# Arithmetic

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## 1. Introduction

So far we have concentrated on numbers and counting. But as we all know, there is a lot more to arithmetic than simply counting; there is also adding, subtracting, multiplying, and dividing. In this chapter, we examine the arithmetic of natural numbers (i.e., the numbers 0, 1, 2, etc.) At the culmination of this chapter, we briefly examine the most famous *theory* of arithmetic, which was originally propounded in the 19th Century by Dedekind and Peano.

The word 'arithmetic' derives from the Greek words *arithmos* [*number*] and *tekhn¶* [*art*]. Arithmetic is accordingly the *art of counting*, and over the centuries humans have developed more and more sophisticated techniques (remember the word *tekhn¶*) to accomplish the task of counting. Among these techniques are, most prominently, the techniques of addition and multiplication.

## 2. Addition

### 1. The Basic Idea

As we learn in elementary school, addition provides a labor-saving technique for counting *sub-divided sets*. For example, suppose we have a collection $C$ of apples and bananas. Suppose there are five apples and seven bananas. Then, how big is our collection $C$? Alternatively, how many apples-***and***-bananas do we have? The answer of course is twelve, which is obtained by adding the number of apples (five) and the number of bananas (seven).

More generally, if we know the number of apples, and we know the number of bananas, we can calculate the number of apples-and-bananas by adding these two numbers together.[1] In particular,

the number of apples-***and***-bananas = the number of apples ***plus*** the number of bananas

or symbolically:

$$\#(A\text{'s and }B\text{'s}) \quad = \quad \#(A\text{'s}) + \#(B\text{'s})$$

Here, the special symbol '#'[2] is short for 'the number of', and the special symbol '+' is the familiar abbreviation for the English word 'plus'.

In order to see that there is a hidden presupposition in the above calculation, let us do a different example. Suppose we have a collection $C$ of actors and bakers.[3] Suppose there are 100 actors, and suppose there are 200 bakers. How big is collection $C$? How many actors-and-bakers are there? Unlike the previous example, in this example we cannot say *exactly* how big $C$ is, although we can deduce that it has *at least* 200 members and *at most* 300 members. How do we reach this conclusion?

Actually, we reach this conclusion the very same way we reached the earlier conclusion about the number of apples and bananas. In both cases, we appeal to the following general principle.

$$\#(A\text{'s and }B\text{'s}) \quad = \quad \#(A\text{'s}) + \#(B\text{'s}) - \#(A\text{'s who are }B\text{'s})$$

In our calculation about apples and bananas, we further *presume* that no apple is a banana, which is a very plausible assumption. In that case, the number of apples who[4] are bananas is zero. So

$$\begin{aligned} \#(\text{apples and bananas}) &= \#(\text{apples}) + \#(\text{bananas}) - \#(\text{apples who are bananas}) \\ &= 5 + 7 - 0 \\ &= 12 \end{aligned}$$

On the other hand, in the case of actors and bakers, we cannot simply presume that there are no actors who are also bakers, so our calculation is considerably more complicated.

$$\begin{aligned} \#(\text{actors and bakers}) &= \#(\text{actors}) + \#(\text{bakers}) - \#(\text{actors who are bakers}) \\ &= 100 + 200 - x \end{aligned}$$

There are limits on the value of $x$. For example, the minimum possible value of $x$ is 0, which corresponds to the scenario in which no one is both an actor and a baker; this gives us:

[1] This observation surely refutes the saying "you cannot add apples and oranges", which must be one of the most inane sayings on this planet.

[2] The symbol '#', which appears on every telephone, is often called the 'pound sign', in reference to the unit of weight, not in reference to the unit of currency, which is symbolized '£'. It is also called the 'number sign' as used in '#1', '#2', etc. Evidently, beginning in the mid-1960's, telephone engineers started calling '#' an 'octotherp' and later 'octothorp', in reference to its eight points. No one however, is quite certain what a "therp" or "thorp" is! At least they got the 'octo' right.

[3] Here, we include amateur actors and bakers, as well as professional actors and bakers. So, if you occasionally bake bread, and you occasionally perform in the local drama company, then you are both an actor and a baker. Also, by 'actor' I mean to include both men and women. As do many people in the drama business, I believe there is no more reason to grammatically distinguish thespians by gender than to distinguish physicians by gender (as in "doctors and doctresses"). Nevertheless, I do accept the classifications in the Academy Awards ("Oscars") – "best actor" versus "best actress".

