for example, 6 can be factored into 2 and 3; i.e., 6 = 2×3. A number is *prime* if and only if it cannot be factored; for example, 7 cannot be factored.
[4] We say that A is a *subset* of B precisely if every member of A is also a member of B. We say that A is a *proper subset* of B precisely if A is a subset of B but B is not a subset of A [in which case $A \neq B$].

> sets A and B are equally-big
> if and only if
> one can set up a one-to-one correspondence between
> elements of A and elements of B,
> **even if A and B are infinitely-big**.

So back to the question whether the following sets are equally-big.

the set of even numbers
the set of numbers

To see that they are equally big, we need merely set up a one-to-one correspondence between them. The following is probably the simplest.

0	↔	0
1	↔	2
2	↔	4
3	↔	6
etc.		

We can do the same thing with all the sets mentioned above.

0	↔	1
1	↔	3
2	↔	5
3	↔	7
etc.		

0	↔	2
1	↔	3
2	↔	5
3	↔	7
etc.		

0	↔	10
1	↔	11
2	↔	12
3	↔	13
etc.		

0	↔	10
1	↔	20
2	↔	30
3	↔	40
etc.		

0	↔	1
1	↔	10
2	↔	100
3	↔	1000
etc.		

4. Denumerable Sets

It is customary to call a set *denumerable* if it has the same size as the set ℕ of natural numbers. In other words:

$$A \text{ is denumerable} \quad =_{df} \quad \#(A) = \#(\mathbb{N})$$

Accordingly, the following sets are all denumerable.

the set of even numbers	{0, 2, 4, 6, …}
the set of odd numbers	{1, 3, 5, …}
the set of prime numbers	{2, 3, 5, 7, 11, 13, …}
the set of numbers bigger than 10	{11, 12, 13, …}
the set of multiples of 10	{10, 20, 30, …}
the set of powers of 10	{1, 10, 100, 1000, …}

Notice, however, that we do not yet have a special symbol for the size of ℕ. We have names of all the finite sizes – 0, 1, 2, … But we have no names for infinite sizes yet. For this purpose, Cantor introduced the following symbol.

$$\aleph_0$$

which is read “aleph naught” or “aleph zero”.[5]

Let us next consider some proper *super*-sets of ℕ.[6] As the reader is doubtless aware, the natural numbers have negative counterparts – the *negative integers* – which are defined as follows.

$$\text{the set of negative integers} \quad =_{df} \quad \{-1, -2, -3, \ldots\}$$

These are not "natural" numbers, because they are not set sizes; for example, “minus four” is not an admissible answer to a how-many question. Nevertheless, the negative integers are valuable mathematical devices for *other* measurement purposes, like measuring the temperature[7] on a very cold day, or measuring your net worth if you are in serious debt!

Next, we obtain the set of *integers* by combining the negative integers with the non-negative integers (i.e., natural numbers).

$$\text{the set of integers} \quad =_{df} \quad \{-1, -2, -3, \ldots\} \cup \{0, 1, 2, 3, \ldots\}$$

How big is the set of integers? It is also denumerable, as seen by the following one-to-one correspondence.

0	↔	0
1	↔	1
2	↔	–1
3	↔	2
4	↔	–2
etc.		

[5] Note that aleph is the first letter of the Hebrew alphabet, or what is sometimes called the ‘alephbet’, since its first two letters are not alpha and beta, but aleph and bet. Perhaps, we should call our alphabet the ‘ay-bee’.
[6] A is a super-set of B if and only if B is a subset of A.
[7] In degrees Fahrenheit, or in degrees Celsius, but not in degrees Kelvin, which does not have negative temperatures.

There are other numbers besides integers, the simplest being the *fractions*. In this context, by a fraction I mean a quantity that may be obtained by dividing one positive integer by a larger positive integer.[8] English provides common words for the simplest of these quantities:

one-half	one-third	one-fourth	…
two-thirds	two-fourths	two-fifths	…
three-fourths	three-fifths	three-sixths	…
…	…	…	…

As with the negative integers, the fractions are not *natural* numbers; for example, "one-third" is not an admissible answer to a how-*many* question. On the other hand, it can serve as an answer to an how-*many-of* question, as in "how many ***of*** the students got an A?" Similarly it can serve as an answer to a how-*much-of* question, as in "how *much of* the pizza did you leave for me?"

