Other Numbers

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1. Introduction

So far, by *number* we have meant *natural number* (in its *cardinal* form, rather than its *ordinal* form). As we have proposed, a natural number is a quantity, specifically a *set size*, more specifically a *finite* set size.[1] As every survivor of grade school (K-12) is aware, there are many other kinds of mathematical objects that go by the name 'numbers'. In particular, in addition to the natural numbers there are also "artificial" numbers, which include the following.

(1)	negative integers	$\{-1, -2, -3, \ldots\}$
(2)	fractions	$\{1/2, 3/2, 2/3, 3/4, 4/3, \ldots\}$
(3)	irrational numbers	$\{\sqrt{2}, \sqrt{3}, e, \pi, \ldots\}$
(4)	imaginary numbers	$\{\sqrt{-1}, \sqrt{-2}, \ldots\}$

We then have various *number systems*, which include the following.

[1] The size of an infinite set is not a natural number, but an infinite cardinal number, what is sometimes called an "aleph". See chapter on infinite sets and infinite sizes.

(1)	natural numbers	{0, 1, 2, 3, …}
(2)	integers	natural numbers *plus* negative integers
(3)	rational numbers	fractions *plus* negative-fractions
(4)	real numbers	rational numbers *plus* irrational numbers
(5)	complex numbers	real numbers *plus* imaginary numbers *plus* sums of these

Also, according to the *usual understanding*, the above systems are increasingly inclusive. In particular:

every natural number is an integer
every integer is a rational number
every rational number is a real number
every real number is a complex number

The obvious philosophical question is – what exactly are these things? For example, we maintain that the word 'two' denotes a property of sets, a property that applies to a set precisely when it is double-membered.[2] By comparison, what does the word 'minus-two' denote? What does the word 'two-thirds' denote?

Another philosophical question comes to mind. In particular, we maintain that the numbers were *discovered*, although the various numerals and numeration systems were *invented*. Can we say the same thing about the non-natural numbers? Did we discover the non-natural numbers, or did we invent them?[3] If we invented them, would we expect other comparably advanced cultures (e.g., on other planets) to invent very similar devices?

In what follows we briefly discuss the most important non-natural numbers – the integers, the fractions, the rational numbers, and the real numbers.

2. The Integers – The Usual Presentation

1. Subtraction

We will begin by describing the non-natural numbers as inventions, since that is the manner in which they are usually presented. Later, we consider whether there are "naturalistic" accounts of at least some of the non-natural numbers.

We begin with the integers, which officially comprise the non-negative integers (alias natural numbers) and the "new kids on the block" – the negative integers. Where do the negative integers come from? There are two inter-related answers to this question. In grade school, in addition to addition and multiplication, we learn about subtraction and division. Let us concentrate on subtraction for the moment. First subtraction is a two-place function sign, written between its arguments (infix notation), where:

$$a - b \quad =: \quad a \textit{ minus } b$$

which may be defined as follows.

[2] Remember, this appears to be circular, but it isn't since 'double-membered' can be defined without reference to the word 'two' or any related word like 'double' or 'twin'.
[3] In this connection, we recall Kronecker's famous dictum – "God made the natural numbers; all else is the work of man". He obviously thought that the non-natural numbers are human inventions.

$a - b \quad =_{df} \quad$ the number x such that $b + x = a$

The *definiens* involves what is known as a *definite description*. Definite descriptions are complex noun phrases involving the word 'the'. On the other hand, not every use of 'the' is descriptive; indeed, there are at least six different ways in 'the' is used in English, listed as follows, along with examples.

'the' used to construct a:	examples:
proper noun	the U.S. the Eiffel Tower
function sign	the square root of x the sum of x and y
quasi-pronoun	if a man respects a woman, then it is likely that the woman respects the man
quasi-demonstrative	the word 'dog' has three letters
generic name	the dodo is now extinct what do you give the man who has everything
description	the woman standing next to the window the phrase directly above

Descriptions basically involve a 'the' followed by a predicate, which may be simple or complex. For example, in the descriptive phrase

the woman standing next to the window

the complex predicate is:

x is a woman who is standing next to the window

or:

x is a woman and x is standing next to the window

Now, the key fact about definite descriptions is that

> definite descriptions may fail to refer (to anything)

Although a description may have a well-defined meaning, it need not denote an actual object.[4] For example, we understand what 'the golden mountain' means; indeed, based on this understanding, we are pretty confident that this phrase doesn't refer to anything.

The fundamental principle about descriptive reference is given as follows.

if there is *exactly one* individual with property $\mathbb{P}$,
then the description 'the $\mathbb{P}$' refers to *that individual*.

otherwise, 'the $\mathbb{P}$' doesn't refer to anything!

If a description succeeds in picking out a unique referent, then we say that the description is *proper*; otherwise, we say that the description is *improper*.

[4] Analogously, a sentence has a meaning in virtue of which we can decide whether it is true or false.

