Numbers and Counting

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1. Introduction

Probably the most fundamental science of all, and perhaps the most crucial to the rise of civilization, is arithmetic – which is oftentimes described as the *science of counting*. Since it is so fundamental both to science and to civilization, any philosophical enquiry into science should start with an enquiry into arithmetic and counting.

Counting is a special way of comparing sizes, the general theory of which we have discussed in an earlier chapter. More specifically, counting is a technique for assessing the *sizes* of sets. For example, the population of India is bigger than the population of Russia.[1]

[1] The latter claim is actually ambiguous; by 'population' do we mean the set of inhabitants, or the number of inhabitants? In this case, however, it turns out that the ambiguity is harmless. Whether we mean the set of inhabitants, or the number of inhabitants, the comparison is the same – one is bigger than the other.

2. Identity

Counting depends on the logical notion of *set*, which in turn depends logically on the notion of *identity*. Even more fundamental, counting directly depends on the logical notion of *identity*; when you count a collection of objects, you must be careful not to count the same thing twice. In order to avoid counting the same object twice, you must be able to recognize that object from one moment to another. This ability involves the logical concept of identity.

There are actually several inter-related notions conveyed by the English words 'identity', 'identify', and 'identical'. But fundamentally, these reduce to the notions of *qualitative identity* and *numerical identity*, which we describe in the next few sections.

1. Qualitative Identity

If we are told that a family has twins, we are naturally curious whether they are *identical* twins or *fraternal* twins. Both situations are noteworthy, but identical twins are considerably *more* noteworthy.[2] Along similar lines we might advertise a photo-copying device as so good that the copy is identical to the original. We might describe this by saying that the copy is *indistinguishable* from the original. This notion of identity is sometimes called 'qualitative identity'.

One key pragmatic feature of qualitative identity is that it *admits of degrees*; in particular, whether two things count as qualitatively identical – or indistinguishable – depends upon the standards we are applying. For example, in real life, no two twins are *exactly* identical, and no photo-copy is *exactly* identical to the original.

On the other hand, according to contemporary physics, all electrons are *intrinsically* exactly alike; in other words, every electron is an exact *duplicate* of every other electron. On the other hand, according to contemporary physics, no two electrons can occupy the same physical state, and so no two electrons are *extrinsically* exactly alike.[3] The intrinsic properties of an electron include its mass and charge; the extrinsic properties of an electron include its location and velocity.[4]

2. Numerical (Logical) Identity

In addition to *qualitative* identity, there is also *numerical* identity, also called *logical* identity. Consider the familiar sort of (classic) Star Trek episode, in which one of the crew members – say, Mister Spock – is "beamed" from one place to another. How does this work? The details are a bit mysterious, but the basic idea is this. On one end, a transmitting device decomposes Spock's molecular makeup into "energy", which is transmitted ("beamed") through space and picked up by a receiving device, which in turn recomposes Spock's molecular makeup – and *voilà*, Spock! Now, here is the key to this process, in virtue of which we are transmitting Spock. The creature who steps out of the receiving device is not merely an accurate *copy* of Spock; it ***IS*** Spock! At least, that is what we are led to believe.[5]

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[2] Note that fraternal twins are simply "litter mates", which is no big deal for dogs.
[3] This is known as the Pauli Exclusion Principle, named after Wolfgang Pauli (1900-1958; Nobel Prize in Physics, 1945).
[4] Modern physics adjusts these ideas considerably, so that the *relativistic mass* of an electron is extrinsic, although its *rest mass* remains intrinsic. All electrons have the same rest mass.
[5] On the other hand, perhaps the beaming machine is simply an exceedingly good copying machine that has the unfortunate side effect that the original is always destroyed in the copying process! I don't know about you, but I personally would never step into one of these machines!

Along similar lines, what if we adjust the Star Trek device so that it instead makes an exact copy of Spock, but leaves the original Spock in place. The creature coming out of the copying machine is qualitatively identical to Spock, but it isn't Spock. Even if a perfect copy were made of Spock, that copy would not ***be*** Spock. There would be *two* creatures – Spock, and Spock-Mark2.

This is the notion of *numerical* identity, or *logical* identity.

3. The Various Uses of 'Is' in English

Logical identity is conveyed in English by the verb 'to be', which conjugates as ⟨I *am*, you *are*, it *is*, etc.⟩. The verb 'to be' is quite versatile; philosophers distinguish at least three uses of the verb 'to be', which are respectively called.

the 'is' of **predication**
the 'is' of **existence**
the 'is' of **identity**

The 'is' of predication is familiar from elementary predicate logic, and is used in expressions such as the following.

Jay **is** tall	one-place predicate
Kay **is** taller than Jay	two-place predicate
Jay **is** between Kay and Elle	three-place predicate

The 'is' of existence is also familiar from elementary predicate logic, and in particular is used in the existential quantifier.

there **is** (i.e., **exists**) at least one person who is happy

The 'is' of existence also figures prominently in Shakespeare's most famous soliloquy.

to **be**, or not to **be**; that is the question

Here, Hamlet is debating whether to commit suicide – in other words, whether or not to exist.