How many fractions are there? If a table has m rows, and n columns, then it has a total of $m \times n$ cells. Now the chart depicted above has infinitely-many rows and infinitely-many columns; what is $\aleph_0 \times \aleph_0$? Surely, a "zillion-zillion" is bigger than a "zillion"! If that is not convincing, consider the following reasoning: between any two fractions, there is another one; for example, between one-third and one-half is five-twelfths. So, when you go to count all the fractions, since they are so densely packed, you can't even get from one to the next! No matter what you want to count next, you have to count something else first!

Notwithstanding these initial intuitions, as it turns out, there are no more fractions than there are natural numbers. The proof of this astonishing fact employs the following counting technique invented by Cantor. Here, for the sake of brevity, we write the fractions in the customary fashion.

1/2	→	1/3		1/4	→	1/5		1/6	→	1/7		…
	↙		↗		↙		↗		↙		↗	
2/3		2/4		2/5		2/6		2/7		2/8		…
↓	↗		↙		↗		↙		↗		↙	
3/4		3/5		3/6		3/7		3/8		3/9		…
	↙		↗		↙		↗		↙		↗	
4/5		4/6		4/7		4/8		4/9		4/10		…
↓	↗		↙		↗		↙		↗		↙	
5/6		5/7		5/8		5/9		5/10		5/11		…
	↙		↗		↙		↗		↙		↗	
…		…		…		…		…		…		…

In order to carry out the counting (pairing), one follows the arrows, starting at 1/2, then going to 1/3, then to 2/3, then to 3/4, and so on.[9]

[8] These are sometimes called 'proper fractions', which are then distinguished from "improper" fractions, which include (e.g.) zero-thirds, three-thirds, and seven-thirds.

[9] Notice that we have made no provision for the minor detail that one-half and two-fourths are presumably the same quantity, but this does not affect the result. If we wanted, we could skip each fraction that has already been counted under a separate guise; for example, when we get to 2/4 we skip it since we have already counted it when we counted 1/2.

5. Uncountable Sets

Next, we introduce the concept *countable*, and its complement *uncountable*.

A is countable	$=_{df}$	A is finite, **or** A is denumerable
A is uncountable	$=_{df}$	A is **not** countable

Recall that a set is denumerable precisely if it has the same size as $\mathbb{N}$.[10]

Are there any uncountable sets. From what has transpired so far, one might naturally conclude that all infinite sets are denumerable, which implies that all infinite sets have the same size, which implies that there is just one infinite size – namely, $\aleph_0$. In other words, infinite is infinite…period! Accordingly, there are no uncountable sets.

Once again, however, our intuitions are trampled; in particular, as first proved by Cantor, we have the following.

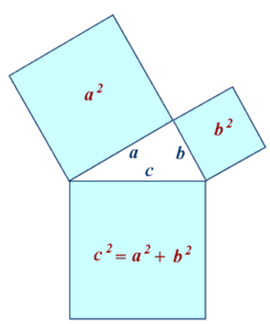
(1) the set of *irrational* numbers is uncountable.
(2) the set of all subsets of $\mathbb{N}$ is uncountable.

First, a *rational number* is not a number that behaves rationally, nor is it a number that believes that all knowledge proceeds from reason. Rather, a rational number is one that can be expressed as a *ratio* of two integers. In other words.

$$x \text{ is rational} \quad =_{df} \quad \exists m \exists n \{ m \text{ and } n \text{ are integers} \quad \text{and} \quad x = m/n \}$$

On the other hand, an *irrational number* is one that cannot be expressed as a ratio. The most famous irrational number is π (pi), which is the fundamental mathematical constant that relates a circle's diameter to its circumference.[11]

The discovery of the irrational numbers was revolutionary. Legend has it that a disciple of Pythagoras[12] took the master's famous theorem, pictured as follows,



