

A Theory of Numbers

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1. Introduction

In previous chapters, we have examined some of the philosophical issues involved in counting and numeration. In the present chapter, we attempt to answer the question “what are numbers?”. The theory we put forth traces to the ground-breaking work of Russell and Whitehead,¹ and may be called a “logical” theory of numbers. Here, the word ‘logical’ is not placed in opposition to ‘illogical’; presumably, no one *intends* to put forth an *illogical* theory of numbers. Rather, ‘logical’ here is associated with a philosophy of mathematics called ‘logicism’, which Russell and Whitehead attempted to defend.² However, note carefully that, although the viewpoint put forth in this chapter employs the logical insights of Russell and Whitehead, it more properly qualifies as a form of mathematical realism.³

Please keep in mind that, for the moment at least, we are concentrating on what are usually called *the natural numbers*.⁴ Basically, a natural number constitutes an answer to a “how many” question. Later, we will also consider what constitutes an answer to a “how much” question.

¹ Russell and Whitehead, *Principia Mathematica* (published 1910-13). Although it has a very similar title to Newton’s monumental work, it is not written in Latin. Nor is it *quite* as influential. Nevertheless, it is worthy of its lofty title – next to Aristotle’s *Organon*, it is regarded as the most influential work on logic ever written. Consult [<http://plato.stanford.edu/entries/principia-mathematica>].

² According to this view, all mathematical truths are ultimately logical truths. For example, see article alluded to in Note 1.

³ In the metaphysics game, *realism* is a viewpoint according to which universals (properties) exist independently of the mind. This is usually contrasted with *conceptualism*, according to which universals exist but *only* in the mind, and nominalism, according to which universals don’t exist at all, but are just words. Interestingly, this usage is nearly opposite to the ordinary usage of the word ‘realist’, which refers to a person whose feet are firmly grounded in the everyday world of facts, a person who would never consider day-dreaming about a world inhabited by abstract objects.

⁴ There seems to be confusion in the educational (i.e., K-12) literature. Many authors tell you that the natural numbers are 1,2,3,... However, the widespread *mathematical* opinion is that the natural numbers are 0,1,2,3,..., whereas the *counting* continued...

2. The Grammar of Number-Words

1. Nouns versus Adjectives

Dictionaries classify words into various parts of speech, including *nouns*, *verbs*, *adjectives*, etc. The classification is not entirely arbitrary; for example, nouns and adjectives are employed quite differently in discourse. In particular, whereas nouns are used *demonstratively*, adjectives are used *attributively*. On the one hand, we use a noun to point at (name) an object or collection of objects. On the other hand, we use an adjective to attribute a property to an object or collection of objects.

Many words have multiple grammatical categories; for example, color-words like ‘red’, ‘green’ and ‘blue’ are used both nominally (as nouns) and adjectively (as adjectives). The following very similar sentences illustrate the two uses of the word ‘blue’.

my favorite shirt is blue
my favorite color is blue

Whereas the first sentence uses ‘blue’ as an adjective, the second sentence uses ‘blue’ as a proper name. Correspondingly, whereas the first sentence uses the ‘is’ of predication, the second sentence uses the ‘is’ of identity. One way to see the difference is to invert the sentences as follows.

blue is my favorite shirt
blue is my favorite color

Whereas the first one sounds odd (or poetic, if you like), the second one sounds just as prosaic as the original sentence from which it was derived.

There are also artificial examples of the adjective-noun phenomenon. As the intro logic student learns, logicians have *invented* a rather artificial usage of the words ‘true’ and ‘false’ in connection with truth-values and truth-tables. The two different uses of ‘true’ are illustrated in the following examples.

my favorite proposition is true
my favorite truth-value is true
the truth-value of a true proposition is true

The first two sentences parallel our earlier sentences involving ‘blue’. The third sentence illustrates the connection between the adjectival use and the nominal use of ‘true’.