Finally, there is the 'is' of identity, which is perhaps the philosophically trickiest, although we have been familiar with it since first-grade at least. The following question-answer dialog illustrates our first exposure to the 'is' of identity.

what ***is*** two plus two?
two plus two ***is*** four!

A logically similar dialog might be:

what ***is*** your favorite number?
my favorite number ***is*** seven!

who ***is*** the president?
Bush ***is*** the president!

4. A Little Grammatical Puzzle

Some words are used both as adjectives and nouns. This can occasionally present grammatical puzzles. Consider the following two sentences.

> my favorite shirt is blue
> my favorite color is blue

How is 'is' used in these sentences? How is 'blue' used in these two sentences? This little puzzle taken up again in a later chapter "A Theory of Numbers".

5. A Few More Examples of Identity

Let us consider some other familiar examples of identity. First of all, people generally need a valid "ID" (short for 'ID card', short for 'identity card') to do certain things, like buy alcoholic beverages.[6] Ideally, an ID card contains information on it, including a picture, that uniquely *identifies* you; to say that the card identifies you is simply to say that you ***are*** the person described by the ID card. Continuing our legal example, you might be asked by the police to "ID" (short for identify) a suspect from a book of mug shots. You might say something like "yes, that ***is*** the person I saw at the scene of the crime". Along similar lines, you might be showing an old family photograph to friends, and you might be asked to *identify* various persons in a photograph; for example, "who ***is*** this person; who ***is*** that person; …?"; to this you might respond "that ***is*** my dad; that ***is*** my mom"[7]

6. Leibniz's Laws

Gottfried Wilhelm Leibniz (1646-1716) was a German philosopher and mathematician, who is probably most famous for inventing the Calculus[8], which was also invented (independently!) by Sir Isaac Newton.[9] Leibniz thought very deeply about the relation between qualitative identity and numerical identity, and in particular proposed the following two laws, which are appropriately called 'Leibniz's Laws'. Unfortunately, both laws are called 'Leibniz's Law'; which one an author means will depend upon the context. We will refer to the two laws as 'Leibniz's Law – First Form', and 'Leibniz's Law – Second Form'.

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[6] Even people in their thirties and forties occasionally get carded at bars and liquor stores. What you have to worry about is when they *stop* carding you, because that means you are "over the hill".

[7] A joke from the fifties goes as follows. Who **was** that lady I saw you with last night? That **was** no lady! That **was** my wife! Although it is sexist, and although we should not laugh at it, this joke does illustrate the distinction between the 'is' of identity and the 'is' of predication. Which uses of 'was' (past tense of 'is') involve the 'is' of identity, and which uses involve the 'is' of predication.

[8] More specifically, the differential and integral calculus. Among mankind's many calculi (i.e., calculation procedures) , the one invented by Leibniz and Newton is regarded as so noteworthy and important that it is called "the" calculus. For the sake of comparison, among mankind's many books, one is regarded as so important that it is called "the" book (i.e., "The Bible").

[9] Newton and Leibniz (rhymes with 'tribe-ritz') never met, but they are forever linked in the history of philosophy and mathematics. In addition to inventing the Calculus, they had a famous debate about the nature of space; Newton defended an absolute conception of space, and Leibniz defended a purely relational conception of space. On a lighter note, they both have cookies named after them! I am a particular fan of the Choco-Leibniz, which is made in Leibniz's home town, Hanover Germany.

Leibniz's Law – First Form

identicals are indistinguishable

i.e.: if x and y are identical , then x and y are indistinguishable

i.e.: if x and y are *numerically* identical ,
then x and y are *qualitatively* identical.

i.e.: if x ***is*** y, then x and y have precisely the same properties

This law is generally regarded as a purely logical truth. After all, if object x ***is*** object y, then when you are talking about x, you are *ipso facto*[10] talking about y, since x and y are the exact same thing!

The other form of Leibniz's Law is given as follows.

Leibniz's Law – Second Form

indistinguishables are identical

i.e.: if x and y are indistinguishable, then x and y are identical

i.e.: if x and y are *qualitatively* identical,
then x and y are *numerically* identical.

i.e.: if x and y have precisely the same properties, then x ***is*** y

Unlike the first form of Leibniz's Law, the second form is not generally regarded as logically true, and accordingly counts as a non-trivial *metaphysical principle*. To clarify it somewhat, consider the following principle.

no two things are *numerically* identical

This simply follows from the meanings of the terms 'two' and 'numerically identical'. In order for *two* things to be numerically identical, they have to be the exact same thing, in which case they are not *two* things at all, but are rather *one* thing.

By contrast, the second form of Leibniz's Law states the following.

no two things are *qualitatively* identical

This is not a logical truth, but is a substantive principle, which may or may not be true. Note carefully that, when we say that two individuals are qualitatively identical, we mean that they share *all* their traits in common, including location in space.