[10] Notice that Denumerable = Infinite + Countable. Sometimes, however, uncountable sets are called 'non-denumerable'. Notice the mildly illogical fact that '*non*-denumerable' does not mean '*not* denumerable'. A set that is *not* denumerable may be either uncountable or finite, whereas a *non*-denumerable set cannot be finite.
[11] The letter 'π' probably comes from the Greek word for 'perimeter' which is 'perimetron' (περιμετρον).
[12] Pythagoras (ca. 569 – ca. 475 BC) was born in Ionia on the west coast of Asia Minor (present day Turkey), and traveled extensively – including to Egypt and Babylonia, where he evidently consulted extensively with the high-priests, who were the guardians of mathematical knowledge. Following the Persian conquest of Asia Minor, Babylonia and Egypt, Pythagoras moved west and founded a philosophical and religious society in Croton (modern day Crotone in southern Italy). Its practices included communal living and vegetarianism, at least for the inner circle, who were known as mathematikoi. On the other hand, the outer circle, known as akousmatics, were permitted to live in their own houses, to have their own possessions, and to eat meat. Consult: [http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Pythagoras.html].

and applied it to the case in which $a = b = 1$, and concluded that the hypotenuse c must have a length of $\sqrt{2}$. He/she[13] went on to demonstrate that this number cannot be expressed as a ratio of integers, and is accordingly irrational. As the story continues, since this discovery seriously undermined the religious teachings of Pythagoras, the disciple was subsequently tossed over the side of a boat in the Adriatic Sea, thereby perpetrating one of the earliest (attempted) cover-ups in history.

The proof that $\sqrt{2}$ is irrational, and the proofs of Cantor's two results, are presented in an appendix (Section 10).

6. There are Infinitely-Many Infinite Sizes

Let A and B be sets. Then A is said to be *included in* B precisely if every member of A is also a member of B. This definition may be symbolically rendered as follow.

$$A \subseteq B \quad =_{\text{df}} \quad \forall x \{ x \in A \rightarrow x \in B \}$$

Alternatively, A is said to be a *subset* of B. Next, the *power set* of A is, by definition, the set of all subsets of A. In symbols:

$$\wp(A) \quad =_{\text{df}} \quad \{ X : X \subseteq A \}$$

The following are examples.

$\wp(\varnothing)$	=	$\{ \varnothing \}$	1 member
$\wp\{1\}$	=	$\{ \varnothing, \{1\} \}$	2 members
$\wp\{1,2\}$	=	$\{ \varnothing, \{1\}, \{2\}, \{1,2\} \}$	4 members
$\wp\{1,2,3\}$	=	$\{ \varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$	8 members

Notice that this is the beginning of a process known as a "combinatorial explosion"; in particular, each set is twice as big as the set before; accordingly, this list does not proceed too long before the set in question cannot be physically encoded by humans or even their machines. For example, to physically write all the subsets of the set of the first 1000 numbers would require more molecules than have ever existed in the universe.

Nevertheless, we can write a simple theorem about the *sizes* of these sets, which is given as follows.

$\wp(A)$ is bigger than A, and in particular:

if $\#(A) = n$, then $\#(\wp(A)) = 2^n$

This pertains to finite sets. Is there a corresponding result for infinite sets? According to the more or less standard view, we have the following.

$\wp(A)$ is bigger than A, and in particular:

if $\#(A) = \aleph_n$, then $\#(\wp(A)) = \aleph_{n+1}$

In particular, there are infinitely-many infinite-sizes, as follows.

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[13] Note that women were permitted to be, and were, members of the Pythagorean Society.

$\aleph_0$ the first infinite size (countably-infinite)
$\aleph_1$ the second infinite size (uncountably-infinite)
$\aleph_2$ the third infinite size (seriously-uncountably-infinite)
etc.