Now, if we adopt a quasi-Germanesque capitalization scheme, according to which we capitalize all *proper* nouns (but not *all* nouns), then we would capitalize ‘blue’ and ‘true’ whenever they are used as proper nouns, in which case we would write the following.

my favorite shirt is blue
my favorite color is Blue
my favorite sentence is true
my favorite truth-value is True
the truth-value of a true proposition is True

numbers are 1,2,3,... After 0,1,2,..., all other numbers – fractions, negative numbers, etc. – are in some sense *artificial*. A much quoted sentence from the mathematician Leopold Kronecker (1823-1891) goes as follows “*Die ganze Zahl schuf der liebe Gott, alles Übrige ist Menschenwerk*”. In other words, God created the *natural numbers*; all else is the work of humans. Thus, humans created the *artificial numbers*.

2. Number-Words: Numerical-Adjectives; Numerical-Nouns

As with color-words, number-words are used both as nouns and as adjectives.⁵ Just as with ‘blue’ and ‘true’, the logically *fundamental* use of number-words is adjectival, as in the following examples.

there is *one* god
 it takes *two* to tango
 Satan has *three* mouths⁶
 there are *seven* liberal arts⁷

In addition to their use as adjectives, number-words are also used as proper names, which are introduced very early in our education. For example, if I ask you any of the following questions,

what is your lucky number?
 what is your favorite number?
 what is your least favorite number?
 what is two-plus-two?

I expect an answer that uses a number-word, *not* as an adjective, but as a *proper name* (proper noun), . The following are examples of grammatically admissible answers.

my lucky number is seven
 my favorite number is sixty⁸
 my least favorite number is two-hundred eighty-eight⁹
 two-plus-two is four

Usually, we elide the introductory restatement of the question, and simply pronounce the underlined material. In any case, notice carefully that the number-words are used here as proper names; they function as predicate nominatives, which is to say that they function as “objects” that could just as easily be “subjects” of the sentence.

Over the next few sections, in trying to answer the question “what are numbers?”, we extensively examine number-words, first as adjectives, and then as nouns.

⁵ The word ‘adjective’ is used somewhat advisedly; a more precise term is ‘quantifier’. But ‘quantifier’ does not count in dictionaries as a part of speech. Also, a quantifier is – after all is said and done – a very special sort of adjective.

⁶ According to Dante (*The Inferno*), Satan occupies the innermost circle of Hell. He moreover has three mouths which respectively contain the villains Judas, Brutus, and Cassius. I think that the 20th Century provided three much better candidates for these positions – Hitler, Stalin, Mao – based on the number of people each of them murdered.

⁷ According to Plato, the liberal arts included Arithmetic, Geometry, Astronomy, and Music (by which he meant music theory; musical performance was regarded as a *practical* art, a trade). To this list, the Medieval universities added the following three – Logic, Grammar, Rhetoric. Whereas the latter three constituted the “trivium”, the former four constituted the “quadrivium”. In this connection, the word ‘trivia’ can be understood as breaking into ‘tri’ [three-fold] and ‘via’ [way]; similarly, ‘quadrivia’ breaks into ‘quadri’ [four-fold] and ‘via’ [way]. The trivium was considered more fundamental or elementary than the quadrivium – hence our modern word ‘trivial’. How a word for ‘elementary’ or ‘fundamental’ ultimately became associated with “unimportant information – for example, Marilyn Monroe’s favorite color” reflects (badly) on modern culture.

⁸ Because, as the Babylonians first noticed, 60 has very many useful factors, including 2,3,4,5,6.

⁹ Because it is two gross (a gross is a base-twelve unit, being a dozen dozen).

3. The Zero-Adjective

The elementary logic student is already familiar with rudimentary numerical adjectives, in the guise of *quantifiers*. In particular, the existential quantifier (\exists) is officially read so that:

$\exists x$ \equiv there is ***at least one*** individual x such that...

Notice that the quantifier concept “at least one” is logically complex:

at least one \approx one ***or*** more \approx one, ***or*** two, ***or*** three, ***or*** ...