[10] **ip· so fac· to** (^{1}p“s½ f^{2}k“t½) *adv*. By the fact itself; by that very fact: *An alien, ipso facto, has no right to a U.S. passport.* [*AHD*]

7. Important Note on Subsequent Usage

Henceforth, when the word ‘identical’ is used without a modifier, it is understood that it means ‘numerically identical’.

8. The Language Problem

One of the chief problems with understanding numerical identity is that the conventions of ordinary language frequently get in the way. Literally speaking, two things cannot be identical, because that would mean that there is only one thing, and not two things. Nevertheless, we use plural talk when talking about identity, even though the whole point of an identity claim is that the subject is singular. For example, Leibniz’s Law (2) has the following logical consequence no matter who a and b are.

> if a and b **are** indistinguishable, then a and b **are** identical

Notice that ‘are’ is plural, which strongly suggests that we are talking about more than one object; for example, one of them is a, and the other one is b. But when we say that *they* – i.e., a and b – are identical, then we are saying that “they” are not a “they” at all, but an “it”.

A similar problem occurs when we make a list of people, or places, or things. The following is an example.

(1) the president
(2) the first lady
(3) Laura Bush

How many items are in the list? Three. How many people are mentioned? Two. The list has three *references* to people, but the list is redundant, since Laura Bush is mentioned twice. We can report this fact by saying the following.

> Laura Bush = the first lady
> Laura Bush ***IS*** the first lady
> Laura Bush and the first lady are **one and the same**
> Laura Bush and the first lady are **the exact same person**
> Laura Bush and the first lady are **identical**

3. Sets

As mentioned earlier, counting depends logically on the notion of identity; without the notion of identity, counting is simply impossible. You cannot count a collection of things unless you can make sure that you haven’t counted the same thing twice. Of course, the other notion that counting depends upon is the notion of a *collection*, or *class*, or *set*. When we count, we count the members of a set. The notion of set in turn depends logically upon the notion of identity.

1. What is a Set?

The basic idea is that sets have *members* (also called elements). In this respect, sets are like clubs. But there is a big difference between clubs and sets. Two clubs – say, the Latin Club, and the Astronomy Club – can have the very same members – say Bill, Ted, and Alice. Nevertheless these are *two* clubs, not *one* club. By Leibniz’s Law (second form), since they are two rather than one, there must be something that distinguishes them; for example, the clubs might meet at different times, or they might have different agendas. Sets are different from clubs; unlike clubs, a set is *completely* determined/identified by its membership – same membership, same set. If set A has precisely the same members as set B, then A and B are in fact the exact same set, which is to say that they are (numerically) identical (i.e., A=B).

The individuation of sets by their members is known as the **Principle of Extensionality**, which is formally stated as follows.

(E) $\forall x(x \in A \leftrightarrow x \in B) \rightarrow A{=}B$

Note that the antecedent of (E) can be expanded by predicate logic as the conjunction of the following two formulas.

(a) $\forall x(x \in A \rightarrow x \in B)$
(b) $\forall x(x \in B \rightarrow x \in A)$

Whereas (a) says that every element of A is an element of B, (b) conversely says that every element of B is an element of A. Accordingly, (E) can be translated into English as follows.

if every element of A is also an element of B,
and every element of B is also an element of A,
then A and B are the exact same set (i.e., A ***is*** B).

2. Set-Abstracts

Sets may be denoted in various ways. In the simplest cases at least, the members are simply listed, and the resulting list is enclosed in curly braces, and the resulting expression denotes the set with precisely that membership. The following are examples.

(1) {Bach}
(2) {Bach, Mozart}
(3) {Bach, Mozart, Beethoven}

The first set has exactly one member – Bach. The second set has Bach and Mozart as members, and nothing else. The third set has Bach, Mozart, and Beethoven as members, and nothing else.

Notice that the order in which we write the members of a set is completely irrelevant. For example, the set {Bach, Mozart} and the set {Mozart, Bach} are the exact same set. In other words:

{Bach, Mozart} = {Mozart, Bach}

The listing technique works for small sets, but becomes unruly for larger sets. Accordingly, it is customary to introduce a special notation – called *set-abstract* notation – which is illustrated as follows.

{x : x lives in the White House}

which reads:

the set of all those x such that x lives in the White House

This denotes the set of all things living in the White House. There is no doubt a huge population of living creatures in the White House, especially if we include one-celled organisms, so this is a very large set indeed! Accordingly, let us narrow our attention as follows.

$\{x : x$ is a human and x lives in the White House$\}$

This is a much smaller set, and consists of precisely those humans living in the White House; I conjecture that this set is identified as follows.

$\{x : x$ is a human and x lives in the White House$\}$ = {George W. Bush, Laura Bush}

On the other hand, the following set

$\{x : x$ is a Democrat and x lives in the White House$\}$

has *no* members. This is alternatively stated as follows.

$\{x : x$ is a Democrat and x lives in the White House$\} = \varnothing =_{\mathrm{df}}$ the empty set

4. Counting

Children learn to count at a very early age, an intellectual feat that (surprisingly) took humanity thousands, or even millions, of years to accomplish. Artifacts discovered in Western Europe – bones into which notches are carved – suggest that humans started counting *at least* 30,000 years ago, and they started tallying at least 20,000 years ago.[11] These are the first known number *tokens*, which are the ancient precursors of numerals.