7. What will Really Cook Your Noodle [14]

Clearly, there are infinitely-many infinite sizes:

$\aleph_0$, $\aleph_1$ $\aleph_2$, …

Unfortunately, this infinite series is just the beginning of a staggering array of infinite sizes.[15] How many infinite-sizes are there? For this, even the word ‘infinitely-many’ is totally inadequate. We simply have no further concept for how many infinite-sizes there are.[16] As a pathetic substitute for a concept, we propose a word – ‘hyper-infinite’. One would think that surely we can figure out what ‘hyper-infinite’ means just as Cantor figured out what ‘infinite’ means. This proves more difficult than anyone might have suspected, as we see in the next section, which will further cook your noodle.

8. Russell’s Paradox

In 1900, the ornate but well-mannered (dare I say "Victorian") world of sets that Cantor had established came crashing to the ground, when Bertrand Russell noticed that Cantor’s theory of sets contained a fatal flaw. Cantor’s theory of sets, further developed by Gottlob Frege,[17] which is now called ‘naïve set theory’, is based on the following two very simple postulates.

(s1) **the axiom of extensionality**;
no two sets have the very same members;
$\forall x \{ x \in S_1 \leftrightarrow x \in S_2 \} \rightarrow S_1 = S_2$

(s2) **the axiom of comprehension**;
for any property, there is a set that comprehends[18] precisely the objects with that property;
$\forall \mathbb{P} \, \exists S \, \forall x \{ x \in S \leftrightarrow x \text{ has } \mathbb{P} \}$

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[14] I unashamedly borrow the expression ‘what will really cook your noodle’ from the movie “The Matrix”. In one of the scenes, the Oracle tells Neo not to worry about the vase near him, as a result of which Neo turns quickly around only to knock the vase off the table, thereby shattering it. He asks how she knew this would happen (well, duh Neo, she’s the Oracle!) She tells him not to worry, and goes on to say that what will really "cook his noodle" is when he wonders whether it would have happened even if she hadn’t warned him. This is a movie that presents numerous philosophical perplexities (in Greek *aporia*).

[15] By ‘staggering’ I mean staggering *even* to mathematicians who are completely comfortable dealing with infinities of infinities of infinities of …

[16] The difficulty is that any attempt to wrangle all the infinite sizes into a set to contain them all involves a complete *logical collapse* of set theory. See Section 8.

[17] Friedrich Ludwig Gottlob Frege (1848-1925) invented modern symbolic logic, and also the entire field of semantics.

[18] The word ‘comprehend’ basically combines ‘com’ [*together*] and ‘prehend’ [*grasp*]. In this connection, note that some monkeys have *prehensile* (i.e., grasping) tails. The first meaning of ‘comprehend’ is *understand*; we even use the word ‘grasp’ as an occasional synonym. Its second meaning, which leads to the derivative word ‘comprehensive’, which means *to take in as a part*. Note that the cognate word ‘comprise’ produces *serious* usage problems, insofar as it is often used in a manner exactly opposite its literal meaning. In particular, the word ‘comprise’ comes from ‘comprisé’, which in French is the past participle of the verb ‘comprendre’, which of course is a cognate of ‘comprehend’. If we use the word according to its strict meaning, the U.S. comprises its 50 states, whereas the 50 states do *not* comprise the U.S.

In this connection, it is useful to introduce a common philosophical notion – *extension*. The *extension* of a property is, by definition, the set of all objects that have that property. In other words:

the extension of $\mathbb{P}$ $=_{df}$ $\{ x : x \text{ has } \mathbb{P} \}$

We can then interpret the Axiom of Comprehension as saying the following.

(s2*) every property has an extension

Thus far, we have taken this principle for granted. In particular, whenever we have written down a set-abstract of the form

$\{ x : \ldots x \ldots \}$

we have naturally presumed that such a set exists.