On the other hand, the negation of ‘at least one’ is comparatively simple.

not at least one \approx none \approx zero

In particular, the compound quantifier ‘ $\sim\exists$ ’ is naturally read so that:

$\sim\exists x$ \equiv there is ***no*** individual x such that

For example, the proposition that there are no unicorns can be expressed by the following formula of predicate logic.

$\sim\exists x\{x \text{ is a unicorn}\}$

which reads:

there is *no* individual x such that x is a unicorn

In discussing how many unicorns there are, we can also employ set-talk and discuss

the set of all unicorns

which may be denoted by the following set-abstract.

$\{x : x \text{ is a unicorn}\}$

The latter expression is usually read as follows.¹⁰

the set of (all and only those individuals) x such that x is a unicorn

Now, it seems fairly clear that the following are equivalent.

the set of unicorns is empty
there are no unicorns
there are zero unicorns
the number of unicorns is zero

Thus, we can understand quantification, in this instance at least, as ascribing a property to sets. Ascribing a property to something is known as *predication*. If we formalize predicate notation in the

¹⁰ The usual reading is not strictly speaking grammatical, since it confuses singular and plural pronouns – the first ‘ x ’ is plural; the second ‘ x ’ is singular. We take this expression to be a lazy way of saying the following more grammatically fastidious formula of English.

the smallest set every element x of which is such that x is a unicorn

style of predicate logic, so that we write predicates first, arguments second, we can write these predications as follows.

EMPTY{ $x : x$ is a unicorn}

NO{ $x : x$ is a unicorn}

ZERO{ $x : x$ is a unicorn}

In each case, we understand the proposition as ascribing a property to the set of unicorns – a property that we have variously *expressed* by the terms ‘empty’, ‘no’ and ‘zero’.

We conclude this discussion by declaring precisely what the *adjectival-zero* object is.¹¹

the numerical-adjective ‘zero’ expresses the *zero-property* (a.k.a. Q), which is the property of *being a memberless set*

This is formally written as follows.

$$Q[A] \leftrightarrow A \text{ is a set } \& \sim \exists x \{ x \in A \}$$

Here, the special symbol ‘ \in ’ abbreviates ‘is a member of’; i.e.:

$$x \in y \quad =: \quad x \text{ is a member (element) of } y$$

Note carefully that, in virtue of the Principle of Extensionality, there is exactly one set with no members; it is usually called ‘**the** empty set’, which is designated as follows.

$$\emptyset \quad =: \quad \text{the set with no elements (a.k.a. the empty set)}$$

We accordingly have the following theorem.

$$Q[A] \leftrightarrow A = \emptyset$$

4. The One-Adjective

We have now shown how ‘zero’ is used as an adjective; it expresses a property that applies to a set precisely if that set is memberless. This property is not widespread, however, since there is only one memberless set, the empty set \emptyset .

We next consider whether this approach can be extended to “one” understood as an adjective. We start by noting that we already have the concept of “at least one”, which is simply the negation of “none”. This concept can be variously conveyed as follows.

$\exists x \{ x \text{ is a unicorn } \}$

NOT-EMPTY{ $x : x$ is a unicorn}

AT-LEAST-ONE{ $x : x$ is a unicorn}

NOT-ZERO{ $x : x$ is a unicorn}

¹¹ We must distinguish between the word ‘zero’, used as a quantifier expression, and the concept or individual this word stands for.

What we want, however, is the narrower concept of “exactly one”, which can be thought of as the conjunction of “at least one” and “at most one”. How do we logically convey this idea? For example, how do we logically analyze the following sentence.

Khufu (a.k.a. Cheops) built ***exactly one*** pyramid (*the* Great Pyramid)

We can re-describe this as follows (omitting the parenthetical information).

the set $\{x : x \text{ is a pyramid and Khufu built } x\}$ has ***exactly one*** element

How do we re-describe this using special predicates or quantifiers. The following are proposed.

$\exists!x \{ x \text{ is a pyramid and Khufu built } x \}$
 EXACTLY-ONE $\{x : x \text{ is a pyramid and Khufu built } x \}$
 $1 \{x : x \text{ is a pyramid and Khufu built } x \}$

The first formula involves a special quantifier ‘ $\exists!$ ’ which is read so that:

$\exists!x \quad \equiv \quad$ there is ***exactly one*** individual x such that ...