Note that there are two ways that a description 'the $\mathbb{P}$' can be improper (i.e., fail to refer):

(1) there are *two or more individuals* with property $\mathbb{P}$;
(2) there are *no individuals* with property $\mathbb{P}$;

So, let us go back now and restate the definition of subtraction.

$$a - b \quad =_{\text{df}} \quad \text{the number } x \text{ such that } b + x = a$$

The property in question is:

$$x \text{ is a number} \quad \text{and} \quad b + x = a$$

First, we observe the following important theorem, without which subtraction would be completely bogus.

$$\text{if } \ a \leqslant b, \ \text{ then } \ \text{there is } \textit{exactly one} \text{ natural number } x \text{ such that } a + x = b$$

This basically follows from the Law of Cancellation:

$$w + x = z \ \ \& \ \ w + y = z \ \ .\!\!\rightarrow \ x = y$$

which is logically equivalent to the following.

the equation '$a + x = b$' has *at most one* solution.

On the other hand, we have the following further theorem.

$$\text{if } \ b < a, \ \text{ then there is no natural number } x \text{ such that } a + x = b$$

Accordingly, all the following descriptions are ill-formed in the arithmetic of natural numbers.

$1-2 \quad 1-3 \quad 1-4 \quad \ldots$
$2-3 \quad 2-4 \quad 2-5 \quad \ldots$
$3-4 \quad 3-5 \quad 3-6 \quad \ldots$
$\ldots$

2. Augmenting the Natural Numbers to form the Integers

The proposed answer to the above problem is to augment the natural numbers with some "new numbers" that serve as the denotations of the above infinite array of ill-defined expressions. These are the *negative integers*. First, let us define positive integer as follows.

a is a *positive integer* $\;=_{\text{df}}\;$ a is a natural number & $a \neq 0$

Next, for each positive integer, we *posit* a corresponding *negative integer*; in particular:

$-m \quad =: \quad$ the negative integer corresponding to m [where $m \geqslant 1$]

For example, the expression '-2' is read "minus two" or "negative two". The former reading is the more common, although it makes for potential confusion with the use of 'minus' for subtraction. We further postulate that no negative integer is a natural number.

$$\sim\exists x \{ x \in \mathbb{N} \ \& \ -m = x \}$$

Thus, the negative integers are genuinely new mathematical objects. Notice that this is sensible, since minus-one, minus-two, etc., cannot be answers to a how-many question.[5]

We further postulate that the negative integers provide referents for all the heretofore ill-defined subtractions [1 – 2, 2 – 3, etc.]. In other words:

if $a < b$, then $a - b$ is a negative integer

Finally, we propose that the system of integers consists of the following items.

(1)	zero	0
(2)	the positive integers	1, 2, 3, …
(3)	the negative integers	–1, –2, –3, …

and nothing else!

We have already declared that a negative integer is not a "how many". So, what is it? The "standard" answer usually offered by mathematicians is the following.

negative integers are *inventions* intended to serve as solutions to equations of the form

$$a + x = b \qquad \text{where } a > b$$

And in particular:

if $a > b$, then there is a positive number m such that $a = b + m$;
the corresponding negative integer, $-m$, is defined so that:

$$a + -m \;=\; b \quad \leftrightarrow \quad a \;=\; b + m$$

3. The Key Mathematical Issue

The mathematical issue is not so much whether we can postulate a new kind of number (the negative numbers), or a new class of numbers (the integers). Rather, the question is whether the proposed entities behave in a mathematically "reasonable" manner. The reasonableness of the integers boils down to whether we can add and multiply them in a manner that makes sense. This requires that we propose both an addition table and a multiplication table, which means we have to propose answers to the following questions.

$2 + -3 \;=\; ?$
$2 \times -3 \;=\; ?$
$-2 + -3 \;=\; ?$
$-2 \times -3 \;=\; ?$
etc.

Are mathematicians permitted simply to make up answers to these questions, which the rest of us simply accept based on their authority. Or, are there strict criteria that restrict what counts as a legitimate addition table and multiplication table? It seems that there are numerous criteria that must be satisfied by the proposed addition and multiplication tables. We propose the following three criteria.

[5] There is a story about the mathematical physicist Paul Dirac (1902-1984; Nobel Prize in Physics, 1933). In grade school, he was given the following sort of word problem to solve: what is the least number of fish that a fisherman can catch that satisfies the following conditions – blah blah blah…? His answer was "minus two"! Later this oddness of thought served him well in propounding the theory of anti-matter.

(c1) **Criterion of Extension:**
addition and multiplication over the class of integers must be an *extension* of these operations as they apply to natural numbers; in other words, when we add or multiply non-negative integers (i.e., natural numbers), the results are the same as before.

(c2) **Criterion of Solution:**
the negative integers must provide solutions to all equations of the form:

$$a + x = b$$

where a and b are arbitrary natural numbers, and $a > b$.