[4] I am "personifying" the apples and bananas, by using 'who', merely for the sake of grammatical simplicity.

$$\begin{aligned}\#(\text{actors and bakers}) &= 100 + 200 - 0 \\ &= 300\end{aligned}$$

This sets the upper limit on the number of actors and bakers – there are *at most* 300 actors and bakers. In order to set the lower limit, we need to calculate the maximum possible value for $x$. In order to calculate this, we notice that every member of the set actors-who-are-bakers is an actor, but there are exactly 100 actors, so the number of actors who are bakers is *at most* 100.[5] This in turn provides the lower limit for the number of actors and bakers, which is calculated as follows.

$$\begin{aligned}\#(\text{actors and bakers}) &= 100 + 200 - 100 \\ &= 200\end{aligned}$$

Putting these two calculations together, we conclude that the number of actors and bakers is at least 200 and at most 300. In other words:

$$200 \leqslant \#(\text{actors and bakers}) \leqslant 300$$

## 2. Set-Theoretic Notation

We can formulate our addition principles more succinctly if we use set-theoretic notation. When we are considering $A$'s and $B$'s, there are several sets worth considering, including the following.

| | | |
|---|---|---|
| $A$ | the set of $A$'s | [e.g., actors] |
| $B$ | the set of $B$'s | [e.g., bakers] |
| $A \cup B$ | the set of $A$'s and $B$'s | [e.g., actors and bakers] |
| $A \cap B$ | the set of $A$'s who are $B$'s | [e.g., actors who are bakers] |

The symbol '$\cup$' is a variant of the letter 'U'; it is the symbol for *set-union*, which is defined so that:

$$x \in A \cup B \;\leftrightarrow.\; x \in A \vee x \in B$$

Here, the wedge symbol '$\vee$' is a variant of the letter 'v', which is short for the Latin word 'vel', which means 'and/or'.[6]

Now, the upside-down counterpart[7] of set-union is *set-intersection*, which is defined so that:

$$x \in A \cap B \;\leftrightarrow.\; x \in A \wedge x \in B$$

Here, '$\cap$' is the upside-down counterpart of '$\cup$', and '$\wedge$' is the upside-down counterpart of '$\vee$', which turns out to be logical conjunction. Although the notation '$\wedge$' for conjunction is visually elegant, and many authors employ it, we do not generally adopt the symbol '$\wedge$' for conjunction, since the modern alphabet already provides a perfectly good symbol for 'and' – namely '&'. We accordingly prefer to write:

$$x \in A \cap B \;\leftrightarrow.\; x \in A \;\&\; x \in B$$

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[5] By parallel reasoning, every actor who is a baker is a also baker; therefore, since there are exactly 200 bakers, there are *at most* 200 actors who are bakers. But this information is implied by the earlier information, since '$x \leqslant 100$' implies '$x \leqslant 200$'.

[6] Logicians call this the *inclusive-or*, which is contrasted with the *exclusive-or*, for which Latin has the word 'aut'.

[7] What mathematicians and logicians call the 'dual'.

Next, we note that to be a member of the set of $A$'s and $B$'s (e.g., actors and bakers) is the be an $A$ (actor) *and/or* a $B$ (baker). On the other hand, to be an $A$ who is a $B$ is to be both an $A$ and a $B$. The set that consists of all actors and bakers is not (generally) the same as the set of actors who are bakers.[8]

We can now state the general rule of addition as follows.

(RA) $\#(A \cup B) \;=\; \#(A) + \#(B) - \#(A \cap B)$

## 3. The Fundamental Principle of Addition; The Definition of Addition

The general rule of addition (RA) logically depends upon having the notions of addition and subtraction *already in hand*. It does not *define* addition or subtraction. For this reason, we back up and formulate a more fundamental principle – the Principle of Addition.