The second formula involves the special predicate ‘exactly-one’ which applies to a set precisely when that set has exactly one element. The third formula is simply a notational variant of the second.

So far, this is mere stenography, unless we can also come up with the logical principles governing these new concepts. As it turns out, ‘exactly-one’ can be logically characterized in a fairly simple manner, based on the concept of logical (numerical) identity.¹² Let us spend a little time understanding this idea.

1. ‘Is the Only’

In order to capture ‘exactly one’ we back up a bit and examine the notion of ‘is the only’ as used in the following sentence.

Kay *is the only* freshman

Here, let us imagine that the domain of quantification is the Philosophy of Science class. So we mean, in effect, that Kay is the only freshman in the Philosophy of Science class.

Now, what can we deduce from this bit of information? Well, it seems that we can deduce the following.

Kay is a freshman, **and**
 no one *else* is (a freshman)

A little further reflection will also convince us that we have the following logical principle.

k is the only F
 \leftrightarrow
 k is F and no one else is F

¹² This was first discovered by Russell and Whitehead near the end of the 19th Century, and is presented in *Principia Mathematica*.

Now, let's examine the bottom formula. The first conjunct is simple, and can be abbreviated in predicate logic as follows.

$$Fk$$

But how do we convey the second conjunct? Many novices write down something like the following.

$$\sim \exists x Fx$$

But this says that no one is a freshman, which contradicts the earlier formula. What we want is that no one *else* is a freshman. The word 'else' alludes to Kay, so we must somehow mention Kay in the second conjunct. Well, as a first approximation, we can write:

no one *other than* Kay is a freshman

which can be paraphrased as:

there is no one who is other than Kay and who is a freshman

which can be logically paraphrased as:

$$\sim \exists x \{ x \text{ is other than Kay, and } x \text{ is a freshman} \}$$

which can be partially abbreviated as:

$$\sim \exists x \{ x \text{ is other than } k \text{ \& } Fx \}$$

Now all we need to do is to figure out how to say 'is other than'. A little reflection reveals that 'is other than' is a variant of 'is distinct from'. A person other than Kay is simply a person who is distinct from Kay, which is to say a person who *isn't* Kay. Notice the reappearance of the 'is' of identity. In this connection, recall the following.

$$\begin{array}{ll} x = y & \text{=: } x \text{ is } y \\ & x \text{ and } y \text{ are the very same thing} \\ & x \text{ and } y \text{ are } \textit{one} \text{ and the same} \end{array}$$

so:

$$\begin{array}{ll} x \neq y & \text{=: } x \text{ isn't } y \\ & x \text{ and } y \text{ are } \textit{not} \text{ the very same thing} \\ & x \text{ and } y \text{ are } \textit{not one} \text{ and the same} \\ & x \text{ and } y \text{ are } \textit{distinct} \text{ (from each other)} \\ & x \text{ is } \textit{other than } y \end{array}$$

Substituting our logical account of "otherness" into the above formula, we obtain:

$$\sim \exists x \{ x \neq k \text{ \& } Fx \}$$

Accordingly, we can express the proposition that Kay is the only freshman using the following formula.

$$Fk \text{ \& } \sim \exists x \{ x \neq k \text{ \& } Fx \}$$

2. A More Succinct Formula

The formula reached at the end of the previous section is an excellent conceptual rendition of the proposition that k is the only F . It can, however, be improved upon from the point of view of conciseness. In this section, we distill the formula down a bit. First, by predicate logic principles, the second conjunct is equivalent to the following.

$$\forall x \{ Fx \rightarrow x = k \}$$

Next, in virtue of Leibniz's Law, the first conjunct is equivalent to the following.

$$\forall x \{ x = k \rightarrow Fx \}$$

Putting these two formulas together, using predicate logic principles, we obtain the following formula.