(c3) **Criterion of Conservation:**
the following fundamental laws of arithmetic must be satisfied by all integers:

(a1)	$x + 0 \;=\; x$	[law of identity]
(a2)	$(x + y) + z \;=\; x + (y + z)$	[law of associativity]
(a3)	$x + y \;=\; y + x$	[law of commutativity]
(a4)	$x + y \;=\; x + z \;\rightarrow\; y = z$	[law of cancellation]
(a5)	$x \times 0 \;=\; 0$	[law of zero]
(a6)	$x \times 1 \;=\; x$	[law of identity]
(a7)	$(x \times y) \times z \;=\; x \times (y \times z)$	[law of associativity]
(a8)	$x \times y \;=\; y \times x$	[law of commutativity]
(a9)	$x \times y = x \times z \;\;\&\;\; x \neq 0 \;\;.\!\rightarrow\; y = z$	[law of cancellation]
(a10)	$x \times (y + z) \;=\; (x \times y) + (x \times z)$	[law of distribution]

Having established certain criteria, the next obvious question is whether the criteria are sufficiently restrictive that they *uniquely* determine the addition and multiplication tables. As it turns out, these criteria succeed in picking out a mathematical system; in particular, they *uniquely* determine the addition and multiplication tables, which are partly presented as follows.[6]

+	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…	–6	–5	–4	–3	–2	–1	0	…
–2	…	–5	–4	–3	–2	–1	0	1	…
–1	…	–4	–3	–2	–1	0	1	2	…
0	…	–3	–2	–1	0	1	2	3	…
1	…	–2	–1	0	1	2	3	4	…
2	…	–1	0	1	2	3	4	5	…
3	…	0	1	2	3	4	5	6	…
…	…	…	…	…	…	…	…	…	…

×	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…	9	6	3	0	–3	–6	–9	…
–2	…	6	4	2	0	–2	–4	–6	…
–1	…	3	2	1	0	–1	–2	–3	…
0	…	0	0	0	0	0	0	0	…
1	…	–3	–2	–1	0	1	2	3	…
2	…	–6	–4	–2	0	2	4	6	…
3	…	–9	–6	–3	0	3	6	9	…
…	…	…	…	…	…	…	…	…	…

[6] See Section 10 for a detailed discussion.

3. Integers – An Axiomatic Account

In a previous chapter, we examined the Peano axiomatization of the arithmetic of natural numbers, which is based on three primitive concepts, and five primitive postulates. Can we do the same thing for the arithmetic of integers. The answer is affirmative. In this section, we briefly examine a Peano-style axiomatization of the arithmetic of integers.

1. Primitive Concepts

First, the primitive concepts are:

(1)	integer	$\mathbb{I}[\alpha]$	$=:$	α is an integer
(2)	positive integer	$\mathbb{P}[\alpha]$	$=:$	α is a positive integer
(3)	negative integer	$\mathbb{N}[\alpha]$	$=:$	α is a negative integer
(2)	zero	0	$=:$	zero
(3)	predecessor	$p(\alpha)$	$=:$	the predecessor of α
(4)	successor	$s(\alpha)$	$=:$	the successor of α

2. Primitive Postulates (Axioms)

The primitive postulates are given as follows.

(a1) zero is an integer;
the successor of any integer is an integer;
the predecessor of any integer is an integer;
nothing else is an integer.

(a2) the predecessor of zero is a negative integer;
the predecessor of any negative integer is a negative integer;
nothing else is a negative integer.

(a3) the successor of zero is a positive integer;
the successor of any positive integer is a positive integer;
nothing else is a positive integer.

(a4) zero is not a positive integer;
zero is not a negative integer.

(a5) no two things have the same successor;
no two things have the same predecessor;

(a6) the successor of the predecessor of any integer is that integer;
the predecessor of the successor of any integer is that integer;

3. Definitions

positive integers:

(d1.1)	1	$=_{\text{df}}$	$s(0)$
(d1.2)	2	$=_{\text{df}}$	$s(1)$
(d1.3)	3	$=_{\text{df}}$	$s(2)$
	etc.		

negative integers:

(d2.1)	-1	$=_{\text{df}}$	$p(0)$
(d2.2)	-2	$=_{\text{df}}$	$p(-1)$
(d2.3)	-3	$=_{\text{df}}$	$p(-2)$
	etc.		

addition:

(d3.1)	$m + 0$	$=_{\text{df}}$	m
(d3.2)	$m + s(n)$	$=_{\text{df}}$	$s(m + n)$
(d3.3)	$m + p(n)$	$=_{\text{df}}$	$p(m + n)$

subtraction:

(d4.1)	$m - 0$	$=_{\text{df}}$	m
(d4.2)	$m - s(n)$	$=_{\text{df}}$	$p(m - n)$
(d4.3)	$m - p(n)$	$=_{\text{df}}$	$s(m - n)$

multiplication:

(d4.1)	$m \times 0$	$=_{\text{df}}$	0
(d4.2)	$m \times s(n)$	$=_{\text{df}}$	$(m \times n) + m$
(d4.3)	$m \times p(n)$	$=_{\text{df}}$	$(m \times n) - m$

order relations:

(d5)	$m \leqslant n$	$=_{\text{df}}$	$\exists x \{ m + x = n \}$
(d6)	$m < n$	$=_{\text{df}}$	$m \leqslant n \ \& \ m \neq n$