(PA) if no $A$'s are $B$'s
then the number of $A$'s and $B$'s ***is*** the number of $A$'s ***plus*** the number of $B$'s

or using set-notation:

(PA) if $A \cap B = \varnothing$
then $\#(A \cup B) \;=\; \#(A) + \#(B)$

First notice that (PA) follows from (RA) once we have a number of further principles about addition and subtraction. Recall that we have already employed this mode of reasoning when we calculated the number of apples-and-bananas. On the other hand, (RA) can also be deduced from (PA) using even fewer principles about addition. Indeed, we can reverse-engineer (PA) to to obtain a *definition* of addition, as follows.

$$m + n \;=_{df}\; \#(A \cup B) \qquad \text{where: } \begin{array}{l} \#(A) = m \\ \#(B) = n \\ A \cap B = \varnothing \end{array}$$

What this definition says is the following.

> Suppose you have two numbers – $m$ and $n$. Find *any* set $A$ whose size is $m$, and find *any* set $B$ whose size is $n$, subject to the further restriction that $A$ and $B$ are disjoint. Then $m$+$n$ is, *by definition*, the size of the set $A \cup B$.

Before continuing, we note that the notion of size (#) must also be logically defined. This is accomplished by the following infinite list.

| | | | |
|---|---|---|---|
| $\#(A) = 0$ | $=_{df}$ | $\underset{\sim}{0}[A]$ | i.e., $A$ is memberless |
| $\#(A) = 1$ | $=_{df}$ | $\underset{\sim}{1}[A]$ | i.e., $A$ is single-membered |
| $\#(A) = 2$ | $=_{df}$ | $\underset{\sim}{2}[A]$ | i.e., $A$ is double-membered |
| etc. | | | |

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[8] $A \cup B = A \cap B$ if and only if $A = B$.

Here, the numerical adjectives – 0, 1, 2, etc. – are officially defined as in an earlier chapter. For example:

$$2[A] \quad =_{df} \quad \exists x \exists y \{ x \neq y \ \& \ A = \{x,y\} \}$$

## 4. Addition Facts

So far we have some theoretical knowledge about addition. But this knowledge is of absolutely no *practical* value unless we also work out some *facts* about addition. If we count eight apples and nine bananas, then we know that we have eight-plus-nine apples-and-bananas, but this is useless unless we also know exactly what eight-plus-nine is. The relevant arithmetic fact of course is:

eight plus nine ***is*** seventeen

The latter fact was *discovered* at some point in the dark recesses of human history; indeed, it was probably *repeatedly* discovered. But eventually, it got added to the collective knowledge of our species, along with myriads of other such facts, including:

one plus one ***is*** two
two plus two ***is*** four
four plus four ***is*** eight
…

## 5. A Rudimentary "Theory" of Addition

At some point early in civilization, these myriad addition facts got summarized into an *addition table*, which children of the ruling classes were then obliged to learn (since knowledge is power). The following is a translation into modern notation of a fragment of such a table.

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| 5 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 6 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 7 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 8 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
| 9 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

The table is read in the familiar way – row-plus-column. For example, taking row-8 and column-9, we read off the following: eight plus nine is seventeen.

Notice that an addition-table qualifies as a rudimentary *theory*, inasmuch as it organizes and systematizes a body of knowledge about a specified domain. One might wonder whether an addition-table provides an *explanation* of the addition facts. I think, at first glance, it does not. On the other hand, it seems that studying this table does help one better understand *how* addition works. We do use the expression 'explain *how* something works'.

### 6. Addition Laws

The addition-facts are nice, especially if you need to govern an ancient civilization in Mesopotamia or the Nile Valley, but there seems to be something important missing, which turns the *art* of counting into the *science* of counting – namely, *laws of arithmetic*!

The study of the *laws* of arithmetic is usually called *algebra*, the invention of which is credited to al-Khowarizmi, who in AD 825 in Baghdad (Mesopotamia again!) published the book *Hidab al-jabr wal-muqubala*. In one act, we got two modern words – the word 'algebra' traces to 'al-jabr', and the word 'algorithm' traces to 'al-Khowarizmi'. In al-Khowarizmi's scheme, 'al-jabr' referred to the act of "re-unifying"; for example, in the formula '$x$–2=3' the number $x$ can be "re-unified" to obtain '$x$=5'.[9]

Fast-forwarding to the present, among the laws of addition, the following are the most prominent.[10]

| | |
|---|---|
| $x + 0 = x$ | [law of identity] |
| $(x + y) + z = x + (y + z)$ | [law of associativity] |
| $x + y = y + x$ | [law of commutativity] |
| $x + y = x + z \rightarrow y = z$ | [law of cancellation] |

These can be used as axioms, on the basis of which *all* the laws of addition can be deduced, including the following, for example.

$$0 + x = x$$
$$x + y = x \rightarrow y = 0$$

However, as we see in a later section, there is an even more sparse reduction of the laws of arithmetic to fundamental principles.