This has never seemed to be an issue. The following seem to be some fairly natural examples, both simple and complex.

property	**corresponding set**
being a natural number	natural numbers
being an even number	even numbers
being a prime number	prime numbers
being a memberless set	memberless sets
being a single-membered set	single-membered sets
being a double-membered set	double-membered sets

The sets all *look* pretty clear-cut, so what could be the problem? Well, basically Russell did to Cantor and Frege's theory of sets what Eubulides[19] did a couple of millennia earlier to Aristotle's theory of truth.[20] In particular, he presented a paradox that showed that the theory is *logically inconsistent*. Now logical inconsistency is to theories what 50 megaton bombs are to cities – *fatal*![21]

Russell asked us to consider a property that we will express by the word 'normal'; in particular, an object is said to be *normal* precisely if it is not a member of itself. In other words:

x is normal $=_{df}$ $x \notin x$

Notice that any object that is not a set is automatically normal; only sets have members. Furthermore, most sets are normal; indeed, it is hard to imagine a set that is not normal. There is nevertheless a prominent example of an *ab*normal set – the *universal set* $\mathbb{U}$ – which is defined as follows.

$\mathbb{U}$ $=_{df}$ $\{ x : x{=}x \}$

[19] Eubulides of Megara (not far from Athens) was a contemporary, and continual philosophical opponent, of Aristotle.
[20] This is the famous "Liar Paradox". Suppose that I say that I am lying, and nothing else; then, am I telling the truth, or am I lying?
[21] Only one such "mega-bomb" has ever been detonated, in 1961 by the Soviet Union. Witnesses report that the blast knocked down people fifty miles away, and that the fireball was visible 600 miles away! For the sake of comparison, the Soviet mega-bomb was 2500 times more powerful than the bombs unleashed on the unfortunate cities of Hiroshima and Nagasaki.

It is easy to show that everything is a member of the universal set, since everything is self-identical. So, $\mathbb{U}$ is a member of $\mathbb{U}$, so $\mathbb{U}$ is *ab*normal.

Now, according to Axiom of Comprehension, every property $\mathbb{P}$ has an extension, which is just the set of objects that have that property. The property of being normal is a property, so it has an extension, which is simply the set of normal objects. Let us call that set $\mathcal{N}$. In other words:

$$\mathcal{N} = \{ x : x \text{ is normal} \}$$

What follows from this and the Axiom of Comprehension is that:

$$\forall x\, (x \in \mathcal{N} \leftrightarrow x \text{ is normal})$$

But this formula applies to everything, so it applies to $\mathcal{N}$, so substituting $\mathcal{N}$ for x yields the following seemingly innocuous claim.

$$\mathcal{N} \in \mathcal{N} \leftrightarrow \mathcal{N} \text{ is normal}$$

In other words $\mathcal{N}$ is an element of the set of normal things if and only if $\mathcal{N}$ is normal. But what does it mean to be normal? Well, we know that

$$\mathcal{N} \text{ is normal} \leftrightarrow \mathcal{N} \notin \mathcal{N}$$

so, putting these together, we get

$$\mathcal{N} \in \mathcal{N} \leftrightarrow \mathcal{N} \notin \mathcal{N}$$

which has the following logical form:

$$P \leftrightarrow {\sim}P$$

The latter, however, is a self-contradiction, since it says that the truth-value of P is the same as the truth-value of ${\sim}P$.

Ouch!

As a result of this crisis, a number of mathematicians and logicians undertook to reformulate Set Theory in a way that avoids logical inconsistency. What was produced was *Axiomatic Set Theory*, as it is called. We will not venture into this subject. Suffice it to say that Axiomatic Set Theory is generally regarded as a reliable foundation for mathematics, and no one has proved that it is inconsistent… *yet*.

9. You Must Remember This

We have gone far enough for the moment. The main thing to keep in mind are the following facts, which are sufficiently mind-boggling in their own right.

(1) there are just as many even numbers as natural numbers!

(2) there are just as many natural numbers as rational numbers!

(3) there are more irrational numbers than rational numbers!

(4) there are more subsets of natural numbers than natural numbers!

10. Appendix; Various Proofs

1. A Proof that Ō2 is Irrational

Below, we offer a proof (in more or less standard mathematical form) that $\sqrt{2}$ is irrational.