4. Examples of Theorems

(t1) $3 + -2 = 1$

(1)	~~SHOW~~: $3 + -2 = 1$	7-13,IL
(2)	$3 = s(2)$	d1.3
(3)	$= ss(1)$	d1.2+IL
(4)	$= sss(0)$	d1.1+IL
(5)	$-2 = p(-1)$	d2.2
(6)	$= pp(0)$	d2.1+IL
(7)	$3 + -2 = sss(0) + pp(0)$	4,6,IL
(8)	$= p[sss(0) + p(0)]$	d3.3
(9)	$= pp[sss(0) + 0]$	d3.3+IL
(10)	$= ppsss(0)$	d3.1+IL
(11)	$= pss(0)$	a6b+IL
(12)	$= s(0)$	a6b+IL
(13)	$= 1$	d1.1

(t2) $2 - 3 = -1$

(1)	~~SHOW~~: $2 - 3 = -1$	4-11,IL
(2)	$2 = ss(0)$	d1.1,d1.2,IL
(3)	$3 = sss(0)$	d1.1,d1.2,d1.3,IL
(4)	$2 - 3 = ss(0) - sss(0)$	2,3,IL
(5)	$= p[ss(0) - ss(0)]$	d4.2
(6)	$= pp[ss(0) - s(0)]$	d4.2,IL
(7)	$= ppp[ss(0) - 0]$	d4.2,IL
(8)	$= pppss(0)$	d4.1,IL
(9)	$= pps(0)$	a6b+IL
(10)	$= p(0)$	a6b+IL
(11)	$= -1$	d2.1

4. Integers – A Philosophical Account

In the previous sections, we have examined a purely mathematical account of the integers. They are postulated to be abstract mathematical objects that satisfy certain principles. Just as there is a philosophical account of the natural numbers, according to which they are *sizes* of sets, there is also a philosophical account of integers, according to which they are *size-differences* between sets. In particular, we propose the following account of integers.

an integer is a set *size-difference*

We already have a *qualitative* size-comparison for sets; we can already say that one set is bigger than another. The proposed notion is a corresponding *quantitative* notion. Whereas a natural number (in its cardinal guise) is a size [a "how-big"] an integer is a size-comparison [a "how-much-bigger"]. Whereas a natural number is a monadic property of sets, expressed by a one-place predicate, an integer is a dyadic relation among sets, expressed by a two-place predicate. It is nevertheless just as objective, just as real, although of course it is also just as abstract.

In an earlier chapter, we introduced the numerical one-place predicates 'one-membered', 'two-membered', etc., which correspond to the adjectival number-words, as in 'I have two brothers'. We now introduce corresponding two-place predicates as follows.

$m[A,B]$ =: A is m-"bigger" than B

The possible values of m are: 0, +1, –1, +2, –2, etc. In particular, each numerical relation has a sign – positive (+), negative (–), or neutral (0). This in turn explains why the word 'bigger' is scare-quoted above. This is because its exact meaning varies according to the *sign* of the integer m. For example:

$-2[A,B]$	=:	A is 2-smaller than B
$0[A,B]$	=:	A is equally big as B
$+2[A,B]$	=:	A is 2-bigger than B

We can also relate these new two-place predicates to our original one-place predicates in the following way.

$$m[A] \quad \leftrightarrow \quad +m[A,\varnothing]$$

In other words, A is m-membered if and only if A is $+m$–bigger than the empty set. For example:

$1[A] \quad \leftrightarrow \quad +1[A,\varnothing]$
$2[A] \quad \leftrightarrow \quad +2[A,\varnothing]$
etc.

1. Are the Natural Numbers Integers?

According to the standard *intuitive* account of numbers, every natural number is also an integer. On the other hand, according to the proposed account of integers, whereas a natural number [e.g., 2] is a *property* of sets, an integer [e.g., +2] is a *relation* between sets. Syntactically speaking, 2 is expressed by a one-place predicate, whereas +2 is expressed by a two-place predicate. Accordingly, technically speaking, no natural number is an integer! In particular:

$1 \neq +1$
$2 \neq +2$
etc.

Nevertheless, every natural number has an obvious integer-counterpart. Indeed, given the integers, we can define the natural numbers as follows.

$$m[A] \quad =_{\text{df}} \quad +m[A,\varnothing]$$

For example,

A is 1-membered iff A is +1-bigger than $\varnothing$
A is 2-membered iff A is +2-bigger than $\varnothing$
etc.

5. Fractions – The Usual Presentation

1. Division

The inverse of addition is subtraction. In order to define subtraction for every pair of natural numbers, we must augment our number system with negative integers, which serve as referents of the otherwise ill-defined descriptions. Alternatively stated, the negative integers arise as proposed solutions to equations of the form:

$$a + x = b \qquad \text{where } a > b$$

This leaves a bit of an asymmetry. What about multiplication? Well, the inverse of multiplication is *division*, which may be defined as follows.

$$a \div b \quad =_{\text{df}} \quad \text{the unique number } x \text{ such that } b \times x = a$$

The definite description presents numerous problems. The first concerns division by zero.