## 3. Multiplication

### 1. Basic Definition

After addition, comes *multiplication*, which is a further technique to aid in counting. The basic concept is quite simple. Suppose you have four baskets of apples, and suppose that each basket has three apples in it. How many apples do you have total? The answer, of course, is that you have twelve apples, this number being obtained by multiplying four times three. The underlying idea in this calculation may be expressed as follows.

four baskets times three apples *per* basket equals twelve apples

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[9] The word 'al' is Arabic for 'the'. Compare this with 'el' in Spanish, and 'il' in Italian. It is worth noting that Latin – which is the common ancestor of Spanish and Italian – does not have a definite article, although the Latin word 'ille' means something like 'this'. We note that Spain was conquered by the Moors in the 8th Century AD. Curiously, the word 'al-jabr' entered Spain at this time, and meant 'bone-setting', and bone-setters were called *algebristas*! The use of words similar to 'algebra' to mean bone-setting also appears in precursors of English. In this connection, keep in mind that the original meaning is 're-unify'.

[10] As in earlier chapters, we omit universal quantifiers for the sake of conciseness. For example, the first one is short for:
$\forall x\, [\, x + 0 = x\, ]$

Notice that you are not *required* to employ multiplication to solve this problem; you *could* instead calculate the total number of apples by using addition – 3+3+3+3 = 12. For that matter, you don't *have to* use addition either; you could simply assess the total number of apples by counting all of them individually! But, if you have addition and/or multiplication, you can spare yourself the trouble.

Let us make the problem a little more abstract. Suppose you have four sets:

$$A_1, A_2, A_3, A_4$$

Furthermore, suppose that each of these sets has exactly three members; i.e.:

$$\#(A_1) \ = \ \#(A_2) \ = \ \#(A_3) \ = \ \#(A_4) \ = \ 3$$

Now consider lumping all these sets together into one big collection $C$, which is their union; i.e.:

$$C \ = \ A_1 \cup A_2 \cup A_3 \cup A_4$$

How big is $C$? If you say "12", you have jumped to a conclusion. You have assumed something about the situation that I have not *explicitly* told you. For example, the following sets satisfy the conditions.

$$A_1 = \{1,2,3\}$$
$$A_2 = \{1,3,4\}$$
$$A_3 = \{2,3,4\}$$
$$A_4 = \{1,2,4\}$$

In particular, there are four (distinct) sets, and each set has three (distinct) members, but

$$C \ = \ A_1 \cup A_2 \cup A_3 \cup A_4 \ = \ \{1,2,3,4\}$$

so:

$$\#(C) \ = \ 4$$

How does this situation differ from the apples situation? Well, in the apples situation we have a further tacit piece of information – that no apple can occupy more than one basket. But sets of apples are not entirely like baskets of apples; the same apple can occupy many different sets, just as a person can belong to many different clubs.

In order to formulate a set-theoretic rule of multiplication, we need to make explicit the presuppositions involved in the apples situation. In particular, we presume that no apple can occupy more than one basket, which is tantamount to presuming that the corresponding sets of apples do not overlap – that they are *mutually disjoint*. This notion has already been employed in connection with addition. Now we make it explicit, as follows.

$$A \text{ and } B \text{ are disjoint} \quad =_{df} \quad A \cap B = \varnothing$$

alt:

$$A \text{ and } B \text{ are disjoint} \quad =_{df} \quad \sim\exists x \, \{ \, x \in A \ \& \ x \in B \, \}$$

Next, to say that sets $A_1, \ldots, A_k$ are *pairwise disjoint* is simply to say that each of the sets is disjoint from every other set in the collection.