But what *is* counting? Let us do a little anthropological thought experiment. Imagine we live in ancient times long before counting was formalized. Imagine we are hunter/gatherers who live in family groups, perhaps even villages. Now, imagine that your particular family chore is to catch exactly one fish for each family member. How do you do this? Well, you could take your family down to the stream and catch one fish for each family member – one fish for this person, one fish for that person, and so on. This works in principle, but it is massively inconvenient. Is there an easier way?

You have a hunch; you formulate an hypothesis! What if you take a leather string, and tie one knot in the string for each family member – one knot for this person, one knot for that person, and so forth. Now, you go down to the stream and catch one fish for each knot – one fish for this knot, one fish for that knot, and so on. Finally, you go back home to test your hypothesis – namely, that your knot procedure will indeed produce one fish for each family member. And it does!

At this point you have *confirmed* your original hypothesis, which roughly goes as follows.

if I tie exactly one knot in the string for each family member,
and I catch exactly one fish for each knot in the string,
then I will have caught exactly one fish for each family member.

At this point, we have a vexing logical problem. We have confirmed that the knot-tying procedure worked *today*. We have not confirmed that it will work *tomorrow* or the next day. Humans want/need the world to be regular; we presume that whatever we learn is completely general. For example, many young children believe briefly that all furry animals are dogs. Adults are no different; it's just that they have a lot more data at their disposal.

[11] Tallying basically involves keeping track of a number by using consecutive marks arranged into groups of five each. More literally it means using sticks for this purpose. The tally system (marked sticks) is still in use today in some parts of the world, and was used in the British Houses of Parliament until the early 19th Century.

And scientists are no different in this regard; their generalization-instinct is always working full-speed. The difference however is attitude; specifically, a generalization suggested by observation is often framed as a *working hypothesis*, which suggests further observations to be carried out. For example, in the knot example, the working hypothesis would probably be:

whenever I apply my special knot procedure, it will work

This in turn can be tested against experience. You can test the hypothesis passively or actively. For example, you can test your hypothesis whenever you are asked by the elders to bring back fish. Or, you can test your hypothesis under controlled circumstances, in which you consider the influence of various factors. For example, does it only work in the day, or does it also work at night? Does it work no matter what string you use? Does it work for other families different from your own. Can other people apply this procedure, or are you somehow endowed with a special power? Lots of questions need to be answered. But as you test your hypothesis against more and more data, you become increasingly confident that it is true.

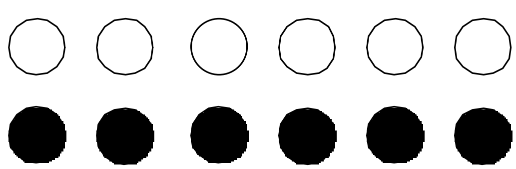
Notice also that you haven't merely discovered some idle piece of existential philosophy (e.g., "I am somebody"). No, you have *invented* a great labor-saving device – a bit of technology! Taking along your knotted string on a fishing expedition is much easier than taking along your whole family. Perhaps this technique will be passed along to other families, and along to future families. Perhaps you will be famous; perhaps not!

Now back to the liberal perspective. You haven't merely invented a valuable tool; you have also discovered an important mathematical fact, though you may not realize it yet (either you the ancient inventor, or you the current student!) Namely, if two sets (e.g., family and fish) stand in a one-to-one correspondence to a third set (e.g., knots) then they stand in a one-to-one correspondence to each other. This idea bears explaining.

A **one-to-one correspondence** between two sets, A and B, is a pairing of members of A with members of B subject to the following two restrictions.

(1) *every* member of A is paired with *exactly one* member of B;
(2) *every* member of B is paired with *exactly one* member of A.

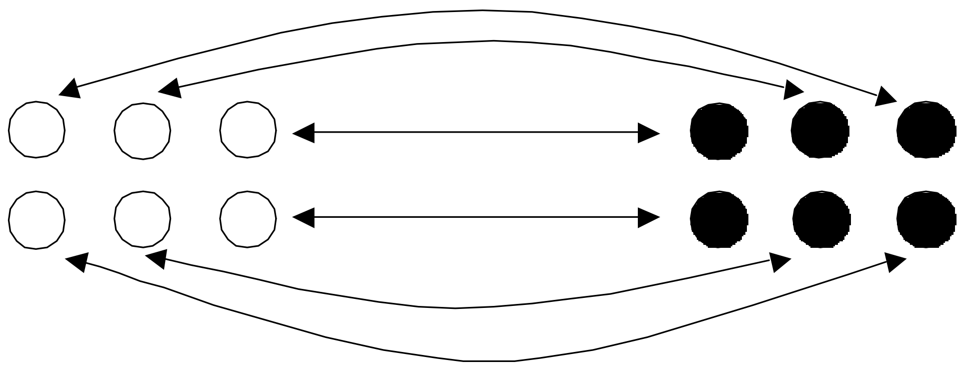
Exactly how a particular pairing is achieved is immaterial. On the one hand, a pairing can be *concrete*. For example, suppose set A consists of white pebbles and set B consists of black pebbles. A pairing of A-members and B-members may then consist of taking white pebbles and black pebbles, and simply physically aligning them, as in the following diagram.