$0 \div 0 \quad =_{\text{df}} \quad$ the unique number x such that $0 \times x = 0$
$1 \div 0 \quad =_{\text{df}} \quad$ the unique number x such that $0 \times x = 1$
$2 \div 0 \quad =_{\text{df}} \quad$ the unique number x such that $0 \times x = 2$
etc.

The problem with the first description is that every number has the property, given the law of zero.

$0 \times x = 0$

The problem with the remaining descriptions is that no number has the property, once again in virtue of the law of zero and the law of commutation ($x \times y = y \times x$).

The second problem is that "most" pairs of natural numbers produce ill-defined descriptions, including the following for example.

$1 \div 2$	$=_{df}$	the unique number x such that $2 \times x = 1$
$2 \div 3$	$=_{df}$	the unique number x such that $3 \times x = 2$
$3 \div 2$	$=_{df}$	the unique number x such that $2 \times x = 3$

2. The Fractions

The mathematical solution is to augment the class of natural numbers with special numbers, which are usually called ‘fractions’. Most natural languages provide special names for these objects; for example, English provides the following.

one-half	two-halves	three-halves	etc.
one-third	two-thirds	three-thirds	etc.
one-fourth	two-fourths	three-fourths	etc.
etc.			

Mathematics also provides a standard collection of fractional numerals, as follows.

0/1	0/2	0/3	…
1/1	1/2	1/3	…
2/1	2/2	2/3	…
3/1	3/2	3/3	…
…			

In general, we have the following.

for any natural number m and any natural number n greater than 0, m/n is a fraction.

Note carefully, however, that there are many duplicate names in the above list. For example:

0	=	0/1	=	0/2	=	…
1	=	2/2	=	3/3	=	…
2	=	2/1	=	4/2	=	…
etc.						
1/2	=	2/4	=	3/6	=	…
1/3	=	2/6	=	3/9	=	…
etc.						

Finally, we have the following definition.

a fraction m/n is *proper* $=_{df}$ $\sim\exists x \{ x \in \mathbb{N} \ \& \ m/n \}$

3. Adding and Multiplying Fractions

As with the integers, the key mathematical question is whether we can add and multiply fractions in a reasonable manner. The answer is affirmative. Without going into the details, we simply note the following rules of addition and multiplication of fractions.

(f1) $(a / b) \times (c / d) \;=\; (a \times c) / (b \times d)$ $\qquad b \neq 0, d \neq 0$
(f2) $(a / b) + (c / d) \;=\; [(a \times d) + (b \times c)] / (b \times d)$ $\qquad b \neq 0, d \neq 0$

For example:

$$1/2 \times 3/4 \;=\; (1 \times 3) / (2 \times 4) \;=\; 3/8$$
$$1/2 + 3/4 \;=\; [(1 \times 4) + (2 \times 3)] / (2 \times 4) \;=\; 10/8 \;=\; 5/4$$

Next, just as the integers enable us to define subtraction, the fractions enable us to define division, so that the following principle obtains.

$$a \div b = c \quad \leftrightarrow \quad a = b \times c \qquad \text{provided } b \neq 0$$

Indeed, one can prove the following "big" theorem about the relation between fractions and division.

$$a \div b \;=\; a / b$$

E.g.:

$2 \div 3 = 2/3$
two divided by three = two-thirds

In other words, fractions seem like they were "designed" to be the answers to division questions. In fact, it is customary to write division using the fraction notation, which obscures the distinction.

6. Fractions – A Philosophical Account

In the philosophical treatment of numbers, fractions are defined very much like the integers are defined, as size comparisons among sets. However, the difference is that the size comparison is based on a *ratio*, not on a *difference*.[7]

a fraction is a *size-ratio*.

In particular, we give the official definition as follows.

$$m[A,B] \quad =: \quad A \text{ is } m \text{ as big as } B$$

Just as with the integers, there are special values for m, given as follows.[8]

[7] Sportscasters can't seem to master the difference between a ratio and a difference. For example, in football, they often refer to a team's "turnover ratio", when they mean 'turnover differential' or 'turnover difference'. The goal of a team is to have a negative turnover differential – i.e., fewer turnovers than the other team.
[8] I have hyphenated all these terms, not to indicate ordinary usage, but simply to indicate that the expressions are *words* morphologically-speaking. They play a role in number-word morphology just like the cardinal number-words 'one', 'two', etc., and the ordinal number-words 'first', 'second', etc.

numeral	alternative	corresponding number-word	alternative
1/2		one-half	half
1/3		one-third	
1/4		one-fourth	
2/3		two-thirds	
4/3		four-thirds	
2/1	$\times 2$	two-times	twice
3/1	$\times 3$	three-times	
4/1	$\times 4$	four-times	
etc.			

The following are examples.

$1/2[A,B]$ =: A is (one-)half as big as B
$\times 2[A,B]$ =: A is two-times (twice) as big as B
$1/3[A,B]$ =: A is one-third as big as B
$\times 3[A,B]$ =: A is three-times as big as B

What is the relation between the natural numbers and the fractions? This is given by the following principle.

$$m[A] \quad \leftrightarrow \quad \times m[A,\{\varnothing\}]$$

In other words:

A has m members $\leftrightarrow$ A is m times as big as $\{\varnothing\}$

Note that the set $\{\varnothing\}$ has exactly one member.