We can now state the Rule of Multiplication.

| | |
|---|---|
| suppose: | $\#\{A_1, \ldots, A_k\} = k$; |
| suppose: | $\#(A_1) = m, \ldots, \#(A_k) = m$; |
| suppose: | $A_1, \ldots, A_k$ are pairwise disjoint; |
| then: | $\#(A_1 \cup \ldots \cup A_k) = m \times k$ |

The Rule of Multiplication can also be reverse-engineered to define multiplication, as follows.

$$m \times n =_{df} \#(A_1 \cup \ldots \cup A_m)$$

where:
$\#\{A_1, \ldots, A_m\} = m$
$A_1, \ldots, A_m$ are pairwise disjoint
$\#(A_1) = n \ldots \#(A_m) = n$

What this definition says:

> Suppose you have two numbers – $m$ and $n$. Find $m$-many pairwise disjoint sets – $A_1, \ldots, A_m$ – each one of which has $n$-many members. Then $m \times n$ is, *by definition*, the size of the set $A_1 \cup \ldots \cup A_m$.

The most practical application of multiplication involves counting arrays of objects. For example, suppose we have a corn field, with 200 rows, and suppose each row has 250 corn plants; then it is a routine matter to calculate the total number of corn plants, which is 200×250 = 50,000. If we assume further that each corn plant has 5 ears of corn on it, then we can calculate that there are a total of 250,000 ears of corn. If we further assume that each ear of corn has 250 kernels, we can calculate that there are 62,500,000 kernels of corn!

## 2. Multiplication Facts

As with addition, the principle of multiplication can be of no practical use unless we have at our command many multiplication facts (as they are called in elementary school). As with addition, the most convenient summary of these facts consists in a table – a multiplication table – a fragment of which is presented as follows.

| × | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 3 | 0 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| 4 | 0 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 5 | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
| 6 | 0 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 |
| 7 | 0 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 |
| 8 | 0 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 |
| 9 | 0 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 |

The table is read in the familiar way – row-plus-column. For example, taking row-8 and column-9, we read off the following: eight times nine is seventy-two.

### 3. Multiplication Laws

Earlier we listed some of the well-known laws of addition. There are also laws of multiplication, and laws of multiplication-addition, a few of which we list as follows.

| | |
|---|---|
| $x \times 0 = 0$ | [law of zero] |
| $x \times 1 = x$ | [law of identity] |
| $(x \times y) \times z = x \times (y \times z)$ | [law of associativity] |
| $x \times y = y \times x$ | [law of commutativity] |
| $x \times y = x \times z \;\;\&\;\; x \neq 0 \;\; .\!\rightarrow \;\; y = z$ | [law of cancellation] |
| $x \times (y + z) = (x \times y) + (x \times z)$ | [law of distribution] |

## 4. The Modern Theory of Arithmetic

### 1. Basic Ideas

Let us begin by reviewing our simple picture of science, which divides the scientific "world" into

(1) data
(2) laws
(3) theories

This demarcation is fairly easy to understood in connection with arithmetic and counting. First, the *data* of arithmetic include the following sorts of propositions, what we called ‘arithmetic facts’ in earlier sections.

(1) $2+3 = 5$
(2) $3\times4 = 12$
(2) $4 > 2$

Second, the *laws* of arithmetic include the following sorts of propositions.

(1) $x + y = y + x$
(2) $x \times (y + z) = (x \times y) + (x \times z)$
(3) $x + y = x + z \;\rightarrow\; y = z$

The third step – to theories – leads us from *discovery* to *invention*. Once again, we recite the fundamental principle in our explanatory framework.

> laws are discovered;
> theories are invented.

For the sake of comparison, mankind *discovered* copper and iron, but *invented* bronze and steel. Of course, numerous discoveries are involved in any given invention – for example, discovering that bronze can be cast to form statues, and discovering that steel can be forged into components that can be assembled into great bridges. Along the same lines, mankind *discovered* the numbers and the laws that govern them, but mankind *invented* numeration systems (the *art* of counting). Along similar but more abstract lines, mankind has also attempted to understand the laws of arithmetic *theoretically*, by inventing conceptual schemes (alias theories) that seek to systematize, organize, and (hopefully!) make sense of the myriad propositions of arithmetic.