$$\forall x \{ Fx \leftrightarrow x = k \}$$

The more or less literal reading of this is:

for any individual x , x is a freshman if and only if x *is* Kay

The most natural reading of this formula into colloquial English is the following.

a person is a freshman if and only if that person *is* Kay (herself)

3. Using Set-Talk to Convey 'Is the Only'

Next, let us use set-talk to convey the idea that Kay is the only freshman. First, consider the set of freshman, which is depicted by the following set-abstract.

$$\{ x : x \text{ is a freshman} \}$$

Next, consider the set that contains Kay and no one else, which is depicted thus.

$$\{ \text{Kay} \}$$

Now, to say that Kay is the only freshman is to say that these two sets are the very same set; i.e.:

$$\{ x : x \text{ is a freshman} \} = \{ \text{Kay} \}$$

It might be instructive to show how this is equivalent to our earlier formulation of 'Kay is the only freshman'. First, we have the following principle about sets.

$$A = B \leftrightarrow \forall x \{ x \in A \leftrightarrow x \in B \}$$

Note that the " \rightarrow " half is a consequence of Leibniz's Law, whereas the " \leftarrow " half is the Principle of Extensionality. Based on this principle we have:

$$\begin{aligned} & \{ x : x \text{ is a freshman} \} = \{ \text{Kay} \} \\ \leftrightarrow & \forall x [x \in \{ x : x \text{ is a freshman} \} \leftrightarrow x \in \{ \text{Kay} \}] \end{aligned}$$

But we also have the following equivalences.

$$\begin{aligned} a \in \{ x : x \text{ is a freshman} \} &\leftrightarrow a \text{ is a freshman} \\ a \in \{ \text{Kay} \} &\leftrightarrow a = \text{Kay} \end{aligned}$$

So we ultimately have:

$$\begin{aligned} \{ x : x \text{ is a freshman} \} &= \{ \text{Kay} \} \\ \leftrightarrow \\ \forall x [x \text{ is a freshman} &\leftrightarrow x = \text{Kay}] \end{aligned}$$

4. From 'Is the Only' to 'Exactly One'

How many elements are in the set $\{\text{Kay}\}$? Well, exactly one – since $\{\text{Kay}\}$ contains Kay and *no one else*! Therefore, supposing that Kay is the only freshman – i.e., supposing that

$$\{ x : x \text{ is a freshman} \} = \{ \text{Kay} \}$$

then the set $\{x : x \text{ is a freshman}\}$ also has exactly one element.

We can partly summarize the above reasoning with the following simple logical implication.

if Kay is the only freshman,
then there is exactly one freshman.

Notice that there is nothing special about Kay here; it could be anyone. This may be summarized as follows.

if *anyone* is the only freshman,
then there is exactly one freshman.

This in turn is equivalent to:

if *someone* is the only freshman,
then there is exactly one freshman.

As it turns out, the converse of the latter conditional also holds¹³; namely:

if there is exactly one freshman,
then someone is the only freshman.

Putting these together, we have the following principle.

there is exactly one freshman
if and only if
there is someone who is the only freshman

We know how to say that Kay is the only freshman.

$$\begin{aligned} &Fk \ \& \ \sim \exists x \{ x \neq k \ \& \ Fx \} \\ \text{or:} \quad &\forall x \{ Fx \leftrightarrow x = k \} \\ \text{or:} \quad &\{ x : Fx \} = \{ k \} \end{aligned}$$

¹³ The oddity of 'anyone' is that whereas these two conditionals are logically equivalent, their converses are not!