This principle can be combined with our earlier principle about the relation between integers and natural numbers to obtain the following.

$$+m[A,\varnothing] \quad \leftrightarrow \quad \times m[A,\{\varnothing\}]$$

Thus, we have the following equivalent sentences, for example.

$2[A]$
$+2[A,\varnothing]$
$\times 2[A,\{\varnothing\}]$

7. The Rational Numbers

In order to perform subtraction on any pair of natural numbers, we must augment the natural numbers with the negative integers. In order to perform division on any pair of natural numbers,[9] we must augment the natural numbers with the proper fractions. If we wish to subtract fractions and divide integers, we must construct a system of numbers that contains negative numbers as well as proper fractions. The resulting system of numbers is customarily called the *rational numbers*. A rational

[9] Where the denominator is not zero.

number is one that can be expressed as a *ratio* of two integers m/n ($n \neq 0$). Alternatively, the rational numbers – denoted ℚ, for quotient – can be constructed so that each rational numeral is a pair of items,

a numerator (any integer)
a denominator (any natural number other than zero)

The following are examples.

+1 / 2	–1 / 2	+1 / 3	–1 / 3	etc.
+2 / 3	–2 / 3	+2 / 5	–2 / 5	etc.
etc.				

The key to making this work mathematically is a logically adequate account of multiplication and addition. Without going into the details how they are achieved, these two operations are given by the following rules, where a and b are any integers, and p and q are any natural numbers other than 0.

(q1) $a / p \times b / q \;=\; a \times b / p \times q$

(q2) $a / p + b / q \;=\; (a \times q) + (b \times p) / p \times q$

For example:

$$-1/2 \times +3/4 \;=\; -\,(1 \times 3) / (2 \times 4) \;=\; -3/8$$
$$-1/2 + -3/4 \;=\; (-1 \times 4) + (-3 \times 2) / (2 \times 4) \;=\; -10/8 \;=\; -5/4$$

8. Irrational Numbers

1. Square Roots

The rational numbers provide a system that enables us to perform addition, subtraction, multiplication, and division. However, it does not enable us to solve all algebraic equations, including the following seemingly innocuous equation.

$$x^2 = 2$$

Here, x^2 – which is read "x squared" – is defined in the customary manner.

$$x^2 \quad =_{\text{df}} \quad x \times x$$

In this connection, the notion of *square-root* is defined as follows.

$\sqrt{m}$ $\quad =_{\text{df}}\quad$ the unique positive number x such that $x^2 = m$

For example, the equation

$$x^2 = 4$$

has two solutions – namely:

$x = +2$
$x = -2$

Accordingly:

$$\sqrt{4} \quad = \quad +2$$

What is the square root of 2? Is it rational? As we have seen in a previous chapter, $\sqrt{2}$ is not rational!

2. The Algebraic Numbers

The case of square roots is a special case of a more general class of equations, which have the following general form.

$$c_1 x^{k1} + c_2 x^{k2} + \ldots + c_m x^{km} \; = \; a$$

where a, c_1, …, c_m, k_1, …k_m, are integers. Notice that if $a = 2$, $c_1 = 1$, $c_2 = \ldots = c_m = 0$, and $k_1 = 2$, then this reduces to

$$x^2 \; = \; 2$$

The numbers that provide solutions to all these equations are called *algebraic numbers*. The problem is that the algebraic numbers are countable, and the irrational numbers are uncountable. This means that most irrational numbers are not algebraic; these numbers are called *transcendental*. The best-known example of a transcendental number is π (pi).

How do we come up with all those extra numbers?

9. The Real Numbers

The *real numbers* (as they are called) include the rational numbers and the irrational numbers. How are they defined? This is very tricky, since they are not defined by reference to equations, as are the integers, fractions, rational numbers, and algebraic numbers. Rather, they are defined by reference to *infinite sums*. This is puzzling in itself. How can one add together infinitely-many numbers and obtain anything other than infinity? For example, the following sequence

$$1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad \ldots$$

sums to infinity. Similarly, the sequence

$$1/2 \quad + \quad 1/2 \quad + \quad 1/2 \quad + \quad 1/2 \quad + \quad \ldots$$

sums to infinity. In general, no matter how small n is, so long as it is not 0, the sequence

$$n \quad + \quad n \quad + \quad n \quad + \quad n \quad + \quad \ldots$$

sums to infinity, or to minus-infinity.[10]

But what happens if we add an infinite sequence of numbers that are decreasing in size? For example, what happens if we add the following sequence?

$$1/2 \quad + \quad 1/4 \quad + \quad 1/8 \quad + \quad 1/16 \quad + \quad \ldots$$

The answer is: 1!

[10] In this context, infinity and minus-infinity are pseudo-concepts. We should more properly say that the sums do not properly exist; they have the exact same status as 1/0, 2/0, etc.