## 2. Peano Arithmetic

Probably the most famous theory of arithmetic is generally credited to the Italian mathematician Giuseppe Peano (1858-1932),[11] although Peano's theory was anticipated by the work of Richard Dedekind (1831-1916).[12] In a brilliant maneuver, reminiscent of Euclid, Peano was able to reduce all of arithmetic to just three primitive concepts and just five postulates (axioms), known as the Peano Postulates. The primitive concepts are:

number
zero
successor

which can be syntactically implemented by the following expressions.

| | |
|---|---|
| zero | [proper name] |
| …is a number | [one-place predicate] |
| the successor of … | [one-place function-sign] |

Note that the successor of a number is simply the number that directly follows it; for example, the successor of zero is one, and the successor of one is two, etc. Next, Peano's postulates are given as follows.

(p1) zero is a number;
(p2) the successor of any number is a number;
(p3) zero is not the successor of any number;
(p4) no two numbers have the same successor;
(p5) if zero has a property, and the successor of a number has that property whenever the number has that property, then every number has that property.

These are stated in ordinary English, and are accordingly somewhat difficult to manipulate logically. For this reason, it is useful to translate them into modern symbolic logic, which we do as follows.

| | | |
|---|---|---|
| $0$ | =: | zero |
| $N[\alpha]$ | =: | $\alpha$ is a number |
| $s(\alpha)$ | =: | the successor of $\alpha$ |

(p1) $N[0]$
(p2) $\forall x\, \{ N[x] \rightarrow N[s(x)] \}$
(p3) $\sim\exists x\, \{ N[x] \;\&\; 0 = s(x) \}$
(p4) $\sim\exists x \exists y\, \{ N[x] \,\&\, N[y] \,\&\, x \neq y \;\&\; s(x) = s(y) \}$
(p5) $\mathbb{P}[0] \;\&\; \forall x\{ N[x] \,\&\, \mathbb{P}[x] \;.\!\rightarrow\; \mathbb{P}[s(x)] \} \;\; .\!\rightarrow\; \forall x\{ N[x] \rightarrow \mathbb{P}[x] \}$

[11] Peano, *Arithmetics principia* (1889). (tr: "Principles of Arithmetic")
[12] Dedekind, "Über die Theorie der ganzen algebraischen Zahlen" (1879). (tr: "on the theory of all the algebraic numbers")

It is amazing that all of arithmetic can be distilled down to the ideas expressed in the above box. All the rest is simply a matter of logic.[13] Peano's technique had a profound effect on Bertrand Russell, probably the greatest philosopher of the 20th Century. For example, in his autobiography, Russell writes of his encounter with Peano at the 1900 International Congress of Philosophy.

> The Congress was the turning point of my intellectual life, because there I met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument on which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. … It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years …

Unfortunately, this may have also marked a turning point in Peano's career. Subsequently, Peano launched a number of projects that were less fruitful. For example, he became very interested in the international language movement, and devoted much effort to this endeavor. For example, he designed a novel language called 'Latino Sine Flexione' (LSF – Latin without inflections).[14] Notwithstanding its numerous advantages over naturally-evolved languages (Latin, Italian, English, etc.), and over rival constructed languages like Esperanto, Peano's language gained very few adherents, let alone speakers or writers. The only major work written in LSF is the last edition of Peano's magnum opus *Formulario Mathematico*.

## 3. Definitions in Peano Arithmetic

One cannot deduce very much from Peano's postulates unless one appends definitions of all the familiar arithmetical terms, including 'one', 'two', etc., 'plus', 'times', and 'less than or equal to'. These are accomplished as follows.

(d1) $1 =_{df} s(0)$
$2 =_{df} s(1)$
$3 =_{df} s(2)$
etc.

(d2) $m + 0 =_{df} m$
$m + s(n) =_{df} s(m + n)$

(d3) $m \times 0 =_{df} 0$
$m \times s(n) =_{df} (m \times n) + n$

(d4) $m \leqslant n =_{df} \exists k\{ \mathrm{m} + k = n \}$

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[13] Note carefully, however, that Peano Arithmetic does not tell us what the numbers are. For a theory of this, the reader is directed to the earlier chapter "A Theory of Numbers". Rather, Peano Arithmetic only provides a means of deducing the various arithmetic facts about them; it only tells us how the numbers are inter-related.