How do we say that *someone* is the only freshman? By existential generalization over the proper name ‘*k*’, which yields the following (be careful to use a new variable!)

$$\begin{aligned} & \exists y \{ Fy \ \& \ \sim \exists x \{ x \neq y \ \& \ Fx \} \} \\ \text{or: } & \exists y \forall x \{ Fx \leftrightarrow x = y \} \\ \text{or: } & \exists y [\{ x : Fx \} = \{ y \}] \end{aligned}$$

Finally, we return to the special ‘exactly one’ quantifier ‘ $\exists!$ ’, which we are now in position to explicate.

$$\exists! x \Phi x \leftrightarrow \exists y \forall x \{ \Phi x \leftrightarrow x = y \}$$

5. The Definition of the One-Predicate

Earlier we defined the zero-predicate so that it satisfies the following principles.

$$\begin{aligned} \mathcal{Q}[A] & \leftrightarrow A \text{ is a set } \& \ \sim \exists x \{ x \in A \} \\ \mathcal{Q}[A] & \leftrightarrow A = \emptyset \\ \mathcal{Q}\{x : \Phi x\} & \leftrightarrow \sim \exists x \Phi x \end{aligned}$$

In other words, the predicate \mathcal{Q} applies to a set precisely if that set is memberless.

We can similarly define a one-predicate so that it satisfies the following principles.

$$\begin{aligned} \mathcal{1}[A] & \leftrightarrow A \text{ is a set } \& \ \exists! x \{ x \in A \} \\ \mathcal{1}[A] & \leftrightarrow \exists x [A = \{x\}] \\ \mathcal{1}\{x : \Phi x\} & \leftrightarrow \exists! x \Phi x \end{aligned}$$

In other words:

the numerical-adjective ‘one’ expresses the *one-property* (a.k.a. $\mathcal{1}$), which is the property of *being a single-membered set*.

Observe that there is a striking difference between \mathcal{Q} and $\mathcal{1}$ – whereas \mathcal{Q} applies to exactly one set, the empty set \emptyset , $\mathcal{1}$ applies to indeterminately-many sets.

6. Sets of Sets

As remarked in the previous section, we have the following.

\mathcal{Q} applies to *exactly one* set
there is *exactly one* memberless set

The latter expression can in turn be formalized as follows.

$$\begin{aligned} & \exists! x \{ x \text{ is a set and } x \text{ is memberless} \} \\ & \mathcal{1}\{x : x \text{ is a set and } x \text{ is memberless}\} \end{aligned}$$

But notice that, in the latter expression, we have formed the following set.

$$\{x : x \text{ is a set and } x \text{ is memberless}\}$$

Notice, in particular, that this is not a set whose members are ordinary individuals (e.g., persons, planets, and positrons). Rather, it is a set whose members are sets; well, actually, it only has one member, but that member is a set. In fact, we have the following identity.

$$\{ x : x \text{ is a set and } x \text{ is empty} \} = \{\emptyset\}$$

One might wonder whether the following is true.

$$\times \quad \{\emptyset\} = \emptyset$$

This claims that the set whose sole element is the empty set is the same as empty set. However, this can be easily dis-proven, using Leibniz's Law. Whereas $\{\emptyset\}$ has one element, \emptyset has no elements; therefore, since they differ in at least one feature, they can't be the same thing.

The moral is this: Counting presupposes collecting, whether we are counting pebbles, or whether we are counting sets.¹⁴ In order to count sets, we have to collect them, at least abstractly, which means that we have to consider sets whose members are themselves sets.

5. Zero, One, Two, Three, ...

We have constructed the zero-predicate, and the one-predicate. The remaining numerical predicates can be similarly constructed. The full list starts as follows, where we presuppose that the object A is a set to begin with.

$$\begin{aligned} 0[A] &\leftrightarrow \sim \exists i \{ i \in A \} \\ &\leftrightarrow A = \emptyset \end{aligned}$$

$$\begin{aligned} 1[A] &\leftrightarrow \exists z \forall i \{ i \in A \leftrightarrow i = z \} \\ &\leftrightarrow \exists z \{ A = \{z\} \} \end{aligned}$$

$$\begin{aligned} 2[A] &\leftrightarrow \exists yz \{ y \neq z \ \& \ \forall i \{ i \in A \leftrightarrow i = y \vee i = z \} \} \\ &\leftrightarrow \exists yz \{ y \neq z \ \& \ A = \{y, z\} \} \end{aligned}$$