Note carefully that, in our ancient scenario, we still do not know *how many* white pebbles there are, or *how many* black pebbles there are; we only know that the set A of white pebbles and the set B of black pebbles are in some sense *equally big*. This is based on the following principle (definition).

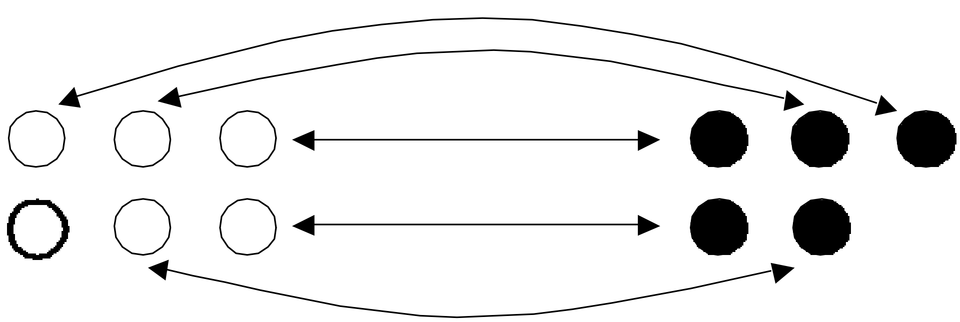
> Two sets A and B are *equally big* if and only if they can be placed in a one-to-one correspondence.

The pairing above is concrete and physical. A pairing may also be more abstract or mental. We might simply mentally pass over the members of A and the members of B and set up a mental pairing, without actually physically aligning them – as in the following diagram.



Here the arrows in the drawing are not actually physically in front of you; rather, you see them with your "mind's eye".

So far, we have discovered an important fact about sameness/equality of size. Recall that in the chapter on theories, we discuss generic size theory. In that theory, we take "bigger" as logically antecedent to "equally big". This does not automatically mean, however, that it is historically antecedent. In our scenario, we have discovered size-sameness first. We have not necessarily discovered the "bigger than" relation just yet. So, next in our ancient road to discovery, we develop an *ordering system* for set sizes. For example, we might recognize that there are *more* family members than knots, or *more* knots than fish, or *more* white pebbles than black pebbles. This is associated with an attempted one-to-one correspondence that fails. This is illustrated in the following diagram.



In this situation, every black pebble is paired with exactly one white pebble, but not every white pebble is paired with a black pebble. There is a white pebble left over, marked by a darker outline. In other words, in the pairing above, every black pebble corresponds to a white pebble, but not every white pebble corresponds to a black pebble. We can summarize this by saying that the set of white pebbles is *bigger than* the set of black pebbles; alternatively there are *more* white pebbles than black pebbles.[12]

The next step is to use this insight to construct a *counting system* based on knots in strings. This system, in particular, consists of a *series* of strings, one with one knot, one with two knots, one with three knots, etc.

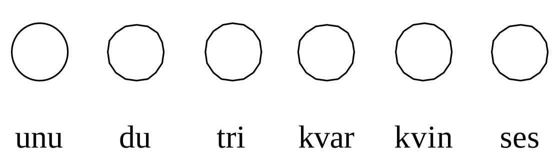
At this point, the strings are physical tokens that stand directly for numbers; we don't have words for numbers yet. On the other hand, we might very well develop names for the various knotted strings. Thus, we have physical tokens for numbers, which are actual physical devices that we employ when we count. We also have words for the tokens; we might even have written symbols for the tokens, either special iconographic symbols that look like the tokens, or phonetic representations for the words we have chosen to name them.

But, amazingly, what we don't yet have are words for the numbers! It is thought that this step in the development of mathematics was very difficult for humankind, and indeed it is still difficult for individual humans. Mesopotamia, an ancient region of southwest Asia between the Tigris and Euphrates rivers in modern-day Iraq, was the home of numerous early civilizations, including Sumer, Akkad, Babylonia, and Assyria. Now, a bright Sumerian scientist got the following idea. The symbol stands for the physical token, which represents the number.[13] Why not let the symbol represent the number directly? Eureka! We have our first case of by-passing the middleman! The difficulty in achieving this simple conceptual breakthrough may be regarded as anthropological evidence of just how hard it is for the ape-like masters of the Earth to master abstract concepts, including numbers. Rather, the numerical tokens seem to take on a special significance *in place of* what they represent.[14]

Now, suppose for the moment that the words[15]

unu, du, tri, kvar, kvin, ses, sep, ok, naux, dek

stand respectively for a one-knotted string, a two-knotted string, etc. By bypassing the middlemen – i.e., the knotted strings – we have a series of words for the first ten numbers themselves. Now, we can start counting in the manner we are all accustomed to. Counting a set of objects – say pebbles – consists in setting up a one-to-one correspondence between the members of that set, on the one hand, and numbers/numerals on the other hand. First, we pick a pebble, and assign 'unu' to it; next, we pick another pebble, and assign 'du' to it; then we pick yet another pebble, and assign 'tri' to it. Furthermore, the process goes on until we have no pebbles left to count. The following diagram illustrates this process.