[14] The term 'Latino sine flexione' comes from the title of his 1903 work *De Latino sine Flexione, Lingua Auxiliare Internationale*. Curiously, this work begins in classical Latin and gradually morphs into the very language Peano is proposing. Check this website for a description: [http://www.homunculus.com/babel/alsf]

## 4. Some Proofs in Peano Arithmetic

In this section, we present three theorems in Peano Arithmetic. The first one proves the (not entirely surprising) result that two plus two is four!

| | | |
|---|---|---|
| (1) | ~~SHOW~~: 2+2 = 4 | identity logic |
| (2) | $2+2 = s(1) + s(1)$ | d1.2 (twice) |
| (3) | $= s(s(0)) + s(s(0))$ | d1.1 (twice) |
| (4) | $= s(\ s(s(0)) + s(0)\ )$ | d2.2 |
| (5) | $= s(\ s(\ s(s(0)) + 0\ )\ )$ | d2.2 |
| (6) | $= s(\ s(\ s(s(0))\ )\ )$ | d2.1 |
| (7) | $= s(\ s(\ s(1)\ )$ | d1.1 |
| (8) | $= s(\ s(\ 2\ )\ )$ | d1.2 |
| (9) | $= s(\ 3\ )$ | d1.3 |
| (10) | $= 4$ | d1.4 |

Next, we prove the Associative Law, which is quite complicated (although not as complicated as the proof of the Commutative Law!) Note that, for the sake of conciseness, we write ‘ $x^+$ ’ in place of ‘ $s(x)$ ’.

| | | |
|---|---|---|
| (1) | ~~SHOW~~: $\forall x \forall y \forall z[(x+y)+z = x+(y+z)]$ | UD×2 |
| (2) | ~~SHOW~~: $\forall z[(a+b)+z = a+(b+z)]$ | 3,8, postulate p5 |
| (3) | ~~SHOW~~: $(a+b)+0 = a+(b+0)$ | DD |
| (4) | $(a+b)+0 = a+b$ | d2.1 |
| (5) | $b+0 = b$ | d2.1 |
| (6) | $a+(b+0) = a+b$ | 5,IL |
| (7) | $(a+b)+0 = a+(b+0)$ | 4,6,IL |
| (8) | ~~SHOW~~: $\forall x\ \{\ (a+b)+x = a+(b+x) \rightarrow (a+b)+x^+ = a+(b+x^+)\ \}$ | UCD |
| (9) | $(a+b)+m = a+(b+m)$ | As |
| (10) | ~~SHOW~~: $(a+b)+m^+ = a+(b+m^+)$ | DD |
| (11) | $(a+b)+m^+ = [(a+b)+m]^+$ | d2.2 |
| (12) | $= [a+(b+m)]^+$ | 9,IL |
| (13) | $b+m^+ = (b+m)^+$ | d2.2 |
| (14) | $a+(b+m^+) = a+(b+m)^+$ | 13,IL |
| (15) | $= [a+(b+m)]^+$ | d2.2 |
| (16) | $(a+b)+m^+ = a+(b+m^+)$ | 12,15,IL |

Finally, we prove that the $\leqslant$ relation, as defined by d4, is transitive. Notice that we appeal to the Associative Law.

| | | |
|---|---|---|
| (1) | ~~SHOW~~: $\forall xyz\{\ x \leqslant y\ \&\ y \leqslant z\ .\!\rightarrow\ x \leqslant z\ \}$ | U3CD |
| (2) | $a \leqslant b$ | As(a) |
| (3) | $b \leqslant c$ | As(b) |
| (4) | ~~SHOW~~: $a \leqslant c$ | DD |
| (5) | $\exists x\ \{\ a + x = b\ \}$ | 2,d4 |
| (6) | $\exists x\ \{\ b + x = c\ \}$ | 3,d4 |
| (7) | $a + m = b$ | 5,∃O |
| (8) | $b + n = c$ | 6,∃O |
| (9) | $(a + m) + n\ =\ c$ | 7,8,IL |
| (10) | $(a + m) + n\ =\ a + (m + n)$ | Associative Law |
| (11) | $a + (m + n)\ =\ c$ | 9,10,IL |
| (12) | $\exists x\ \{\ a + x\ =\ c\ \}$ | 11,∃I |
| (13) | $a \leqslant c$ | 12,d4 |