Similarly, if we add

9/10 + 9/100 + 9/1000 + 9/10000 + …
.9 + .09 + .009 + .0009 + …

we get 1!

What about the following sequence?

1/2 + 1/8 + 1/32 + 1/128 + …

The sum is: 2/3!

What about:

6/10 + 6/100 + 6/1000 + 6/10000 + …
.6 + .06 + .006 + .0006 + …

Again, the sum is: 2/3!

Note however that some decreasing sequences do not sum; the following is an example.

1/2 + 1/3 + 1/4 + 1/5 + …

Probably the simplest way to think of real numbers is by way of their decimal encodings, the following being some examples.[11]

1/2	0.500000000000000…
1/3	0.333333333333333…
1/4	0.250000000000000…
2/3	0.666666666666666…
3/4	0.750000000000000…
7/22	0.318181818181818…
1/π	0.318309886183791…?
1/√2	0.707106781186547…?

Some of the sequences "terminate", which means that at some point the remaining digits are all 0's. When a sequence terminates with an infinite sequence of zero's, we of course omit them in the pronunciation – for example we say "point five", not "point five zero zero zero …".[12] Other sequences do not terminate, but are nevertheless completely regular, in the sense that the digits eventually group into repeating units, as in the encodings of 1/3, 2/3, and 7/22. Still others are "helter-skelter", like 1/π 1/√2, which have no obviously discernable pattern. These, basically, are the irrational fractions.

[11] Note that the algorithm does not automatically produce a unique encoding for each fraction. The problem is that "point five" is identical to "point four nine nine nine …"! This can be seen by subtracting the latter from the former and noticing that the result is zero. In our encoding, we mean to stringently exclude the goofy ones; in particular, we do not count as *admissible* any encoding that has a sub-sequence of endlessly repeating 9's.
[12] Unless we are trying to make a point about *precision*, which pertains to measurement, which is not a part of *pure* mathematics, but of *applied* mathematics.

10. Appendix – There is only one Way to Implement Addition and Multiplication for the Integers

Earlier we proposed the following criteria for assessing the adequacy of proposed addition and multiplication tables.

(c1) **Criterion of Extension:**
addition and multiplication over the class of integers must be an *extension* of these operations as they apply to natural numbers; in other words, when we add or multiply non-negative integers (i.e., natural numbers), the results are the same as before.

(c2) **Criterion of Solution:**
the negative integers must provide solutions to all equations of the form:
$a + x = b$
where a and b are arbitrary natural numbers, and $a > b$.

(c3) **Criterion of Conservation:**
the following fundamental laws of arithmetic must be satisfied by all integers:

(a1)	$x + 0 = x$	[law of identity]
(a2)	$(x + y) + z = x + (y + z)$	[law of associativity]
(a3)	$x + y = y + x$	[law of commutativity]
(a4)	$x + y = x + z \;\rightarrow\; y = z$	[law of cancellation]
(a5)	$x \times 0 = 0$	[law of zero]
(a6)	$x \times 1 = x$	[law of identity]
(a7)	$(x \times y) \times z = x \times (y \times z)$	[law of associativity]
(a8)	$x \times y = y \times x$	[law of commutativity]
(a9)	$x \times y = x \times z \;\;\&\;\; x \neq 0 \;\;.\!\!\rightarrow\; y = z$	[law of cancellation]
(a10)	$x \times (y + z) = (x \times y) + (x \times z)$	[law of distribution]

In this appendix, we examine how these criteria uniquely determine the addition and multiplication tables. To see this, we consider the various criteria in turn, to see what restrictions they place on the addition and multiplication tables. We first concentrate on addition, and we start by considering criterion (c1). This forces the following entries.

+	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…								…
–2	…								…
–1	…								…
0	…				0	1	2	3	…
1	…				1	2	3	4	…
2	…				2	3	4	5	…
3	…				3	4	5	6	…
…	…	…	…	…	…	…	…	…	…

We next consider criterion (c2), which maintains that the negative integers are solutions to equations of the form:

$$a + x = b \qquad \text{where } a,b \in \mathbb{N} \;\&\; a > b$$

As stated earlier, every negative integer has the form $-m$ where m is a positive integer. What does it mean for $-m$ to be a solution to the above equation? It amounts to the following biconditional being true.

(LN) if $a,b,m \in \mathbb{N}$ & $a > b$,
then $a + -m = b \leftrightarrow a = m + b$ [law of negation]

We can immediately deduce the following important corollary to the law of negation.

(t1) $m + -m = 0$ [simple law of negation]

To see that (t1) follows from (LN), set a=m, and b=0. Then we have:

$$m + -m = 0 \quad \leftrightarrow \quad m = m + 0$$

But the second constituent is simply the law of identity (a1), which we already know is satisfied by the natural numbers.

Finally, we consider criterion (c3), which requires that addition and multiplication satisfy the laws (a1)-(a10). Based on this, we can deduce the following corollary to (LN).