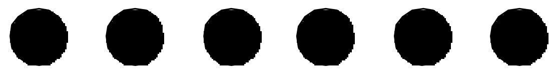


[12] These ideas have to be very carefully reformulated when we talk about *infinite* sets, but in our story that discovery is 30,000 years down the road! More about this later.
[13] Check at [www.sumerian.org/tokens.htm] for a picture of Sumerian number tokens made of stone.
[14] It's as if, as a species, we are inclined to worship false idols.
[15] These number-words come from Esperanto, the most prominent "planned" language, which was devised in 1887 by L.L. Zamenhof of Warsaw (then in Russia). It is based on elements of the major Western European languages. It is estimated that about two million people speak and/or write Esperanto. Go here for a website written in Esperanto [http://www.esperanto.net/veb/].

In counting, what we have actually done is to set up a one-to-one correspondence between the white pebbles, and an *initial series* of numerals/numbers. We declare the result of our counting as follows.

the number of white pebbles is “ses”
or: there are “ses” white pebbles

We can also count the black pebbles in our earlier example, and that would go as follows.



unu du tri kvar kvin ses

We now declare the result of our counting as follows.

the number of black pebbles is "ses"
or: there are "ses" black pebbles

At this point, we can further declare that

the number of white pebbles and the number of black pebbles is *the same*

This could be symbolized as follows.

$$\#(A) = \#(B)$$

This is a logical consequence of the following two facts.

$$\#(A) = \text{ses}$$
$$\#(B) = \text{ses}$$

5. Some Important Distinctions

We have used both the term ‘number’ and the term ‘numeral’. When we count, are we using numbers, or are we using numerals? What is the difference?

1. Numerals versus Numbers

Although ordinary language is very lax with regard to the distinction between numerals and numbers, there is a crucial philosophical distinction, which is described as follows.

Numbers are quantities; numerals are symbols; numerals are names of numbers;

The connection between numbers and numerals, and more generally numerical expressions, is quite straightforward, and is fundamental to understanding arithmetic. Arithmetic is the “science of counting”; more specifically, it is a mathematical theory of (about) the natural numbers (0, 1, 2, …) In particular, arithmetic *uses* names of the various numbers to *mention* them. These names are numerals, which include the atomic numerals (e.g., ‘0’, ‘1’, ‘2’, …) and the molecular numerals (e.g., ‘10’, ‘42’, ‘101’).

Now, there is exactly one system of numbers (supposing mathematical realism). On the other hand, there are many quite distinct numeral systems, including Egyptian, Babylonian, Mayan, Hebrew, Greek, Roman, Arabic[16], etc., although nearly everyone calls these “number systems”. Furthermore, within the Arabic numeral system, currently adopted by most of the world, there are many numeral *sub*systems. – decimal, binary, octal, hexadecimal, etc.

Of course, having made a critical philosophical distinction, which is called the number-numeral distinction, we must resign ourselves to the fact that we cannot legislate usage in natural language. Ordinary language uses the word ‘number’ for virtually any numerical expression or concept. The following are a few examples.

room numbers
building numbers
street numbers
house numbers
page numbers
telephone numbers
social-security numbers
credit card numbers
lottery numbers
flight numbers
etc.

In most cases, these items are not numbers – i.e., quantities – but rather *numerical labels*. The advantage of using numerical labels rather than ordinary names – like ‘Bill’ and ‘Sue’ – is that there is a simple and reliable technique for generating zillions of numerical labels, unlike ordinary names.

Here is a basic rule of thumb, to tell whether number-like objects are *really* numbers – i.e., quantities – ask yourself whether it makes any sense to add or subtract them.

2. Cardinal Numbers Versus Ordinal Numbers

In describing numbers as discrete quantities (set sizes), we have been concentrating on *cardinal numbers*,[17] which are distinguished from *ordinal numbers*.[18] The basic idea is that, whereas cardinal numbers are quantities, ordinal numbers are *rankings*. Many languages distinguish these concepts

[16] The so-called "Arabic" numerals entered Europe in about 1200 A.D. (C.E.) from the Islamic World (Middle East, Northern Africa). However, it is generally acknowledged now that they entered the Islamic World (specifically Baghdad) from India, around AD 771. They are accordingly more properly referred to as the “Hindu-Arabic numerals”. Also, don’t expect to see “Arabic numerals” in Cairo or Baghdad; the numerals used in Arab countries don’t look very much like the ones we use.

[17] The word ‘cardinal’ has several uses in English, interestingly related to one another. The fundamental use means ‘foremost’ or ‘paramount’. The cardinal sins are the really big ones! Along the same vein (or is that vain?), the cardinals of the Roman Catholic Church are prominent officials just below the Pope (from ‘Il Papa’, the Holy Father). Cardinals dress in very flashy red garments, a lot like the birds named after them.