> In order to prove that $\sqrt{2}$ is irrational, we proceed by *reductio ad absurdum* (alias indirect derivation). In particular, we assume that $\sqrt{2}$ is rational, and show that this leads to contradiction. So, suppose to the contrary that $\sqrt{2}$ *is* rational. Then by definition, it is equal to a ratio of integers. Let us suppose those integers are m and n; i.e., $\sqrt{2} = m/n$. By a well-known theorem of arithmetic, every fraction can be "reduced" to a fraction in which the two integers have *no common factors* [for example, 3/6 reduces to 1/2]. Let us accordingly reduce m/n to its simplest form, which we suppose is p/q. Accordingly, we have:
>
> (1) $\sqrt{2} = p/q$ where p and q have no common factors
>
> By a few simple algebraic manipulations, we can deduce the following equation.
>
> (2) $p^2 = 2q^2$
>
> From this it follows that p^2 is even, from which it follows that p is even [since odd $\times$ odd = odd].
>
> It now follows that q is odd. For if q is even, then both p and q are even, in which case they share 2 as a common factor, which is contrary to (1).
>
> As stated earlier, p is even, so we have the following – for some integer, say, r.
>
> (3) $p = 2r$
>
> Squaring both sides yields:
>
> (4) $p^2 = 4r^2$
>
> Combining (4) with (2), we obtain:
>
> (5) $2q^2 = 4r^2$
>
> from which we immediately obtain:
>
> (6) $q^2 = 2r^2$
>
> from which it immediately follows that q^2 is even, and so q is even. But this contradicts our earlier claim that q is odd.

2. A Proof that the Subsets of $\mathbb{N}$ are Uncountable

In what follows, we offer a proof in more or less standard mathematical form:

> By way of constructing a *reductio ad absurdum* argument, we suppose to the contrary that $\wp(\mathbb{N})$ is countable, in which case $\wp(\mathbb{N})$ is either finite or denumerable. It is fairly easy to dismiss the possibility that $\wp(\mathbb{N})$ is finite (exercise!), so we concentrate on the second possibility, that $\wp(\mathbb{N})$ is denumerable. It follows that there is a one-to-one correspondence between $\wp(\mathbb{N})$ and $\mathbb{N}$. Define f so that:
>
> $f(n)$ = the subset of $\mathbb{N}$ uniquely paired with n according to the alleged correspondence
>
> Call a number n "proper" if it is *not* an element of its counterpart set $f(n)$. Consider the subset $\mathcal{P}$ of all proper numbers, so understood. In other words,
>
> (1) $\mathcal{P} = \{ n : n \notin f(n) \}$

By hypothesis, every subset of $\mathbb{N}$ corresponds to some number; so $\mathcal{P}$ corresponds to some number – call it p. Given the definition of f, we have:

(2) $f(p) = \mathcal{P}$

Now, of course, either p is proper, or p is improper. In either case, we have a contradiction, as we show below.

Suppose (case 1) that p is proper, in which case:

(3) $p \in \mathcal{P}$

Combining (3) with (1) yields:

(4) $p \notin f(p)$

But combining (3) with (2) yields:

(5) $p \in f(p)$

which contradicts (4).

On the other hand, suppose (case 2) that p is improper, in which case:

(6) $p \notin \mathcal{P}$

Combining this with (1) yields:

(7) $p \in f(p)$

And combining this with (2) yields:

(8) $p \in \mathcal{P}$

which contradicts (6).

3. A Proof that the Irrational Numbers are Uncountable

The set $\mathbb{R}$ of *real numbers* is defined as follows.

$\mathbb{R} =_{df}$ the set of rational numbers $\cup$ the set of irrational numbers

Earlier we defined a fraction to be a ratio of integers m/n where $m<n$. Let us henceforth call these *ordinary fractions*. More generally, by a "fraction" we mean a non-negative real number that is strictly less than 1.

x is a fraction $=_{df}$ x is a real number, and $0 \leq x < 1$

Some fractions are ordinary (i.e., rational) fractions; others are extraordinary (i.e., irrational) fractions. In what follows, we argue that the set of fractions, both ordinary and extraordinary, is uncountable. We employ a technique, originally devised by Cantor, called a ‘diagonal argument’.

Suppose, to the contrary, that the set of fractions is countable. Then it is either finite or denumerable. The first case can be quickly dismissed, and is left as an exercise. In the second case, there is a one-to-one correspondence between the fractions and natural numbers. Define f as follows.