(t2) $-(m + n) = -m + -n$ [law of negative-distribution]

To see that (t2) follows from (t1) and hence (LN), we consider the following proof. First we note that by (t1), we have:

(1) $(m + n) + -(m + n) = 0$

So by a3, we have:

(2) $-(m + n) + (m + n) = 0$

Adding $-n$ to both sides yields:

(3) $[-(m + n) + (m + n)] + -n = 0 + -n$

By a combination of a2 and t1, the left item is:

$$-(m + n) + m$$

By a1 and a3, the right item is:

$$-n$$

So: (4) $-(m + n) + m = -n$

Adding $-m$ to both sides yields:

(5) $[-(m + n) + m] + -m = -n + -m$

By a combination of a2 and t1, the left item is:

$$-(m + n)$$

By a3, the right item is:

$$-m + -n$$

Thus: $-(m + n) = -m + -n$

Once we have (t2), we can fill in the following entries.

+	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…	–6	–5	–4					…
–2	…	–5	–4	–3					…
–1	…	–4	–3	–2					…
0	…				0	1	2	3	…
1	…				1	2	3	4	…
2	…				2	3	4	5	…
3	…				3	4	5	6	…
…	…	…	…	…	…	…	…	…	…

Next, the law of identity (a1) forces us to fill in the following entries.

+	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…	–6	–5	–4	–3				…
–2	…	–5	–4	–3	–2				…
–1	…	–4	–3	–2	–1				…
0	…	–3	–2	–1	0	1	2	3	…
1	…				1	2	3	4	…
2	…				2	3	4	5	…
3	…				3	4	5	6	…
…	…	…	…	…	…	…	…	…	…

Next, the law of negation forces the following lower left entries, and subsequently the law of commutation (a3) forces the following upper right entries.

+	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…	–6	–5	–4	–3			0	…
–2	…	–5	–4	–3	–2		0	1	…
–1	…	–4	–3	–2	–1	0	1	2	…
0	…	–3	–2	–1	0	1	2	3	…
1	…			0	1	2	3	4	…
2	…		0	1	2	3	4	5	…
3	…	0	1	2	3	4	5	6	…
…	…	…	…	…	…	…	…	…	…

The remaining entries can be computed one by one using a number of principles and earlier results. For example, to compute –3+1, we reason as follows.

By an earlier result (et),

$$-3 = -2 + -1$$

Adding 1 to both sides yields:

$$-3 + 1 \;=\; (-2 + -1) + 1$$

But:

$$(-2 + -1) + 1 \;=\!(a2)\!=\; -2 + (-1 + 1) \;=\!(et)\!=\; -2 + 0 \;=\!(a1)\!=\; -2$$

Thus:

$$-3 + 1 = -2$$

Similar reasoning allows us to fill in the remainder of the table, as follows.

+	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…	–6	–5	–4	–3	–2	–1	0	…
–2	…	–5	–4	–3	–2	–1	0	1	…
–1	…	–4	–3	–2	–1	0	1	2	…
0	…	–3	–2	–1	0	1	2	3	…
1	…	–2	–1	0	1	2	3	4	…
2	…	–1	0	1	2	3	4	5	…
3	…	0	1	2	3	4	5	6	…
…	…	…	…	…	…	…	…	…	…

This takes care of addition; we still need to construct a corresponding table for multiplication. The key to the construction is proving the following theorems.

$$-m \times n = -(m \times n)$$
$$-m \times -n = m \times n$$

The first one can be proved by induction, as follows.

(1)	~~SHOW~~: $-m \times 1 = -m$	a6+c3
(2)	$-m \times n = -(m \times n)$	As
(3)	~~SHOW~~: $-m \times (n+1) = -(m \times (n+1))$	4-7
(4)	$-m \times (n+1) = (-m \times n) + (-m \times 1)$	a10+c3
(5)	$= -(m \times n) + -m$	1,2,IL
(6)	$= -[(m \times n) + m]$	t2
(7)	$= -[m \times (n + 1)]$	ET about ℕ

The second one can be shown directly, as follows.

(1)	$-m \times 0 = 0$	a5
(2)	$n + -n = 0$	t1
(3)	$-m \times (n + -n) = 0$	1,2IL
(4)	$(-m \times n) + (-m \times -n) = 0$	3,a10
(5)	$-m \times n = -(m \times n)$	ET
(6)	$-(m \times n) + (-m \times -n) = 0$	4,5,IL
(7)	$-(m \times n) + (m \times n) = 0$	t1+a3
(8)	$-m \times -n = m \times n$	6,7,a4

Given the law of zero (a5) and the law of identity (a6), once we have these two theorems, the entries in the multiplication table are completely determined, as follows.

×	…	–3	–2	–1	0	1	2	3	…
…	…	…	…	…	…	…	…	…	…
–3	…	9	6	3	0	–3	–6	–9	…
–2	…	6	4	2	0	–2	–4	–6	…
–1	…	3	2	1	0	–1	–2	–3	…
0	…	0	0	0	0	0	0	0	…
1	…	–3	–2	–1	0	1	2	3	…
2	…	–6	–4	–2	0	2	4	6	…
3	…	–9	–6	–3	0	3	6	9	…
…	…	…	…	…	…	…	…	…	…