$$\begin{aligned} 3[A] &\leftrightarrow \exists xyz \{ x \neq y \ \& \ x \neq z \ \& \ y \neq z \ \& \ \forall i \{ i \in A \leftrightarrow i = x \vee i = y \vee i = z \} \} \\ &\leftrightarrow \exists xyz \{ x \neq y \ \& \ x \neq z \ \& \ y \neq z \ \& \ A = \{x, y, z\} \} \end{aligned}$$

$$\begin{aligned} 4[A] &\leftrightarrow \exists wxyz \{ w \neq x \ \& \ w \neq y \ \& \ w \neq z \ \& \ x \neq y \ \& \ x \neq z \ \& \ y \neq z \ \& \\ &\quad \forall i \{ i \in A \leftrightarrow i = w \vee i = x \vee i = y \vee i = z \} \} \\ &\leftrightarrow \exists wxyz \{ w \neq x \ \& \ w \neq y \ \& \ w \neq z \ \& \ x \neq y \ \& \ x \neq z \ \& \ y \neq z \ \& \ A = \{w, x, y, z\} \} \end{aligned}$$

¹⁴ Counting also presupposes the notion of logical identity, but we know exactly what the identity conditions are for sets; these are given by the Principle of Extensionality.

6. Number-Words as Proper Nouns

So far, we have discussed the logically fundamental notion of numbers, which is associated with the use of number-words as adjectives, which are interpreted as properties of sets. For example, 0 applies to a set precisely when it is memberless, 1 applies to a set precisely when it is single-membered, 2 applies to a set precisely when it is double-membered, and so forth. This accounts for those uses of number-words such as the following.

there are no (zero -many) unicorns	$0\{x : x \text{ is a unicorn}\}$
there is exactly one unicorn	$1\{x : x \text{ is a unicorn}\}$
there are exactly two unicorns	$2\{x : x \text{ is a unicorn}\}$
there are exactly three unicorns	$3\{x : x \text{ is a unicorn}\}$
etc.	

As mentioned earlier, number-words are also used as proper names, as in the following examples.

my lucky number is seven
 my favorite number is sixty
 my least favorite number is two-hundred eighty-eight
 two-plus-two is four

Also, as mentioned earlier, whereas nouns are used *demonstratively*, adjectives are used *attributively*. Whereas we use a noun to point at (name) an object or collection of objects, we use an adjective to attribute a property to an object or collection of objects. For example, it makes no sense to ask what ‘tall’ is the name of; we do not use ‘tall’ in this manner. On the other hand, whenever a proper name is employed in discourse, we can naturally and rightfully ask what particular object is being picked out by that name. For example, if I say ‘Homer’, do I mean the author of *The Iliad*, or do I mean the American painter,¹⁵ or do I mean the patriarch of the Simpson family.¹⁶ To *whom* am I referring?¹⁷

As mentioned earlier, many words function both as adjectives and as proper names, including the words ‘blue’, ‘true’, and ‘two’. However, when these words are used as proper names, they are used demonstratively, and so we might naturally ask what particular objects they point at; if they are proper names, what do they name?

Probably the easiest and most economical answer to this question is that an adjective-like noun simply *names* what the original adjective *expresses*. For example:

the noun ‘Blue’ *names* what the adjective ‘blue’ *expresses*;
 the noun ‘True’ *names* what the adjective ‘true’ *expresses*;
 the noun ‘Two’ *names* what the adjective ‘two’ *expresses*.

What do the adjectives ‘blue’, ‘true’, and ‘two’ express? We understand the idea of adjective-expression in terms of rules of application. For example: The adjective ‘blue’ *applies* to an object precisely when that object is, well, blue! The adjective ‘true’ applies to a proposition precisely when

¹⁵ Specifically, Winslow Homer (1836-1910), who painted many well-known scenes of New England, especially New England fishing. See, e.g., [<http://www.boston.com/mfa/homer/jpgfog.html>].

¹⁶ One of many great paintings by Rembrandt is “Aristotle Contemplating a Bust of Homer” Check here for an image and also an “updated” version [<http://faculty.washington.edu/smcohen/arihomer.html>]. The page is part of Marc Cohen’s website, where you may also find other useful links to Ancient Philosophy.