$f(n)$ = the fraction uniquely paired with n in the alleged one-to-one correspondence

Next, we appeal to the theorem that every fraction can be expressed as ("encoded by") an infinite decimal sequence, the following being familiar examples.[22]

```
1/2     0.500000000000000…
1/3     0.333333333333333…
1/4     0.250000000000000…
2/3     0.666666666666666…
3/4     0.750000000000000…
7/22    0.318181818181818…
1/π     0.318309886183791…?
1/√2    0.707106781186547…?
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Some of the sequences "terminate", which means that at some point the remaining digits are all 0’s. When a sequence terminates with an infinite sequence of zero’s, we of course omit them in the pronunciation – for example we say "point five", not "point five zero zero zero …", unless we are trying to make a point about precision (about which we will later speak). Other sequences do not terminate, but are nevertheless completely regular, in the sense that the digits eventually group into repeating units, as in the encodings of 1/3, 2/3, and 7/22. Still others are "helter-skelter", like 1/π 1/√2, which have no obviously discernable pattern. These, basically, are the irrational fractions.

Now suppose that we have a one-to-one correspondence between the natural numbers and the set of real fractions, thought of as their decimal encodings. Define f so that:

$f(n)$ = the fraction uniquely paired with n according to the alleged correspondence

Construct a decimal-encoding $\mathcal{E}$ as follows.

take the nth digit of $f(n)$; call it δ;
if δ is 0, then $\mathcal{E}_n = 1$;
otherwise, $\mathcal{E}_n = \delta - 1$.

Notice that $\mathcal{E}_n$ will never be 9; this constraint is included in order to avoid producing an inadmissible encoding (one with a terminal infinite sequence of 9’s). The following is an example, where the relevant digit in each sequence is highlighted.

[22] Note that the algorithm does not automatically produce a unique encoding for each fraction. The problem is that "point five" is identical to "point four nine nine nine …"! This can be seen by subtracting the latter from the former and noticing that the result is zero. In our encoding, we mean to stringently exclude the goofy ones; in particular, we do not count as *admissible* any encoding that has a sub-sequence of endlessly repeating 9’s.

number	fraction	resulting Ɛ
1	.412535286438453485...	3
2	.582938752320309532...	7
3	.340859403890349589...	1
4	.928347584349857348...	2
5	.839453002439549350...	4
6	.483974583945834953...	3
7	.897405723293409855...	6
8	.284634530498503934...	2
....	....	...

We next check to see whether the resulting encoding Ɛ [in this case .37124362…] is one of the items in the list of fractions. Well, it is not the first one, since Ɛ disagrees in the first spot, and it is not the second one, since Ɛ disagrees in the second spot, and so forth. Thus, Ɛ does not appear on the list. It follows that there is a fraction that we have failed to count.[23]

Thus, no matter how we *attempt* to set up a one-to-one correspondence between fractions and natural numbers, we can always produce a fraction that has been omitted. It follows that there is *no* one-to-one correspondence between ℕ and the fractions, and therefore the fractions are uncountable.

This proves that the fractions are uncountable. We next use this to show that the irrational fractions are uncountable. The fractions are given as follows.

fractions = {0} ∪ ordinary fractions ∪ irrational fractions

Although we will not prove it here, we appeal to the following theorem.

if A and B are both countable, then $A \cup B$ is also countable.

We have already shown that the ordinary fractions are countable; it is obvious that {0} is countable; so, if the irrational fractions are countable, then the fractions are countable. But we have just proved that the fractions are not countable. Accordingly, the irrational fractions are not countable either.

Next, since the irrational fractions are a subset of the set of all irrational numbers, the set of all irrational numbers is uncountable.

Finally, since the set of irrational numbers is a subset of the set of real numbers, the set of real numbers is uncountable.

[23] There is a potential problem; we might end up producing an inadmissible sequence, that is, a sequence with an infinite repeating sequence of 9's. But this is ruled out, since our technique generates a decimal encoding in which there are *no* 9's whatsoever.