[18] Like the word ‘cardinal’, the word ‘ordinal’ has several uses, although they are all technical. The fundamental meaning refers to *order*, as in ‘batting order’, [not as in ‘law and order’, or ‘may I take your order’] . Other uses of ‘ordinal’ pertain to ordination, a process by which one is ordained, as in a church.

syntactically. For example, in English, whereas the cardinal number-words are ‘one’, ‘two’, ‘three’, etc., the corresponding ordinal number-words are ‘first’, ‘second’, ‘third’, etc.[19]

On the other hand, ordinary English usage conflates these concepts in numerous places; for example, sports fans might repeatedly chant “we’re number one!” [which can be very annoying!] This expression uses a cardinal number-word ‘one’ to convey an ordinal number concept. The phrase ‘we’re number one’ is not a statement about our size, but about our ranking (say, in the world of college basketball). In this usage, ‘we’re number *one*’ is synonymous with ‘we’re *first*’ or ‘we’re the best’.[20]

The ordinal concept of number accounts for the expression ‘page number’. The page "number" at the top or bottom of a page, by which I mean the actual page *label*, is a numeral that indicates the relative *ordinal position* of the page within the document; thus, page "numbers" are ordinal-numerals. On the other hand, in the header of this document, you will find an expression of the form ‘page x of y’. The first numeral – ‘14’ – is used ordinally; it indicates the relative location of the page within the overall document. On the other hand, the second numeral – ‘16’ – is used cardinally; it indicates the size (number of pages) of the overall document.

Street numbers are also ordinal numbers. For example, it would be safe to presume that Forty-seven<u>th</u> Street and Fif<u>th</u> Avenue (e.g., in NYC) are related in an <u>order</u>ly (i.e., systematic) fashion to other streets similarly named. Whereas NYC uses street-avenue nomenclature, Washington D.C. labels streets by numerals and letters – for example ‘K Street’. But the ordinal use of letters is just as simple as the ordinal use of numerals – that is, so long as we agree on what the alphabetic order is. But for this we have a song![21]

Room numbers are analogous to street numbers, although their orderliness is not always entirely obvious. You initially expect (or at least hypothesize) that Bartlett 363 to be *between* Bartlett 362 and Bartlett 364. This may not work out, however, as you are doubtless aware from you dealings with room numbers. Other rules are often in use in the "design" and "evolution" of room numbers – for example, the even-odd rule. This applies equally to house numbers, which are also ordinal numbers. Again, there is usually *some* method to the madness, but you have to work out the exact details for each street you happen upon.[22]

We will return to the philosophical and scientific differences between cardinal numbers and ordinal numbers again when we discuss *measurement* in a later chapter.

[19] The basic rule is – to obtain an ordinal-word, one takes a cardinal-word and suffixes ‘th’, but there are numerous exceptions. English is an *evolved* language, not a *designed* language, as evidenced by all the exceptions to the rules, in spelling, morphology, and grammar. By contrast, Esperanto is a *designed* language – no exceptions to any rule. For example, the ordinal-word is obtained from the cardinal-word simply by suffixing ‘a’. Thus: *unua* (first), *dua* (second), *tria* (third) *kvardek-nauxa* (forty-ninth).

[20] This is not to say that the ranking might be cardinally based. We might be number one in size. But that is not the same as saying that we are one in size, which would mean that the set $\{x : x \text{ is one of us}\}$ has just one element.

[21] Twinkle, Twinkle, E, F, G; yes sir, yes sir, L,M,N,O,P!

[22] When I built my house I had to submit a house-number proposal to the Town of Amherst; given the location of the house, quite a few numbers were admissible.

3. Numerals versus Letters

The confusion between numbers and numerals begins at an early age. For example, sometimes children simply recite the first few numerals, just as they recite the letters of the alphabet.[23] But numerals and letters are very different. Whereas numerals represent quantities, letters by and large represent the sounds we use in forming spoken words.[24] A word represents a concept; its letters represent the sounds used to pronounce the word.[25] By contrast, any string of numerals counts as a word.[26]

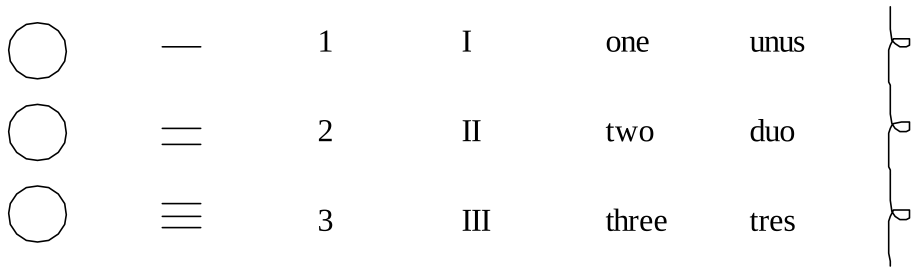
Since the numerals represent quantities, they have a natural order. By contrast, the order of the letters of the alphabet is completely conventional. Also conventional are the phonetic rules for these letters. For example, the "r" sound is represented by the letter 'R' in the Roman alphabet, but by the letter 'P' (rho) in both the Cyrillic and Greek alphabets. Even languages that share the Roman alphabet do not entirely agree on the phonetic rules for the letters.[27] Sounds that are commonplace in one language may simply not occur in other languages, which accordingly have no letters for these sounds. More profoundly, concepts may be represented in one language but not another; for example, what is the *English* word for 'burrito'?[28]

Not all items of the world get equal billing in all languages. Nevertheless, there is one very important exception – every modern language has number-words that stretch to infinity![29] And every civilization has chosen to name the very same numbers, although the words and symbols they choose vary greatly.