¹⁷ Or using a number-set example, if I say ‘the population of China’, do I mean the residents of China, taken as a whole, or do I mean the size of this set?

that proposition obtains. The adjective ‘two’ applies to a collection precisely when that collection is double-membered. We can summarize this by saying:

the <i>adjective</i>	‘blue’	<i>expresses</i>	the <i>property</i> of being blue
the <i>adjective</i>	‘true’	<i>expresses</i>	the <i>property</i> of being true
the <i>adjective</i>	‘two’	<i>expresses</i>	the <i>property</i> of being two(-membered)

Accordingly, we can say:

the <i>noun</i>	‘Blue’	<i>names</i>	the <i>property</i> of being blue
the <i>noun</i>	‘True’	<i>names</i>	the <i>property</i> of being true
the <i>noun</i>	‘Two’	<i>names</i>	the <i>property</i> of being two(-membered)

Some people don’t like reference to properties, and prefer to talk about sets. In that case, we must rewrite the latter sentences as follows.

the <i>noun</i>	‘Blue’	<i>names</i>	the <i>set</i> of objects to which the adjective ‘blue’ applies;
the <i>noun</i>	‘True’	<i>names</i>	the <i>set</i> of objects to which the adjective ‘true’ applies;
the <i>noun</i>	‘Two’	<i>names</i>	the <i>set</i> of objects to which the adjective ‘two’ applies.

Or, to state things in the material mode:

Blue	=	{ x : x is blue }
True	=	{ x : x is true }
Two	=	{ x : x is two }

Here, using ‘two’ as a singular-adjective is very odd sounding; the more natural-sounding rendering is ‘two-membered’. Also, the objects to which ‘two’ (‘two-membered’) apply are not ordinary objects, but are rather sets. For example,

{Bach, Beethoven}	is	two	(i.e., two-membered)
{Bach, Mozart}	is	two	(i.e., two-membered)
{Mozart, Beethoven}	is	two	(i.e., two-membered)

And accordingly:

{Bach, Beethoven}	∈	Two
{Bach, Mozart}	∈	Two
{Mozart, Beethoven}	∈	Two

Or in property-talk,

{Bach, Beethoven}	has the property	Two	(i.e., being two-membered)
{Bach, Mozart}	has the property	Two	(i.e., being two-membered)
{Mozart, Beethoven}	has the property	Two	(i.e., being two-membered)

7. What is a Number?

Finally, we consider the original question – what are numbers? What is the number zero, the number one, the number two, etc.? The simple answer is that they are *quantities*; more specifically, they are *quantitative properties*; more specifically, they are quantitative properties that apply to sets. The properties include:

the property of being memberless
 the property of being single-membered
 the property of being double-membered
 etc.

On the one hand, these properties are *expressed* by the number-adjectives ‘zero’, ‘one’, ‘two’, etc. On the other hand, they are *named* by the number-nouns – ‘Zero’, ‘One’, ‘Two’, etc. For example,

the <i>adjective</i>	‘zero’	<i>expresses</i>	the property of being memberless
the <i>noun</i>	‘Zero’	<i>names</i>	the property of being memberless
the <i>adjective</i>	‘one’	<i>expresses</i>	the property of being single-membered
the <i>noun</i>	‘One’	<i>names</i>	the property of being single-membered
the <i>adjective</i>	‘two’	<i>expresses</i>	the property of being double-membered
the <i>noun</i>	‘Two’	<i>names</i>	the property of being double-membered
			etc.

In short:

The numbers Zero, One, Two, ..., are *properties of sets*; in particular:

Zero *is* the property of being memberless;
 One *is* the property of being single-membered;
 Two *is* the property of being double-membered;
 etc.

If, we are averse to property-talk, we can formulate the latter collection of principles as follows.

The numbers Zero, One, Two, ..., are *sets of sets*; in particular:

Zero *is* the set of all memberless sets;
 One *is* the set of all single-membered sets;
 Two *is* the set of all double-membered sets;
 etc.