6. When we Count, are we Using Numbers or Numerals?

The answer is that it doesn't really matter! No matter how we count, we are setting up a one-to-one correspondence between a *target set* and an appropriately large set of *known objects*, whether it is the set of numbers themselves {1, 2, 3}, or the English number-names {'one', 'two', 'three'}, or the Latin number-names {'unus', 'duo', 'tres'}, or the Roman numerals {'I', 'II', 'III'}, or the Arabic numerals {'1', '2', '3'}, or even the set of knots in our number-cord. The following illustrates the correspondence.

[23] One can even sing the first seven English number-words to the famous alphabet song; "one, two, three, four, five six, seven…" after that things break down rather badly.
[24] Sometimes, of course, a single letter constitutes a word, as in 'a' and 'I'. Notice that I am ignoring languages, such as Chinese, which do not have a phonetically-written language.
[25] Ideally, a language is almost completely phonetic, like Italian. English, on the other hand, is riddled with goofy phonetic rules; for example, the string 'ough' is pronounced in five different ways – tough, through, though, bough, ought.
[26] What number a given string – say '111' – represents depends upon the base one is using. In base-ten, '111' stands for the number one-hundred-and-eleven; in base-two, '111' stands for the number seven.
[27] My favorite phonetics "joke" is that German has no 'j' sound; think about it!
[28] This is a trick question. The word 'burrito' is an English word. If English lacks a decent translation for a given foreign word, the best thing to do is simply to adopt the foreign word, and it will eventually become English; why make up a word for 'burrito' when there is already a perfectly good one available? This also works the other way; look at foreign-language web sites to see how many "foreign" words you already know!
[29] Needless to say, most number words have never been uttered, and never will be uttered. Nevertheless, they are all there in our language, a discovery that brings awe, wonder, and puzzlement to a child when s/he discovers this amazing fact.



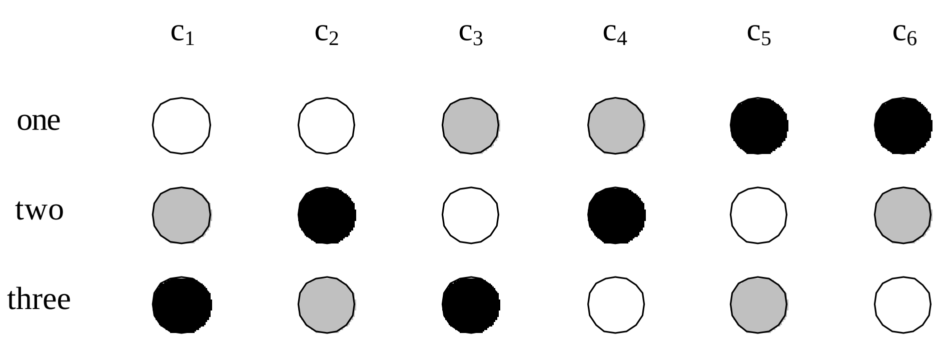
Here, I am *pictographically* representing all the objects under consideration, including the pebbles, the numbers, the knots, as well as the various numerical symbols and words. Note that there is a one-to-one correspondence is between the set of pebbles and *any* of the following sets.

{1, 2, 3}	the numbers themselves
{'1', '2', '3'}	Arabic numerals
{'I', 'II', 'III'}	Roman numerals
{'one', 'two', 'three'}	English number-words
{'unus', 'duo', 'tres'}	Latin number-words
$\{\text{knot}_1, \text{knot}_2, \text{knot}_3\}$	the physical knots in our string

Based on this counting procedure, we conclude that there are three pebbles.

7. A Remaining Issue

The above example may hide a very important feature of counting – namely, there are generally many different ways of counting a given target set. In order to see this, we adjust our example so that the collection under consideration consists of one white pebble, one gray pebble, and one black pebble. In this case, there are six different ways to count this set, which are illustrated as follows.



For example, in the first counting – c_1 – we count the white pebble first, the gray pebble second, and the black pebble third. On the other hand, in the sixth counting – c_6 – we count in the exact opposite order. All told, there are six ways to count set A. On the other hand, irrespective of which order we choose, the result is the same – we have a one-to-one correspondence between the set of pebbles and the set {1,2,3}. This is summarized in the following fundamental principle of counting.

> The order in which we count the elements of a set is irrelevant to the result.[30]

[30] This principle applies to *finite* sets only. *Infinite* sets present unexpected difficulties.